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Twisted Alexander Invariants Detect Trivial Links

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Abstract. It follows from earlier work of Silver and Williams and the authors that twisted Alexander polynomials detect the unknot and the Hopf link. We now show that twisted Alexander polynomials also detect the trefoil and the figure-8 knot, that twisted Alexander polynomials detect whether a link is split and that twisted Alexander modules detect trivial links. We use this result to provide algorithms for detecting whether a link is the unlink, whether it is split, and whether it is totally split.

1 Introduction and Main Results

An *(oriented) m-component link* $L = L_1 \cup \cdots \cup L_m \subset S^3$ is a collection of m disjoint smooth oriented closed circles in S^3. Given such link L we denote by ϕ_L the canonical epimorphism $\pi_1(S^3 \smallsetminus L) \to \langle t \rangle$, which is given by sending each meridian to t. Given a representation $\alpha\colon \pi_1(S^3 \smallsetminus L) \to \mathrm{GL}(k, \mathbb{C})$ we will introduce in Section 2.1 the corresponding twisted Alexander $\mathbb{C}[t^{\pm 1}]$-module $H_1^{\alpha\otimes\phi_L}(S^3 \smallsetminus L; \mathbb{C}[t^{\pm 1}]^k)$.

The purpose of this paper is to discuss to what degree the collection of twisted Alexander modules detects various types of links. The model example is the following. We can extract information from these modules by looking at their order; in particular, following Lin [Lin01] and Wada [Wa94] we can define the one-variable twisted Alexander polynomial $\Delta_L^\alpha \in \mathbb{C}[t^{\pm 1}]$. Silver and Williams [SW06] proved that the collection of twisted Alexander polynomials detects the trivial knot among 1-component links, *i.e.*, knots. More precisely, if $L \subset S^3$ is a knot, then L is the unknot if and only if $\Delta_L^\alpha = 1$ for all representations $\alpha\colon \pi_1(S^3 \smallsetminus L) \to \mathrm{GL}(k, \mathbb{C})$.

We thus see that twisted Alexander polynomials detect the unknot, and in a similar vein, we showed in [FV07] that twisted Alexander polynomials detect the Hopf link. It is natural to ask whether twisted Alexander modules characterize other classes of knots and links. The purpose of this paper is to discuss a number of cases where the answer is affirmative. We will now present the main results, referring to the following sections for the precise statements. Our first result is Theorem 3.1, which significantly improves upon [FV07, Theorem 1.3] and can be summarized as follows.

Theorem 1.1 *Twisted Alexander polynomials detect the trefoil and the figure-*8 *knot.*

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The second result asserts that twisted Alexander modules detect split links (recall that a link L is *split* if there exists a 2-sphere $S \subset S^3$ such that each component of $S^3 \setminus S$ contains at least one component of L).

In order to state the result we need two more definitions. First, we denote by $\operatorname{rk}(L, \alpha)$ the rank of the twisted Alexander module, *i.e.*,

$$\operatorname{rk}(L, \alpha) := \operatorname{rk}_{\mathbb{C}[t^{\pm 1}]} H_1^{\alpha \otimes \phi_L}\big(S^3 \setminus L; \mathbb{C}[t^{\pm 1}]^k\big).$$

Secondly, in this paper we say that a representation $\alpha\colon \pi_1(S^3 \setminus L) \to \mathrm{GL}(k, \mathbb{C})$ is an *almost-permutation representation* if given any g the matrix $\alpha(g)$ has precisely one non-zero value in each row and each column, and each non-zero entry is a root of unity. We now have the following result.

Theorem 1.2 *If a link L is split, then for any representation $\alpha\colon \pi_1(S^3 \setminus L) \to \mathrm{GL}(k, \mathbb{C})$, we have $\operatorname{rk}(L, \alpha) > 0$. Conversely, if L is not split, then there exists a representation $\alpha\colon \pi_1(S^3 \setminus L) \to \mathrm{GL}(k, \mathbb{C})$ with $\operatorname{rk}(L, \alpha) = 0$. Furthermore, the representation can be assumed to be an almost-permutation representation.*

(A more detailed result, relating $\operatorname{rk}(L, \alpha)$ with the splittability of L, is presented in Section 2.1.)

Note that the condition $\operatorname{rk}(L, \alpha) > 0$ is equivalent to the vanishing of Δ_L^α. The first statement of the theorem thus also asserts that twisted Alexander polynomials cannot distinguish inequivalent split links; in particular, they fail to characterize the trivial link with more than one component. However, whenever the twisted Alexander module is not torsion, we can define a secondary invariant, defined as the order of the torsion part of the twisted Alexander module. More precisely, we consider the following invariant:

$$\widetilde{\Delta}_L^\alpha := \operatorname{ord}_{\mathbb{C}[t^{\pm 1}]}\big(\operatorname{Tor}_{\mathbb{C}[t^{\pm 1}]} H_1^{\alpha \otimes \phi_L}(S^3 \setminus L; \mathbb{C}[t^{\pm 1}]^k)\big).$$

(We refer to Section 2.1 for details.) We can now formulate our third main result.

Theorem 1.3 *An m-component link L is trivial if and only if for any almost-permutation representation $\alpha\colon \pi_1(S^3 \setminus L) \to \mathrm{GL}(k, \mathbb{C})$ we have $\operatorname{rk}(L, \alpha) = k(m-1)$ and $\widetilde{\Delta}_L^\alpha = 1$.*

In order to prove the theorems above we will build on the results of [FV13, FV15], where we showed that twisted Alexander polynomials determine the Thurston norm and detect the existence of fibrations for irreducible 3-manifolds with non-empty toroidal boundary. These results in turn rely on the virtual fibering theorem of Agol [Ag08] and the work of Wise and Przytycki [Wi09, Wi12a, Wi12b, PW12].

Remark Note that if L is non-split or non-trivial, there exists not only an almost-permutation presentation $\pi_1(S^3 \setminus L) \to \mathrm{GL}(k, \mathbb{C})$ that has the desired property, but there also exists a rational representation $\pi_1(S^3 \setminus L) \to \mathrm{GL}(k, \mathbb{Q})$ with the desired twisted Alexander module. This is an immediate consequence of the proofs and of [FV15, Remark 2, p. 2]. We leave the straightforward verification to the reader.

In Section 5 we will show that the invariants $\mathrm{rk}(L,\alpha)$ and $\widetilde{\Delta}_L^\alpha$ can be computed efficiently for almost-permutation representations. We will use this result to then show that Theorems 1.2 and 1.3 give rise to algorithms for detecting split links and for detecting unlinks. We will also indicate how these algorithms can be used for determining the splitting number of a link as defined by Batson and Seed [BS15].

We conclude this introduction with some observations tying in the results above with some group-theoretic aspects. First, the fact that twisted Alexander polynomials detect the unknot and the Hopf link is perhaps not entirely surprising, as these are the only links whose fundamental group is abelian. Instead, the fundamental group of any non-trivial knot is non-abelian, hence detection of the trefoil and the figure-8 knot requires far deeper results. Similarly, the unlink is characterized by the fact that $\pi_1(S^3 \smallsetminus L)$ is a free group, but in general it is difficult to distinguish a non-cyclic free group from other non-abelian groups. (We refer the reader to [AFW15] and references therein for a survey on 3-manifold groups from which these observations can be easily deduced.)

Convention Unless specified otherwise, all spaces are assumed to be compact and connected, and links are assumed to be oriented. Furthermore, all groups are assumed to be finitely presented.

2 Preliminaries

2.1 The Definition of Twisted Alexander Modules and Polynomials

In this section we quickly recall the definition of the twisted Alexander modules and polynomials for links, referring to [Tu01, Hi02, FV10] for history, details and generalizations.

Let $L \subset S^3$ be an oriented m-component link. Consider the canonical morphism $\phi_L\colon \pi_1(S^3 \smallsetminus L) \to \mathbb{Z} = \langle t \rangle$ sending the meridian of each component to t and let $\alpha\colon \pi_1(S^3 \smallsetminus L) \to \mathrm{GL}(k,\mathbb{C})$ be a representation. Using the tensor representation

$$\begin{aligned} \alpha \otimes \phi_L\colon \pi_1(S^3 \smallsetminus L) &\longrightarrow \mathrm{GL}\big(k, \mathbb{C}[t^{\pm 1}]\big) \\ g &\longmapsto \alpha(g) \cdot \phi_L(g), \end{aligned}$$

we can define the homology groups $H_*^{\alpha\otimes\phi_L}(S^3 \smallsetminus L; \mathbb{C}[t^{\pm1}]^k)$ of $S^3 \smallsetminus L$ with coefficients in $\mathbb{C}[t^{\pm1}]^k$, which inherit from the system of coefficients an action of $\mathbb{C}[t^{\pm1}]$ and, as $\mathbb{C}[t^{\pm1}]$ is a PID, are finitely presented as $\mathbb{C}[t^{\pm1}]$-modules. We refer to these modules as *twisted Alexander modules of* (L,α).

We now recall that any finitely generated $\mathbb{C}[t^{\pm1}]$-module H can be written as

$$H = \mathbb{C}[t^{\pm1}]^r \oplus \bigoplus_{i=1}^{s} \mathbb{C}[t^{\pm1}]/p_i(t)$$

with $p_i(t) \neq 0$, $i = 1,\dots,s$. We then refer to $\mathrm{rk}_{\mathbb{C}[t^{\pm1}]}(H) := r$ as the *rank of* H and to $\mathrm{ord}_{\mathbb{C}}[t^{\pm1}](H) := \prod_{i=1}^{s} p_i(t)$ as the *order of* H. Returning to the twisted Alexander

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modules, we now define

$$\Delta^\alpha_{L,i} := \operatorname{ord}_{\mathbb{C}[t^{\pm1}]} H_i^{\alpha\otimes\phi_L}(S^3 \smallsetminus L; \mathbb{C}[t^{\pm1}]^k),$$
$$\widetilde{\Delta}^\alpha_{L,i} := \operatorname{ord}_{\mathbb{C}[t^{\pm1}]} \operatorname{Tor}_{\mathbb{C}[t^{\pm1}]} H_i^{\alpha\otimes\phi_L}(S^3 \smallsetminus L; \mathbb{C}[t^{\pm1}]^k),$$
$$\operatorname{rk}(L,\alpha,i) := \operatorname{rk}_{\mathbb{C}[t^{\pm1}]} H_i^{\alpha\otimes\phi_L}(S^3 \smallsetminus L; \mathbb{C}[t^{\pm1}]^k).$$

We refer to $\Delta^\alpha_{L,i}$ as the *i-th twisted Alexander polynomial of* (L,α). Note that $\Delta^\alpha_{L,i} \in \mathbb{C}[t^{\pm1}]$ and $\widetilde{\Delta}^\alpha_{L,i} \in \mathbb{C}[t^{\pm1}]$ are well defined up to multiplication by a unit in $\mathbb{C}[t^{\pm1}]$. Throughout the paper, whenever we have an equation of the form $\Delta^\alpha_{L,i} = f(t)$ or $\Delta^\alpha_{L,i} = f(t)$ for some $f(t) \in \mathbb{C}[t^{\pm1}]$, this equality is understood up to the indeterminacy of the left-hand side, *i.e.*, up to multiplication by a unit in $\mathbb{C}[t^{\pm1}]$.

(Throughout this paper we drop the i from the notation when $i = 1$, and drop α from the notation if α is the trivial one-dimensional representation over $\mathbb{C}$.)

We conclude this section with an elementary observation. Let $\alpha\colon \pi_1(S^3 \smallsetminus L) \to \mathrm{GL}(k,\mathbb{C})$ and $\beta\colon \pi_1(S^3 \smallsetminus L) \to \mathrm{GL}(l,\mathbb{C})$ be two representations. We can then also consider the diagonal sum representation $\alpha\oplus\beta\colon \pi_1(S^3\smallsetminus L) \to \mathrm{GL}(k+l,\mathbb{C})$. It follows immediately from the definitions that

$$\Delta^{\alpha\oplus\beta}_{L,i} = \Delta^\alpha_{L,i}\cdot\Delta^\beta_{L,i}. \tag{2.1}$$

2.2 Degrees of Twisted Alexander Polynomials and the 0-th Twisted Alexander Polynomial

We will make use of the following lemma.

Lemma 2.1 *Let $L \subset S^3$ be a link and let $\alpha\colon \pi_1(S^3\smallsetminus L) \to \mathrm{GL}(k,\mathbb{C})$ be a representation. Then $H_0^{\alpha\otimes\phi_L}(S^3 \smallsetminus L; \mathbb{C}[t^{\pm1}]^k)$ is $\mathbb{C}[t^{\pm1}]$-torsion and* $\deg(\Delta^\alpha_{L,0}) \le k$.

Proof Recall that if X is a space and $\gamma\colon \pi_1(X) \to \operatorname{Aut}(V)$ a representation, then it is well known (see, *e.g.*, [HS97, Section VI]) that

$$H_0^\gamma(X;V) = V/\big\{ (\gamma(g)-\mathrm{id}_k)v \mid g \in \pi_1(X) \text{ and } v \in V \big\}. \tag{2.2}$$

In particular in our case, we pick $g \in \pi_1(S^3 \smallsetminus L)$ such that $\phi_L(g) = t$. It then follows from (2.2) and the definition of the Alexander polynomial that

$$\Delta^\alpha_{L,0} \mid \det\big((\alpha\otimes\phi_L)(g) - \mathrm{id}_k \big).$$

Note that $(\alpha\otimes\phi_L)(g) = \alpha(g)t$; in particular,

$$\det\big((\alpha\otimes\phi_L)(g)-\mathrm{id}_k\big) = \det\big(\alpha(g)t-\mathrm{id}_k\big) = \det\big(\alpha(g)\big)\,t^k + \cdots + (-1)^k$$

is a polynomial of degree k. It now follows that $\Delta^\alpha_{L,0} \neq 0$ and that

$$\deg \Delta^\alpha_{L,0} \le \deg\big(\alpha(g)t - \mathrm{id}_k\big) = k. \qquad \blacksquare$$

2.3 Almost-permutation Representation

A matrix in $\mathrm{GL}(k,\mathbb{C})$ is called an *almost-permutation matrix* if in each row and each column it has precisely one value that is non-zero and if all non-zero entries are roots

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of unity. We then say that a representation $\alpha\colon \pi \to \mathrm{GL}(k,\mathbb{C})$ is an *almost-permutation representation* if given any g the matrix $\alpha(g)$ is an almost-permutation matrix.

Lemma 2.2 *Any almost-permutation representation factors through a finite group.*

Proof Let $\alpha\colon \pi \to \mathrm{GL}(k,\mathbb{C})$ be an almost-permutation representation. We first pick a finite generating set for π. We denote by n the least common multiple of the orders of the roots of unity that appear as the non-zero entries of α applied to the generating set. It is straightforward to see that any non-zero entry of any $\alpha(g)$ is now an n-th root of unity.

Given $g \in \pi$ we denote by $\beta(g)$ the matrix that is given by replacing all non-zero entries in $\alpha(g)$ by 1. It is straightforward to see that $g \mapsto \beta(g)$ also defines a representation with $\ker(\alpha) \subset \ker(\beta)$. Note that the image of β is a subgroup of the permutation group S_k.

Summarizing, we have a short exact sequence

$$1 \longrightarrow K \longrightarrow \pi/\ker(\alpha) \longrightarrow \pi/\ker(\beta) \longrightarrow 1,$$

where $\pi/\ker(\beta)$ is a subgroup of the permutation group S_k and where K is a subgroup of the group of all diagonal $k\times k$-matrices whose entries are n-th roots of unity. Thus, we see that $\pi/\ker(\alpha)$ is a finite group whose order is bounded above by $n^k \cdot k!$. ∎

2.4 The Thurston Norm, Fibered Classes, and Twisted Alexander Polynomials

Let $L \subset S^3$ be an oriented m-component link. Recall that the link L is *fibered* if its complement can be fibered over S^1 by Seifert surfaces of the link. (Note that when $m \geq 2$, this is stronger than the requirement that $S^3 \smallsetminus L$ admits a fibration; precisely, it is equivalent to requiring that the class of $H^1(S^3 \smallsetminus L;\mathbb{Z})$ determined by the canonical morphism $\phi_L\colon \pi_1(S^3 \smallsetminus L) \to \langle t\rangle$ is fibered.)

In the sequel, given a class $\phi \in H^1(S^3 \smallsetminus \nu L;\mathbb{Z})$ we denote by $\|\phi\|_T$ its Thurston norm [Th86]. Recall that this is defined as the minimal complexity of a surface dual to ϕ; more precisely, it is defined as

$$\|\phi\|_T := \min\left\{ \sum_{i=1}^{m} \max\{0,-\chi(S_i)\} \;\middle|\; \begin{array}{l} S_1\cup\cdots\cup S_m \text{ properly embedded surface} \\ \text{dual to } \phi \text{ with } S_1,\dots,S_m \text{ connected} \end{array} \right\}.$$

For example, if K is a non-trivial knot and $\phi_K \in H^1(S^3 \smallsetminus K;\mathbb{Z})$ is a generator, then

$$\|\phi_K\|_T = 2\,\mathrm{genus}(K) - 1.$$

For a link we can thus view $\|\phi_L\|_T$ as a generalization of the notion of the genus of a knot.

The following theorem is a consequence of [FK06, Theorems 1.1 and 1.2, Proposition 2.5 and Lemma 2.8] (see also [Fr14] for an alternative proof).

Theorem 2.3 *Let $L \subset S^3$ be an oriented m-component link and let $\alpha\colon \pi_1(S^3 \smallsetminus L) \to \mathrm{GL}(k,\mathbb{C})$ be a representation such that $\Delta_L^\alpha \neq 0$. Then*

$$\max\{0, \deg\Delta_L^\alpha - \deg\Delta_{L,0}^\alpha\} \leq k\|\phi_L\|_T. \tag{2.3}$$

Furthermore, if L is a fibered link, then $\Delta_L^\alpha \neq 0$, and (2.3) is an equality.

The above theorem thus says that degrees of twisted Alexander polynomials give lower bounds on the Thurston norm of $\|\phi_L\|_T$ and that they determine it for fibered links. Using the work of Agol [Ag08], Liu [Liu13], Przytycki–Wise [PW14,PW12], and Wise [Wi09,Wi12a,Wi12b], the authors proved in [FV13, Theorem 1.1] and [FV15, Theorem 5.9] that twisted Alexander polynomials decide the fiberability and determine the Thurston norm of $\|\phi_L\|_T$ of a non-split link. Specifically we have the following theorem.

Theorem 2.4 *Let $L \subset S^3$ be an oriented m-component link that is non-split. Then there exists an almost-permutation representation $\alpha\colon \pi_1(S^3 \setminus L) \to \mathrm{GL}(k, \mathbb{C})$ such that $\Delta_L^\alpha \neq 0$ and such that*

$$\max\{0, \deg \Delta_L^\alpha - \deg \Delta_{L,0}^\alpha\} = k\|\phi_L\|_T.$$

Furthermore, if L is not fibered, there exists an almost-permutation representation

$$\alpha'\colon \pi_1(S^3 \setminus L) \longrightarrow \mathrm{GL}(k, \mathbb{C})$$

such that $\Delta_L^{\alpha'} = 0$.

Proof Let $L \subset S^3$ be an oriented m-component link which is non-split. Note that this assumption implies that $S^3 \setminus L$ is irreducible. By [FV15, Theorem 5.9] there exists an "extended character α" such that for the corresponding twisted Reidemeister torsion τ_L^α we have $\deg \tau_L^\alpha = k\|\phi_L\|_T$. The first statement of the theorem now follows from the observation that an "extended character" is an almost-permutation matrix and the discussion in [FV10, Section 3.3.1] relating twisted Reidemeister torsions to twisted Alexander polynomials. The second statement follows immediately from [FV13, Theorem 1.1], and the observation that a representation $\alpha'\colon \pi_1(S^3 \setminus L) \to \mathrm{GL}(k, \mathbb{C})$ induced by a homomorphism $\pi_1(S^3 \setminus L) \to G$ to a group with $|G| = k$ is in fact an almost-permutation matrix. ∎

This theorem has the following corollary, whose second part refines one of the main theorems of [FV07] inasmuch as it asserts the sufficiency of the use of one-variable twisted Alexander polynomials.

Corollary 2.5 (i) *Let $K \subset S^3$ be a knot. If K is trivial, then for any representation $\alpha\colon \pi_1(S^3 \setminus K) \to \mathrm{GL}(k, \mathbb{C})$ we have $\Delta_K^\alpha = 1$. Conversely, if K is non-trivial, then there exists an almost-permutation representation $\alpha\colon \pi_1(S^3 \setminus K) \to \mathrm{GL}(k, \mathbb{C})$ such that $\Delta_K^\alpha \neq 1$.*

(ii) *Let $L \subset S^3$ be a 2-component link. If L is the Hopf link, then for any representation $\alpha\colon \pi_1(S^3 \setminus L) \to \mathrm{GL}(k, \mathbb{C})$, we have*

$$\tau_L^\alpha := \Delta_L^\alpha(\Delta_{L,0}^\alpha)^{-1} = 1.$$

Conversely, if L is not the Hopf link, then there exists an almost permutation representation $\alpha\colon \pi_1(S^3 \setminus L) \to \mathrm{GL}(k, \mathbb{C})$ such that $\tau_L^\alpha \neq 1$.

The reader may have noticed that the invariant τ_L^α introduced in the statement of the corollary is, in fact, the twisted Reidemeister torsion; see *e.g.*, [FV10, Section 3.3.1] for a discussion of this point of view.

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Proof Let $K \subset S^3$ be a knot. If K is trivial, then all first twisted homology modules are zero, hence all twisted Alexander polynomials are equal to 1. Conversely, if K is non-trivial, then the genus is greater than zero, and it then follows immediately from Theorem 2.4 that there exists an almost-permutation representation with corresponding non-constant twisted Alexander polynomial.

Now let $L \subset S^3$ be a 2-component link. Then it is well known that the following are equivalent:

(a) L is the Hopf link;
(b) $S^3 \setminus L \cong T^2 \times I$;
(c) L is fibered with $\|\phi_L\|_T = 0$.

It follows easily from the implication (a) $\Rightarrow$ (b) that the twisted Alexander modules of the Hopf link are the homology groups of the infinite cyclic cover of $T^2 \times I$ determined by ϕ_L, *i.e.*, homotopically a copy of S^1. Given any representation $\alpha: \pi_1(S^3 \setminus L) \to \mathrm{GL}(k, \mathbb{C})$ it follows that $\tau_L^\alpha = 1$ (we refer to [KL99, p. 644] for details). Now suppose that L is not the Hopf link. Then ϕ_L is either not fibered or $\|\phi_L\|_T > 0$. It follows from Theorem 2.4 that there exists an almost-permutation representation $\alpha: \pi_1(S^3 \setminus L) \to \mathrm{GL}(k, \mathbb{C})$ such that either $\Delta_L^\alpha = 0$ or $\deg(\Delta_L^\alpha) - \deg(\Delta_{L,0}^\alpha) > 0$. Either way, $\tau_L^\alpha \neq 1$. ∎

3 Proofs of the Main Results

3.1 Twisted Alexander Polynomials Detect the Trefoil and the Figure-8 Knot

The following theorem is the promised more precise version of Theorem 1.1.

Theorem 3.1 *Let K be a knot. Then K is equivalent to the trefoil knot (resp. figure-8 knot) if and only if the following conditions hold:*

(i) $\Delta_K = 1 - t + t^2$ *(resp.* $\Delta_K = 1 - 3t + t^2$*)*
(ii) *for any almost permutation representation* $\alpha: \pi_1(S^3 \setminus K) \to \mathrm{GL}(k, \mathbb{C})$*, we have*

$$\Delta_K^\alpha \neq 0 \quad \textit{and} \quad \deg \Delta_K^\alpha \leq 2k.$$

Proof Let K be the trefoil knot or the figure-8 knot. It is well known that in the former case, $\Delta_K = 1 - t + t^2$ and that in the latter case, $\Delta_K = 1 - 3t + t^2$. Note that in either case K is a fibered genus one knot. It now follows from Theorem 2.4 that for any almost-permutation representation $\alpha: \pi_1(S^3 \setminus K) \to \mathrm{GL}(k, \mathbb{C})$, we have $\Delta_K^\alpha \neq 0$ and that

$$\deg \Delta_K^\alpha - \deg \Delta_{K,0}^\alpha = k(2\,\mathrm{genus}(K) - 1) = k.$$

We deduce from Lemma 2.1 that $\deg \Delta_{K,0}^\alpha \leq k$. We thus obtain the desired inequality

$$\deg \Delta_K^\alpha \leq 2k.$$

This concludes the proof of the "only if" direction of the theorem.

Now suppose that K is a knot such that for any almost-permutation representation $\alpha: \pi_1(S^3 \setminus K) \to \mathrm{GL}(k, \mathbb{C})$ we have

$$\Delta_K^\alpha \neq 0 \quad \text{and} \quad \deg \Delta_K^\alpha \leq 2k.$$

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It follows from Theorem 2.4 that K is fibered and that the genus of K equals one. From [BZ85, Proposition 5.14] we deduce that K is equivalent to either the trefoil knot or the figure-8 knot. The 'if' direction of the theorem now follows from the fact mentioned above that the ordinary Alexander polynomial distinguishes the trefoil knot from the figure-8 knot. ∎

3.2 Split Links

We say that a link L is *s-splittable* if there exist s disjoint 3-balls $B_1, \dots, B_s \subset S^3$ such that each B_i contains at least one component of L and such that $S^3 \setminus (B_1 \cup \dots \cup B_s)$ also contains a component of L. Furthermore, we say that L is *s-split* if L is s-splittable but not $(s+1)$-splittable.

Theorem 1.2 is a consequence of the following result.

Theorem 3.2 *Let $L \subset S^3$ be an oriented m-component link. Then the following hold:*

(i) *If L is s-splittable, then for any representation $\alpha\colon \pi_1(S^3 \setminus L) \to \mathrm{GL}(k, \mathbb{C})$ we have*
$$\mathrm{rk}(L, \alpha) \geq sk.$$

(ii) *If L is s-split, then there exists an almost-permutation representation*
$$\alpha\colon \pi_1(S^3 \setminus L) \to \mathrm{GL}(k, \mathbb{C})$$
such that $\mathrm{rk}(L, \alpha) = sk$.

Proof As usual, denote by $\phi_L\colon \pi_1(S^3 \setminus L) \to \langle t \rangle$ the map that is given by sending each meridian to t. By slight abuse of notation, we will also denote by ϕ_L the restriction of ϕ_L to any subset of $S^3 \setminus L$.

Suppose that $L \subset S^3$ is an s-splittable link. We pick disjoint 3-balls $B_1, \dots, B_s \subset S^3$ such that each B_i contains at least one component of L and such that $B_0 := S^3 \setminus (B_1 \cup \dots \cup B_s)$ also contains a component of L. For $i = 1, \dots, s$ we write $S_i := \partial B_i$ and for $i = 0, \dots, s$ we write $L_i := L \cap B_i$. By assumption, L_i is non-empty for any i.

Now let $\alpha\colon \pi_1(S^3 \setminus L) \to \mathrm{GL}(k, \mathbb{C})$ be a representation. We consider the following Mayer–Vietoris sequence
$$\begin{aligned}\bigoplus_{i=1}^{s} H_1(S_i; \mathbb{C}[t^{\pm 1}]^k) \longrightarrow \bigoplus_{i=0}^{s} H_1(B_i \setminus L_i; \mathbb{C}[t^{\pm 1}]^k) \longrightarrow H_1(S^3 \setminus L; \mathbb{C}[t^{\pm 1}]^k) \longrightarrow \\ \bigoplus_{i=1}^{s} H_0(S_i; \mathbb{C}[t^{\pm 1}]^k) \longrightarrow \bigoplus_{i=0}^{s} H_0(B_i \setminus L_i; \mathbb{C}[t^{\pm 1}]^k) \longrightarrow \cdots\end{aligned}$$
where the representation is given by $\alpha \otimes \phi_L$ in each case. Note that the restriction of $\alpha \otimes \phi_L$ to $\pi_1(S_i)$, $i = 1, \dots, s$ is necessarily trivial, but that the restriction of ϕ_L to $\pi_1(B_i \setminus L_i)$, $i = 0, \dots, s$ is non-trivial, since L_i consists of at least one component. It follows immediately from the definition of homology with coefficients that for $i = 1, \dots, s$, we have $H_0(S_i; \mathbb{C}[t^{\pm 1}]^k) \cong \mathbb{C}[t^{\pm 1}]^k$ and $H_1(S_i; \mathbb{C}[t^{\pm 1}]^k) \cong 0$.

Finally, note that for $i = 0, \dots, s$ and $j = 0, 1$ we have inclusion induced isomorphisms
$$H_j\big(B_i \setminus L_i; \mathbb{C}[t^{\pm 1}]^k\big) \xrightarrow{\cong} H_j\big(S^3 \setminus L_i; \mathbb{C}[t^{\pm 1}]^k\big).$$

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This entails, by Lemma 2.1 that for $i = 0, \dots, s$ the modules $H_0(B_i \setminus L_i; \mathbb{C}[t^{\pm 1}]^k)$ are torsion $\mathbb{C}[t^{\pm 1}]$-modules. We thus see that the above Mayer-Vietoris sequence gives rise to an exact sequence

$$0 \longrightarrow \bigoplus_{i=0}^{s} H_1(S^3 \setminus L_i; \mathbb{C}[t^{\pm 1}]^k) \longrightarrow H_1(S^3 \setminus L; \mathbb{C}[t^{\pm 1}]^k) \longrightarrow \mathbb{C}[t^{\pm 1}]^{ks} \longrightarrow T \tag{3.1}$$

where T is a torsion $\mathbb{C}[t^{\pm 1}]$-module. In particular we now deduce that

$$\mathrm{rk}(L, \alpha) = \mathrm{rk}_{\mathbb{C}[t^{\pm 1}]}\big(H_1(S^3 \setminus L; \mathbb{C}[t^{\pm 1}]^k)\big) \geq \mathrm{rk}_{\mathbb{C}[t^{\pm 1}]}\, \mathbb{C}[t^{\pm 1}]^{ks} = ks.$$

This concludes the proof of (i).

We now suppose that L is in fact an s-split link. Note that we have a canonical homeomorphism

$$S^3 \setminus L \cong S^3 \setminus L_0 \# \cdots \# S^3 \setminus L_s.$$

The links $L_i \subset S^3$, $i = 0, \dots, s$, are non-split by definition of an s-split link. It follows from Theorem 2.4 that for $i = 0, \dots, s$ there exists an almost-permutation representation $\alpha_i : \pi_1(S^3 \setminus L_i) \to \mathrm{GL}(k_i, \mathbb{C})$ such that $\Delta_{L_i}^{\alpha_i} \neq 0$. We now denote by k the greatest common divisor of the k_i. After replacing α_i by the diagonal sum of k/k_i-copies of the representation α_i we can in light of (2.1) assume that, in fact, $k = k_i$, $i = 0, \dots, s$. We now denote by

$$\alpha \colon \pi_1(S^3 \setminus L) \longrightarrow \mathrm{GL}(k, \mathbb{C})$$

the unique representation which has the property that for $i = 0, \dots, s$ the restriction of α to $\pi_1(B_i \setminus L_i)$ agrees with the restriction of α_i to $\pi_1(B_i \setminus L_i)$. Note that α is again an almost-permutation representation. By the above, the modules $H_1(S^3 \setminus L; \mathbb{C}[t^{\pm 1}]^k)$ are $\mathbb{C}[t^{\pm 1}]$-torsion modules. It now follows from (3.1) that

$$\mathrm{rk}(L, \alpha) = \mathrm{rk}_{\mathbb{C}[t^{\pm 1}]}\big(H_1(S^3 \setminus L; \mathbb{C}[t^{\pm 1}]^k)\big) = \mathrm{rk}_{\mathbb{C}[t^{\pm 1}]}\, \mathbb{C}[t^{\pm 1}]^{ks} = ks.$$

This concludes the proof of (ii). ∎

3.3 Detecting Unlinks

We finally turn to the problem of detecting unlinks. The following well-known lemma gives a purely group-theoretic characterization of unlinks.

Lemma 3.3 *A link L is trivial if and only if $\pi_1(S^3 \setminus L)$ is a free group.*

Proof The "only if" direction is obvious. So suppose that $L = L_1 \cup \cdots \cup L_m$ is an m-component link such that $\pi_1(S^3 \setminus L)$ is a free group. We have to show that each L_i bounds a disk in the complement of the other components. We denote by T_i the torus that is the boundary of a tubular neighborhood around L_i. It is well known that the kernel of $H_1(T_i) \to H_1(S^3 \setminus L)$ is spanned by the longitude λ_i of L_i. Since $\pi_1(S^3 \setminus L)$ is a free group and since every abelian subgroup of a free group is cyclic, it now follows easily that the longitude also lies in the kernel of $\pi_1(T_i) \to \pi_1(S^3 \setminus L)$. By Dehn's lemma (see [He76, Chapter 4]), longitude bounds an embedded disk in $S^3 \setminus L$. ∎

Note that if a finitely presented group is free, then one can show this using Tietze moves. On the other hand, there is, in general, no algorithm for showing that a finitely presented group is not a free group. Our main theorem now gives, in particular, an algorithm for showing that a given link group is not free.

Theorem 3.4 *An m-component link L is the trivial link if and only if for any almost-permutation representation $\alpha\colon \pi_1(S^3 \smallsetminus L) \to \mathrm{GL}(k,\mathbb{C})$, we have* $\mathrm{rk}(L,\alpha) = k(m-1)$ *and* $\widetilde{\Delta}_L^\alpha = 1$.

Proof The proof of the "only if" statement is very similar to the proof of Theorem 3.2(i). In fact it follows easily from (3.1) that for the m-component trivial link L and a representation $\alpha\colon \pi_1(S^3 \smallsetminus L) \to \mathrm{GL}(k,\mathbb{C})$, we have $H_1(S^3 \smallsetminus L;\mathbb{C}[t^{\pm1}]^k) \cong \mathbb{C}[t^{\pm1}]^{k(m-1)}$. In particular, $\mathrm{rk}(L,\alpha) = k(m-1)$ and $\widetilde{\Delta}_L^\alpha = 1$.

We now suppose that $L = L_0\cup\cdots\cup L_{m-1}$ is an m-component link such that for every almost-permutation representation $\alpha\colon \pi_1(S^3 \smallsetminus L) \to \mathrm{GL}(k,\mathbb{C})$ we have $\mathrm{rk}(L,\alpha) = k(m-1)$. It follows immediately from Theorem 3.2(ii) that L is an $(m-1)$-split link. We can therefore pick disjoint 3-balls $B_1,\dots,B_{m-1} \subset S^3$ such that each B_i contains a component of L and such that $B_0 := S^3 \smallsetminus (B_1\cup\cdots\cup B_s)$ also contains a component of L. Without loss of generality, we can assume that for $i = 0,\dots,m-1$ we have $L_i = L\cap B_i$. For $i = 1,\dots,m-1$ we furthermore write $S_i := \partial B_i$.

It remains to show that if one of the components L_i is not the unknot, then there exists an almost-permutation representation $\alpha\colon \pi_1(S^3 \smallsetminus L) \to \mathrm{GL}(k,\mathbb{C})$ with $\widetilde{\Delta}_L^\alpha \neq 1$. So we now suppose that L_0 is not the unknot. It follows from Theorem 2.4 and Corollary 2.5 that for $i = 0,\dots,m-1$ there exists an almost-permutation representation $\alpha_i : \pi_1(S^3 \smallsetminus L_i) \to \mathrm{GL}(k_i,\mathbb{C})$ such that $\Delta^{\alpha_i}_{S^3\smallsetminus L_i} \neq 0$ and such that $\Delta^{\alpha_0}_{S^3\smallsetminus L_0}$ is not a constant. As in the proof of Theorem 3.2, we can assume that $k := k_0 = \cdots = k_{m-1}$. We then denote by

$$\alpha\colon \pi_1(S^3 \smallsetminus L) \to \mathrm{GL}(k,\mathbb{C})$$

the unique representation that has the property that for $i = 0,\dots,m-1$ the restriction of α to $\pi_1(B_i \smallsetminus L_i)$ agrees with the restriction of α_i to $\pi_1(B_i \smallsetminus L_i)$. Note that α is again an almost-permutation representation.

It now follows from (3.1) that

$$\mathrm{Tor}_{\mathbb{C}[t^{\pm1}]}\big(H_1(S^3 \smallsetminus L;\mathbb{C}[t^{\pm1}]^k)\big) \cong \mathrm{Tor}_{\mathbb{C}[t^{\pm1}]}\Big(\bigoplus_{i=0}^{m-1} H_1(S^3 \smallsetminus L_i;\mathbb{C}[t^{\pm1}]^k)\Big).$$

We now conclude that

$$\begin{aligned}
\widetilde{\Delta}_L^\alpha &= \mathrm{ord}_{\mathbb{C}[t^{\pm1}]}\big(\mathrm{Tor}_{\mathbb{C}[t^{\pm1}]}(H_1(S^3 \smallsetminus L;\mathbb{C}[t^{\pm1}]^k))\big)\\
&= \mathrm{ord}_{\mathbb{C}[t^{\pm1}]}\Big(\mathrm{Tor}_{\mathbb{C}[t^{\pm1}]}\big(\bigoplus_{i=0}^{m-1} H_1(S^3 \smallsetminus L_i;\mathbb{C}[t^{\pm1}]^k)\big)\Big)\\
&= \prod_{i=0}^{m-1} \mathrm{ord}_{\mathbb{C}[t^{\pm1}]}\Big(\mathrm{Tor}_{\mathbb{C}[t^{\pm1}]}\big(H_1(S^3 \smallsetminus L_i;\mathbb{C}[t^{\pm1}]^k)\big)\Big)
\end{aligned}$$

$$= \prod_{i=0}^{m-1} \operatorname{ord}_{\mathbb{C}[t^{\pm 1}]}\big(H_1(S^3 \smallsetminus L_i; \mathbb{C}[t^{\pm 1}]^k)\big)$$
$$= \prod_{i=0}^{m-1} \Delta^{\alpha}_{L_i} = \prod_{i=0}^{m-1} \Delta^{\alpha_i}_{L_i}.$$

But this is not a constant, since $\Delta^{\alpha_0}_{L_0}$ is not a constant. ■

4 Extending the Results

Let L be an s-split. We pick disjoint 3-balls $B_1, \ldots, B_s \subset S^3$ such that each B_i contains a component of L and such that $B_0 := S^3 \smallsetminus (B_1 \cup \cdots \cup B_s)$ also contains a component of L. For $i = 0, \ldots, s$ we write $L_i := L \cap B_i$. We then view $L_0, \ldots, L_s$ as links in S^3. These links are called the *split-components* of L. It is well known that the set of split-components is well defined and does not depend on the choice of the $B_1, \ldots, B_s$.

As a consequence of the proofs of Corollary 2.5, Theorems 1.3 and 3.1, it is rather straightforward to see that twisted Alexander modules determine any s-split link such that each of the split-components is either the unknot, the trefoil, the figure-8 knot or the Hopf link.

This result now begs the following question.

Question 4.1 *Are there any other links that are determined by twisted Alexander modules?*

In fact, we propose the following conjecture.

Conjecture 4.2 *Any torus knot is detected by twisted Alexander polynomials.*

Note that torus knots are fibered and that twisted Alexander polynomials detect fibered knots. It thus remains to detect torus knots among the class of fibered knots. A positive answer to [Ko12, Question 7.1] would come close to proving the conjecture.

5 An Algorithm for Detecting Unlinks and Split Links

In this section we will first outline how the invariants $\widetilde{\Delta}^{\alpha}_{L,i}$ and $\operatorname{rk}(L, \alpha, i)$ for $i = 0, 1$ can be calculated efficiently for almost-permutation representations of link groups. We will then show that Theorems 3.4 and 3.2 give rise to algorithms for detecting whether a given link is the unlink or a split link. Finally we outline some applications for determining the unlinking and the splitting number of a link.

5.1 Computing the Invariants for Almost-permutation Representations

Let L be a link and let $\alpha: \pi := \pi_1(S^3 \smallsetminus L) \to \mathrm{GL}(k, \mathbb{C})$ be an almost-permutation representation. We denote by $\phi: \pi \to \mathbb{Z}$ the canonical epimorphism sending each meridian to 1. In the proof of Lemma 2.2 we saw that there exists an n such that α takes values in $\mathrm{GL}(k, \mathbb{F})$ with $\mathbb{F} = \mathbb{Q}(e^{2\pi i/n})$. Note that $\mathbb{C}[t^{\pm 1}]$ is flat over $\mathbb{F}[t^{\pm 1}]$; *i.e.*,

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we have a canonical isomorphism

$$H_i^{\alpha\otimes\phi}\big(S^3\smallsetminus L;\mathbb{C}[t^{\pm1}]^k\big)\cong H_i^{\alpha\otimes\phi}\big(S^3\smallsetminus L;\mathbb{F}[t^{\pm1}]^k\big)\otimes_{\mathbb{F}[t^{\pm1}]}\mathbb{C}[t^{\pm1}]$$

of $\mathbb{C}[t^{\pm1}]$-modules. It thus follows that

$$\widetilde{\Delta}^{\alpha}_{L,i}=\operatorname{ord}_{\mathbb{F}[t^{\pm1}]}\operatorname{Tor}_{\mathbb{F}[t^{\pm1}]}H_i^{\alpha\otimes\phi}(S^3\smallsetminus L;\mathbb{F}[t^{\pm1}]^k),$$
$$\operatorname{rk}(L,\alpha,i)=\operatorname{rk}_{\mathbb{F}[t^{\pm1}]}H_i^{\alpha\otimes\phi}(S^3\smallsetminus L;\mathbb{F}[t^{\pm1}]^k).$$

Let $\langle g_1,\dots,g_m\mid r_1,\dots,r_n\rangle$ be a presentation for π. After possibly adding trivial relators we can and will assume that $n\geq m-1$. We denote by X the corresponding 2-complex with one 0-cell, k 1-cells, and n 2-cells a,nd we identify $\pi_1(X)$ with π. In the following we extend the tensor representation $\alpha\otimes\phi\colon\pi=\pi_1(X)\to\mathrm{GL}(k,\mathbb{F}[t^{\pm1}])$ to a representation $\mathbb{Z}[\pi]\to M(k,\mathbb{F}[t^{\pm1}])$, which we also denote by $\alpha\otimes\phi$. Furthermore, given an $r\times s$-matrix A over $\mathbb{Z}[\pi]$ we denote by $(\alpha\otimes\phi)(A)$ the $rk\times sk$-matrix over $\mathbb{F}[t^{\pm1}]$ which is given b,y applying $\alpha\otimes\phi$ to each entry of A. For $i=1,\dots,m$ we now denote by

$$\frac{\partial}{\partial i}\colon\mathbb{Z}[\pi]\longrightarrow\mathbb{Z}[\pi]$$

the i-th Fox derivative (where we follow the convention of [Ha05, Section 6]). The twisted chain complex X with coefficients provided by $\alpha\otimes\phi$ is then isomorphic to the chain complex

$$0\to\mathbb{F}[t^{\pm1}]^{nk}\xrightarrow{(\alpha\otimes\phi)\left(\frac{\partial r_h}{\partial g_i}\right)}\mathbb{F}[t^{\pm1}]^{mk}\xrightarrow{(\alpha\otimes\phi)(1-g_j)}\mathbb{F}[t^{\pm1}]^k\to0,\tag{5.1}$$

where $h=1,\dots,n$, $i=1,\dots,m$ and $j=1,\dots,m$. In the sequel we refer to the boundary matrix on the left as B_1 and to the boundary matrix on the right as B_0. It is well known that twisted homology modules in dimensions 0 and 1 only depend on the fundamental group. We can thus use the chain complex (5.1) to calculate $\widetilde{\Delta}^{\alpha}_{L,i}$, $i=0,1$ and $\operatorname{rk}(L,\alpha,1)$.

Since $\mathbb{F}[t^{\pm1}]$ is a PID we can appeal to standard algorithms to find a matrix $P_1\in\mathrm{GL}(mk,\mathbb{F}[t^{\pm1}])$ such that

$$B_0P_1=\begin{pmatrix}0 & A_0\end{pmatrix},$$

where A_0 is a $k\times k$-matrix. It follows from the theory of modules over PIDs that

$$\Delta^{\alpha}_{L,0}=\det(A_0).$$

Note that by Lemma 2.1 we have $\det(A_0)\neq0$. Also note that the fact that

$$(B_0P_1)(P_1^{-1}B_1)=B_0B_1=0$$

implies that the last k row of $P_1^{-1}B_1$ are zero. Again, using standard algorithms over a PID we can find a matrix $P_2\in\mathrm{GL}(nk,\mathbb{F}[t^{\pm1}])$ such that

$$P_1^{-1}B_1P_2=\begin{pmatrix}A_1 & 0\\ 0 & 0\end{pmatrix},$$

where A_1 is a diagonal $(m-1)k\times(m-1)k$-matrix over $\mathbb{F}[t^{\pm1}]$ with diagonal entries $d_1,\dots,d_{(m-1)k}$. It then follows from the definitions that

$$\widetilde{\Delta}^{\alpha}_{L,1}=\prod_{d_i\neq0}d_i\quad\text{and}\quad\operatorname{rk}(L,\alpha,1)=\#\{i\mid d_i=0\}.$$

Finally, we point out that since $\mathbb{F}$ is a finite extension of $\mathbb{Q}$, all these base changes can be performed by a computer without difficulty.

5.2 The Algorithms

Theorem 5.1 *There exists an algorithm that takes as input a diagram for a link in S^3 and decides after finitely many steps whether L is the unlink or not.*

Note that there are various other ways of detecting the unlink. For example, Ozsváth and Szabó [OS08] showed that Link Floer Homology detects the unlink, and the combinatorial description of Link Floer Homology in [MOST07] then gives an algorithm for detecting the unlink.

In a similar vein, Hedden–Ni [HN13, Theorem 1.3] showed that an m-component link is the unlink if and only if the Khovanov module is isomorphic to

$$\mathbb{F}_2[x_0,\dots,x_{m-1}]/(x_0^2,\dots,x_{m-1}^2).$$

In general, at least it is difficult though to check whether or not two $\mathbb{F}_2[x_0,\dots,x_{m-1}]$-modules are isomorphic.

Proof Let $L = L_1 \cup \cdots \cup L_m \subset S^3$ be a link. We start out with a few observations.

(i) If L is the unlink, then it follows from Reidemeister's theorem that any diagram of L can be turned into the standard diagram of the unlink, using a finite sequence of Reidemeister moves.

(ii) If L is the unlink, then it follows from Theorem 3.4 that given any almost-permutation representation $\alpha\colon \pi_1(S^3 \setminus L) \to \mathrm{GL}(k,\mathbb{C})$, we have $\mathrm{rk}(L,\alpha) = m(k-1)$ and $\widetilde{\Delta}_L^\alpha \neq 1$.

(iii) If L is not the unlink, then it follows from Theorem 3.4 that there exists an almost-permutation representation $\alpha\colon \pi_1(S^3 \setminus L) \to \mathrm{GL}(k,\mathbb{C})$ such that either $\mathrm{rk}(L,\alpha) \neq m(k-1)$ or $\widetilde{\Delta}_L^\alpha \neq 1$.

The algorithm consists of two programs running simultaneously.

(a) The first program goes systematically over all finite sequences of Reidemeister moves applied to the given diagram. We terminate this program once it turns the given diagram of L into the standard diagram of the unlink. By the above discussion this program will terminate after finitely many steps if L is the split link.

(b) The second program first determines a Wirtinger presentation

$$\langle g_1,\dots,g_k \mid r_1,\dots,r_l\rangle$$

for $\pi_1(S^3 \setminus L)$ from the given link diagram. The program then systematically goes through all almost-permutation representations of $\pi_1(S^3 \setminus L)$. This can be done by going through all assignments of almost-permutation matrices to the g_i and verifying that the relations hold. As we discussed in Section 5.1, it is possible to calculate $\mathrm{rk}(L,\alpha) \neq m(k-1)$ and $\widetilde{\Delta}_L^\alpha \neq 1$ for any such representation α. We terminate the program once we find an almost-permutation representation $\alpha\colon \pi_1(S^3 \setminus L) \to \mathrm{GL}(k,\mathbb{C})$ such that either $\mathrm{rk}(L,\alpha) \neq m(k-1)$ or $\widetilde{\Delta}_L^\alpha \neq 1$. It follows from the above discussion that this program will terminate only if L is

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the unlink, and it will terminate after finitely many steps if the link is not the unlink. ∎

We also have the following theorem.

Theorem 5.2 *There exists an algorithm that takes as input a link in S^3 and decides after finitely many steps whether L is split or not.*

Proof The proof is very similar to the proof of Theorem 5.1. We thus only outline the changes one has to make in the proof. So let $L = L_1 \cup \cdots \cup L_m \subset S^3$ be a link. We again start out with three observations.

(i) If L is a split link, then it follows from Reidemeister's theorem that any diagram of L can be turned into a split diagram, using a finite sequence of Reidemeister moves. Here we say that a diagram for the link L is split if it is contained in two disjoint disks such that each disks contains a non-empty diagram.

(ii) If L is a split link, then L is 1-splittable. It follows from Theorem 3.2 that given any almost-permutation representation $\alpha\colon \pi_1(S^3 \smallsetminus L) \to \mathrm{GL}(k, \mathbb{C})$, we have $\mathrm{rk}(L, \alpha) > 0$.

(iii) If L is not a split link, then L is 0-split. It follows from Theorem 3.2 that there exists an almost-permutation representation $\alpha\colon \pi_1(S^3 \smallsetminus L) \to \mathrm{GL}(k, \mathbb{C})$ such that $\mathrm{rk}(L, \alpha) = 0$.

As in the proof of Theorem 5.1 we now run two programs, with obvious modifications, one of which will terminate after finitely many steps precisely if L is a split link, and the other will terminate after finitely many steps precisely if L is not a split link. ∎

We now say that an m-component link $L = L_1 \cup \cdots \cup L_m \subset S^3$ is *totally split* if it is $(m-1)$-split, *i.e.*, if it is the split union of its components. An obvious modification of the proof of Theorem 5.2 now gives us the following result.

Theorem 5.3 *There exists an algorithm taking as input a link in S^3 and decides after finitely many steps whether L is totally split or not.*

With our present understanding of representations of link groups it is impossible to give a rigorous estimate for how efficient these algorithms are. But from our experience (see, *e.g.*, [FK06, DFJ12]), in practice, twisted Alexander polynomials tend to be extremely efficient at detecting fiberedness and the Thurston norm. We are thus quite confident that twisted Alexander polynomials and modules are very efficient at showing that a non-trivial link is indeed non-trivial and at showing that a non-split link is indeed non-split.

5.3 The Splitting Number

In a recent paper Batson–Seed [BS15] defined the *splitting number* $\mathrm{sp}(L)$ of a link L to be the minimal number of crossing changes between different components that are needed to turn L into a totally split link. (Note that this differs from the notion of "splitting number" used in [Ad96, Sh12] where crossing changes between the same component are allowed.)

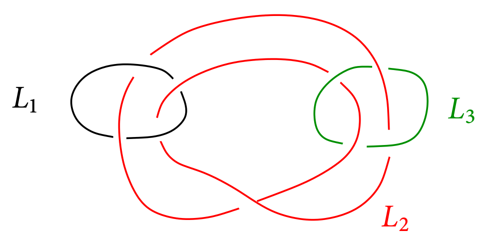
The splitting number of a link is usually determined by finding upper and lower bounds on the splitting number. The upper bounds are obtained by performing crossing changes until one obtains a totally split link. This makes it necessary to have an efficient algorithm for detecting whether or not a given link is totally split.

The lower bounds on the splitting number usually come from invariants, *e.g.*, Khovanov homology [BS15], linking numbers of covering links and Alexander polynomials in [CFP13]. We now quickly recall a further lower bound on the splitting number which was introduced in [CFP13] and which turns out to be very efficient for many links.

A sublink of a link is called *obstructive* if it is not totally split and if all the linking numbers are zero. Given a link L we then define $c(L)$ to be the maximal size of a collection of distinct obstructive sublinks of L, such that any two sublinks in the collection have at most one component in common. In [CFP13, Lemma 2.1] it is shown that for any link $L = L_1 \cup \cdots \cup \Lambda_m$ we have

$$\mathrm{sp}(L) \geq \sum_{i>j} |\mathrm{lk}(L_i, L_j)| + 2c(L). \tag{5.2}$$

For example, consider the link $L = L_1 \cup L_2 \cup L_3$ shown in the figure. The sublinks $L_1 \cup L_2$ and $L_2 \cup L_3$ are non-split links, which can be seen by the observation that their Alexander polynomials are non-zero. Since $L_1 \cup L_3$ is a split link it now follows that $c(L) = 2$. It thus follows from (5.2) that $\mathrm{sp}(L) \geq 4$. In order to apply the inequality



(5.2), one once again needs an efficient algorithm for determining whether or not a given link is totally split.

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