AN ARITHMETIC CHARACTERIZATION OF THE PARABOLIC POINTS OF $G(2\cos\pi/5)$

by DAVID ROSEN†

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By the group $G(2\cos\pi/q)$ we mean the group of linear fractional transformations of the complex plane onto itself, generated by $V(z) = -1/z$ and $U(z) = z+\lambda_q$, where $\lambda_q = 2\cos(\pi/q)$, q being a positive integer greater than 2. In this paper we shall be concerned only with the group given by $q = 5$, and we shall therefore omit the subscript 5 on the λ. We note that $\lambda = \lambda_5$ satisfies the equation

$$x^2 - x - 1 = 0; \tag{1}$$

hence $\lambda = (1+5^{\frac{1}{2}})/2$.

It is well known [1] that $G(\lambda)$ is a real zonal horocyclic group (see [4] for these terms); i.e. $G(\lambda)$ is a fuchsian group of the first kind. We let $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $(ad-bc = 1)$ represent the transformation $z' = T(z) = (az+b)/(cz+d)$, and notice that T and $-T$ represent the same transformation. If U and V are the corresponding matrices of $U(z)$ and $V(z)$, it is easy to verify in this notation that the generators satisfy the relations

$$V^2 = I = (VU)^5, \tag{2}$$

where I represents the identity transformation.

As a consequence of (2) we can write the transformation $T(z)$ as a *word* in U and V, namely: $T = U^{r_0}VU^{r_1}\cdots VU^{r_n}$, where the r_i are rational integers. In [5] it was shown that these words with certain conventions could be made unique. The unique word in turn led naturally to a continued fraction representation of the transformation, and hence a continued fraction representation of the parabolic points—the transforms of ∞. We shall use the fact, which can be deduced from the theorems of [5], that a parabolic point is a unique finite λ-fraction which we write in the form

$$(r_0\lambda;\ e_1/r_1\lambda, \ldots, e_n/r_n\lambda), \tag{3}$$

where the r_i $(i > 0)$ are positive integers, while r_0 may be a positive or negative integer or zero, and $e_i = \pm 1$. We shall also assume whatever theorems on continued fractions are necessary, and especially the results in [5]. We shall however use the current term approximant for convergent.

If P_m/Q_m is the mth approximant of a λ-fraction (3), it is a consequence of (1) that P_m and Q_m are algebraic integers in the field $R(5^{\frac{1}{2}})$, which when expressed in terms of the basis $(1, \lambda)$ have the form $m+n\lambda$, where m and n are rational integers. The parabolic points therefore, as finite λ-fractions, are quotients of integers in the field. We shall denote a typical one by a/b where $a = a_1+a_2\lambda$, $b = b_1+b_2\lambda$.

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The motivation for this study arose in connection with the problem of finding the Fourier coefficients of automorphic forms and functions belonging to $G(\lambda)$. These are given explicitly by Petersson [3], and involve sums of the type $\sum |c|^{-k}$ taken over all substitutions of the group. The characterization will therefore be directed toward describing c, the third element of the transformation.

The more interesting problem is the converse one, namely, which rational elements of the field are parabolic points or cusps. It seems, as a conjecture, that every rational element of the field is a cusp, as numerical calculations in abundance have not turned up counterexamples. For the groups $G(\lambda_q)$ with q even, the conjecture is false, since it is shown† in [5, p. 558] that 1 has an infinite λ-fraction representation and hence cannot be a cusp. We shall show that the units λ^m are all parabolic points.

We recall the essential property of the reduced λ-fraction. If $r_i\lambda+e_{i+1}<1$ (so that $r_i=1$, $e_{i+1}=-1$), then $r_{i+1}\geqq 2$, and either $e_i=1$ or $r_{i-1}\geqq 2$. As a matter of notation we shall write the numerator and denominator of the mth approximant as

$$P_m = P_{1m}+P_{2m}\lambda;\quad Q_m = Q_{1m}+Q_{2m}\lambda, \tag{4}$$

where the subscript 1 denotes the rational component, and 2 the λ-component. The following general formula will be useful:

$$Q_{k+1} = \lambda r_{k+1}Q_k+e_{k+1}Q_{k-1} = Q_{1(k+1)}+Q_{2(k+1)}\lambda. \tag{5}$$

We deduce from (1), (4), and (5) that

$$Q_{1(k+1)} = r_{k+1}Q_{2k}+e_{k+1}Q_{1(k-1)},\quad Q_{2(k+1)} = r_{k+1}(Q_{1k}+Q_{2k})+e_{k+1}Q_{2(k-1)}. \tag{6}$$

In particular, $Q_{11}=0$, $Q_{21}=r_1$, $Q_{12}=r_1r_2+e_2$, $Q_{22}=r_1r_2$.

LEMMA 1. (i) *$Q_{12}>Q_{11}=0$; (ii) $Q_{22}\geqq Q_{21}>0$; (iii) $Q_{22}>Q_{12}$ or $Q_{12}=Q_{22}+1$.*

Proof. Since $Q_{12}=r_1r_2+e_2$, (i) is obvious if $e_2=1$. If $e_2=-1$, then either r_1 or $r_2\geqq 2$, and the conclusion is immediate. Part (ii) is obvious. If $e_2=1$, then clearly $Q_{12}=Q_{22}+1$. If $e_2=-1$, either r_1 or $r_2\geqq 2$, so that part (iii) follows easily.

LEMMA 2. (i) *Either $Q_{13}\geqq Q_{12}$ or $Q_{12}=Q_{13}+1$; (ii) $Q_{23}\geqq Q_{22}$; (iii) $Q_{23}\geqq Q_{13}>0$.*

Proof. We prove part (iii) first. By (6) we have $Q_{23}=Q_{13}+r_3Q_{12}+e_3Q_{12}$, where we use the fact that $Q_{11}=0$, so that $Q_{13}=r_3Q_{22}$. We must now show that $r_3Q_{12}+e_3Q_{21}\geqq 0$, or more explicitly that $r_1r_2r_3+r_3e_2+r_1e_3\geqq 0$. A careful analysis of the possibilities such as $e_3=-1$, $r_3=1$, which forces $r_2\geqq 2$, gives the desired conclusion. We omit the details and remark that the alternatives of Lemma 1 part (iii) do not cause any difficulty.

To prove part (ii), we must show that $Q_{23}=2r_1r_2r_3+e_2r_3+e_3r_1>r_1r_2=Q_{22}$. This follows, however, from the similar inequality in (iii).

† There are a few errors in [5] which we correct at this time. (i) In (4.3) $r_{\nu-1}$ should be $r'_{\nu-1}$. (ii) On p. 557, line 3, $\alpha_{\nu-1,n}$ should be $\alpha'_{\nu-1,n}$. (iii) On p. 558, line 14 from bottom, $\zeta=(B(h+1), -1/\xi)$. (iv) On line 2 of the proof of Theorem 4, replace the first $\lambda>2$ by $\lambda<2$. (v) Theorem 7, 6th line of proof, should read: $a_i\leqslant a_{i-1}\lambda/2$ or $a_{i-1}/a_i\lambda\geqslant 2/\lambda^2=U_0$. (vi) P. 561, line 1, should read $a_{i-1}/a_i\lambda\geqslant 2/\lambda^2\ldots$. (vii) In he line above (7.7), replace K by k.

We prove part (i) by considering cases. Since $Q_{13} = r_1 r_2 r_3$, it is obvious that if $e_2 = -1$, then $r_1 r_2 r_3 \geqq r_1 r_2 - 1$. If $e_2 = 1$, then $r_1 r_2 r_3 > r_1 r_2 + 1$ when $r_3 \geqq 2$, while $Q_{12} = Q_{13} + 1$ when $r_3 = 1$.

We shall say that a finite λ-fraction has length m if P_m/Q_m is the value of the λ-fraction.

LEMMA 3. (i) *Either* $Q_{14} \geqq Q_{13}$ *or* $Q_{13} = Q_{14} + 1$; (ii) $Q_{24} \geqq Q_{23}$; (iii) $Q_{24} \geqq Q'_{14}$.

Proof. Part (i): From (6) we have $Q_{14} = r_4 Q_{23} + e_4 Q_{12}$. We consider three cases.

(*a*) $e_4 = 1$. Since $Q_{23} \geqq Q_{13}$, by Lemma 2, the conclusion is obvious.

(*b*) $e_4 = -1, r_4 \geqq 2$. Clearly $Q_{14} \geqq 2Q_{23} - Q_{12}$. Let $A = Q_{13} + 2r_3 Q_{12} + 2e_3 Q_{21} - 2e_3 Q_{11}$. From (6) we deduce that $2Q_{23} - Q_{12} = Q_{13} + A - Q_{12}$. We show that $A - Q_{12} \geqq 0$. If $e_3 = 1$, this is obvious, since $Q_{21} > Q_{11}$. If $e_3 = -1$, $r_3 \geqq 2$, straightforward estimation produces the result. We remark that in this case $Q_{13} > Q_{12}$. If $e_3 = -1$, $r_3 = 1$, we must have $r_2 \geqq 2$, and the required inequality follows easily.

(*c*) $e_4 = -1, r_4 = 1$. Let $B = Q_{12}(r_3 - 1) + e_3 Q_{21}$. We deduce from (6) that $Q_{14} = Q_{13} + B$. If $e_3 = 1$, then $B > 0$. If $e_3 = -1$ and $r_3 \geqq 3$, then $B \geqq 2Q_{12} - Q_{21} = 2r_1 r_2 + 2e_2 - r_1$. The right side is positive if either r_1 or $r_2 \geqq 2$ when $e_2 = -1$, but this is the case for a reduced λ-fraction. If $r_3 = 2$, $B > 0$ if $e_2 = 1$, and $B = -1$ if $e_2 = -1, r_2 = 1$. Hence part (i) is proved, since r_3 cannot be 1.

Part (ii): From (6) we see that the inequality is proved if

$$Q_{24} = r_4(Q_{13} + Q_{23}) + e_4 Q_{22} \geqq Q_{23}.$$

(*a*) $e_4 = 1$. The inequality is obvious.

(*b*) $e_4 = -1, r_4 \geqq 2$. By Lemma 2, $Q_{23} > Q_{22}$ and the inequality is easily deduced.

(*c*) $e_4 = -1, r_4 = 1$. Now $Q_{24} = Q_{13} + Q_{23} - Q_{22}$. If we substitute $Q_{13} = r_3 Q_{22} + e_3 Q_{11}$ in the right side, the desired inequality follows easily since $r_3 \geqq 2$.

Part (iii): Let $D = Q_{24} - Q_{14} = r_4 Q_{13} + e_4(Q_{22} - Q_{12})$; we show that $D \geqq 0$.

(*a*) $e_4 = 1$. Obviously $D \geqq Q_{13} + Q_{22} - Q_{12}$. If $Q_{13} \geqq Q_{12}$, then $D > 0$ trivially. If $Q_{12} = Q_{13} + 1$, then $D \geqq Q_{22} - 1 \geqq 0$.

(*b*) $e_4 = -1, r_4 \geqq 2$. Since $2Q_{13} - Q_{22} > 0$, it follows that $D > 0$.

(*c*) $e_4 = -1, r_4 = 1$. Now $D = Q_{13} + Q_{12} - Q_{22}$, which is easily seen to be positive.

This proves the lemma.

Using exactly the same type of argument, we can prove that for $m = 5$, only the possibility $Q_{15} \geqq Q_{14}$ occurs. Hence if the λ-fraction has length $m \geqq 5$, we can prove

THEOREM 1. *For* $m=5, 6, \ldots$, (i) $Q_{1m} \geqq Q_{1(m-1)} > 0$, (ii) $Q_{2m} \geqq Q_{2(m-1)} > 0$, *and* (iii) $Q_{2m} \geqq Q_{1m}$.

Proof. The proof is by induction and requires exactly the same type of arguments as we used in the previous lemmas. We shall therefore omit the details, but we state the induction lemma: If (i), (ii), (iii) are valid for $m = 5, 6, \ldots, k$, then (i), (ii), (iii) are valid for $m = k+1$.

We are now in a position to say something about the third element of the transformation $T(z)$.

COROLLARY 1. *Let* $c = c_1 + c_2\lambda$, *be the third element of* $T(z)$. *Either* $c_2 \geqq c_1 \geqq 0$, *or* $c_1 = c_2 + 1$.

Proof. We assume that the transformation has been reduced to minimum length [5], so that the λ-fraction $T(\infty) = a/c$ is a reduced λ-fraction. Since a and c are numerator and denominator of the λ-fraction, c is some Q_n. If $T = U^k$, then $c = 0$. If $T = V$, $c = 0\lambda + 1$. If the λ-fraction has length 1, $c = r_1\lambda + 0$. If the λ-fraction has length 2, part (iii) of Lemma 1 applies. If the λ-fraction has length $n \geqq 3$, part (iii) of Lemmas 2 and 3 and Theorem 1 apply.

We next define a *section* of the continued fraction as

$$\alpha_{j,m} = (r_j\lambda;\, e_{j+1}/r_{j+1}\lambda, \ldots, e_m/r_m\lambda). \tag{7}$$

If the λ-fraction is finite and of length m, we shall refer to $\alpha_{j,m}$ as the *tail.* We put $\alpha_{j,m} = P_{j,m}/Q_{j,m}$ and see that $Q_{j-1,m} = P_{j,m}$. Hence the numerator of a section can be expressed in terms of the denominators

$$P_{j,m} = Q_{j-1,m} = r_j\lambda Q_{j,m} + e_{j+1}Q_{j+1,m} \quad (1 \leqq j < m). \tag{8}$$

We note in particular that $P_{m,m}/Q_{m,m} = r_m\lambda$, so that $Q_{m,m} = 1$, and also that $Q_{0,m} = Q_m$. We write $Q_{-1,m} = P_{0,m} = P_m$, so that a special case of (8) is

$$P_m = r_0\lambda Q_m + e_1 Q_{1,m}\,. \tag{9}$$

In Lemmas 1, 2, 3, and Theorem 1, we calculated from the front of the continued fraction. In order to obtain information about the numerators we shall use (9). It is convenient therefore to calculate the tails. The results we obtain are completely analogous in content to the previous ones, and the proofs use identical arguments, so that we only state the results in

THEOREM 2. *Let* $Q_{j,n} = Q_{j,1n} + Q_{j,2n}\lambda \quad (j = n, n-1, \ldots, 0)$.

(i) $Q_{j,1n} \geqq Q_{j+1,1n} > 0$ *or* $Q_{j,1n} + 1 = Q_{j+1,1n}$ *if* $j = n-3, n-4$, *while*

$$Q_{j,1n} > Q_{j+1,1n} > 0 \text{ if } j = n-2, n-5, \ldots, 0.$$

(ii) $Q_{j,2n} \geqq Q_{j+1,2n} > 0 \quad (j = n-1, n-2, \ldots, 0)$.

(iii) $Q_{j,2n} \geqq Q_{j,1n}\, (j = n-1, n-2, \ldots, 1)$; *for* $j = n-2$ *the alternative*

$$Q_{n-2,1n} = Q_{n-2,2n} + 1$$

is possible.

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We point out that if the continued fraction $\alpha \geqq 2/\lambda$, then $r_0 \geqq 1$, and if $r_0 = 1$, $e_1 = -1$, then $r_1 \geqq 2$; i.e. α is reduced from r_0 instead of from r_1 as required in the definition of a λ-fraction [5, p. 555]. Consequently, Theorem 2 is valid for $j = -1$, and from (9) we deduce information about the numerators.

THEOREM 3. *If* $a/c \geqq 2/\lambda$, $a = a_1 + a_2\lambda$, $c = c_1 + c_2\lambda$, *then* (i) $a_1 > c_1$ *or* $a_1 + 1 = c_1$, (ii) $a_2 \geqq c_2$, (iii) $a_2 \geqq a_1$ *or* $a_1 = a_2 + 1$, (iv) $c_2 \geqq c_1$ *or* $c_2 + 1 = c_1$.

Proof. Since $a/c \geqq 2/\lambda$, Theorem 2 is valid for $j = -1$. We point out that the two alternatives in (iii) and (iv) cannot occur at the same time. This is evident from examining λ-fractions of length 1 and 2.

The restriction $a/c \geqq 2/\lambda$ is no serious loss of generality, since it amounts to a restriction to certain members of a coset decomposition of $G(\lambda)$ with respect to the subgroup generated by $U(z) = z + \lambda$. We denote by G the group of matrices that contains $-I$ and is such that $(az+b)/(cz+d) \in G(\lambda)$ if and only if

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G \quad (ad - bc = 1). \tag{10}$$

THEOREM 4. *If* (10) *holds and*

$$S = \begin{pmatrix} x & y \\ c & d \end{pmatrix},$$

where $x = a + ct$, $y = b + dt$, *then* $S \in G$ *if and only if* $t = m\lambda$, *where* m *is a rational integer.*

Proof. If $t = m\lambda$, then $S = U^m T$, which belongs to G, since U does. Conversely, if $S \in G$ then so does

$$ST^{-1} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix},$$

which is possible only if $t = m\lambda$.

We point out that we make no claims as to the existence of solutions to the diophantine equation with fixed integers c, d in $R(5^{\frac{1}{2}})$ which would give a transformation of $G(\lambda)$. Hutchinson [2] investigated a wide class of automorphic groups in which the coefficients a, b, c, d were integers in quadratic fields. It can be shown that $G(\lambda)$ does not belong to this class. What we can assert however, is this: If d/c is a finite λ-fraction which we write as $d'/c' = P_m/Q_m$, where $d' = d\lambda^n$ and $c' = c\lambda^n$, for some n, then by Theorem 1 of [5], the penultimate approximant and d'/c' provide a substitution of $G(\lambda)$, and Theorem 4 gives all substitutions with the same (c, d).

The next theorem gives us information on the parities of the coefficients of a and c, and seems to have a curious property relating to λ_5.

THEOREM 5. *Let a/c be a finite λ-fraction, let $+$ denote an even integer, $-$ an odd integer. The parities of a_1, a_2, c_1, c_2 occur in the following combinations only:*

	a_2	a_1	c_2	c_1
1.	$-$	$+$	$+$	$-$
2.	$+$	$+$	$+$	$-$
3.	$-$	$+$	$-$	$+$
4.	$+$	$-$	$+$	$+$
5.	$+$	$-$	$-$	$+$

Proof. We consider the tails $\alpha_{j,n}$. We see that $\alpha_{n,n} = (r_n\lambda+0)/(0\lambda+1)$ trivially has the form of 1 or 2 in the table. Also, by definition we have $\alpha_{j,n} = r_j\lambda + e_{j+1}/\alpha_{j+1,n}$, so that we can express the components of $\alpha_{j,n}$ in terms of the components of $\alpha_{j+1,n}$.

$$\left.\begin{aligned} P_{j,2n} &= r_j(P_{j+1,1n}+P_{j+1,2n})+e_{j+1}Q_{j+1,2n},\\ P_{j,1n} &= r_jP_{j+1,2n}+e_{j+1}Q_{j+1,1n},\\ Q_{j,1n} &= P_{j+1,1n},\\ Q_{j,2n} &= P_{j+1,2n}.\end{aligned}\right\} \tag{11}$$

It can be seen that if $\alpha_{j+1,n}$ has one of the forms listed in the table, then $\alpha_{j,n}$ also has one of the listed forms. In fact the change of form from $\alpha_{j,n}$ to $\alpha_{j+1,n}$ is given by the permutation (15)(24) if r_j is even, and by (13524) if r_j is odd.

If f is a tail of the form i, ($i = 1, 2, 3, 4, 5$), then it turns out that $f = \lambda^3 f/\lambda^3$, $\lambda^2 f/\lambda^2$, $\lambda f/\lambda$ determine three mutually exclusive classes, each class consisting of five distinct possible arrangements. This means that every rational element in $R(5^{\frac{1}{2}})$ can be expressed as in Theorem 5.

COROLLARY 2. *If c is the denominator of a parabolic point, then at least one of the components is even.*

Proof. Each row in the table of Theorem 3 has at least one $+$ in the c_2 and c_1 columns. It is clear that a similar conclusion holds for the numerators.

LEMMA 5. *Let $c = c_1 + c_2\lambda$; then $c_1 = c_2$ only if $c_1 = 2$, 4 or 6.*

Proof. We assume that a/c is a λ-fraction of the form (3), and that $a = P_m$, $c = Q_m$. Hence $Q_1 = r_1\lambda$, and if the fraction has length 1, then $c \neq c_2$. Suppose that $m = 2$; then $c = Q_2 = r_1r_2\lambda + (r_1r_2+e_2)$. Clearly, $c_1 = c_2$ implies that $e_2 = 0$, which is impossible. The interesting situations arise with $m = 3, 4$. These we examine in detail.

To have $Q_{13} = Q_{23}$, we must have $r_1r_2r_3 + e_2r_3 + e_3r_1 = 0$. Clearly, at least one e must be -1, so that $r_1r_2r_3 = r_1 - r_3$, $r_3 - r_1$, or $r_1 + r_3$. We consider the case $r_1 - r_3 = \pm k$, $k \geqq 0$, and look for integral solutions of $(\pm k + r_3)r_2r_3 = \pm k$. Since $k/(k+r_3)$ is not an integer, the

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upper sign is impossible; the lower sign gives the solutions $k = 1, r_2 = 4, r_3 = \frac{1}{2}$; $k = 2, r_2 = 2, r_3 = 1, r_1 = -1$; and $k = 4, r_2 = 1, r_3 = 2, r_1 = -2$. None of these are permissible λ-fractions. The case $r_1 + r_3 = k$, however, gives two solutions. Indeed, we find that these are

$$(0;\, 1/2\lambda,\, -1/\lambda,\, -1/2\lambda) = (2\lambda+1)/(4\lambda+4) = \lambda/4$$

and

$$(0;\, 1/\lambda,\, -1/2\lambda,\, -1/\lambda) = (2\lambda+1)/(2\lambda+2) = \lambda/2.$$

If the λ-fraction has length 4, we find that $Q_{14} = Q_{24}$ gives the nice condition $r_1 r_2 r_3 r_4 = e_2 e_4$. Since $r_i > 0$ $(1 \leqq i \leqq 4)$, we get $r_1 = r_2 = r_3 = r_4 = 1$, and, for the λ-fraction to be reduced, we must have $e_2 = e_4 = 1$. These conditions yield

$$(0;\, 1/\lambda,\, 1/\lambda,\, 1/\lambda,\, 1/\lambda) = (4\lambda+1)/(6\lambda+6).$$

We next prove that for $m \geqq 5$, $Q_{2m} = Q_{1m}$. In Theorem 1 (iii) we proved that $Q_{2m} \geqq Q_{1m}$. If in the inductive assumption we can replace $\geqq$ by $>$, we obtain the desired result. Hence we prove that the inequality is strict for $m = 5$. The induction is then exactly as in Theorem 1. We suppose that $Q_{15} = Q_{25}$. The equations of (6) lead to the condition

$$r_5 Q_{14} = e_5(Q_{13} - Q_{23}). \tag{12}$$

Since $Q_{14} > 0$, and $Q_{23} \geqq Q_{13}$, we must have $e_5 = -1$. If $Q_{23} = Q_{13}$, then we see that $Q_{14} = 0$ or $r_5 = 0$, which is impossible. We now have $r_5 Q_{14} = Q_{23} - Q_{13} > 0$, and we replace Q_{14} by $r_4 Q_{23} + e_4 Q_{12}$ from (6). The equation in (12) becomes

$$Q_{23} - Q_{13} = r_5(r_4 Q_{23} + e_4 Q_{12}) \geqq r_5(Q_{23} - Q_{12}).$$

From Lemma 2, either $Q_{13} \geqq Q_{12}$ or $Q_{13} + 1 = Q_{12}$. The first alternative leads to a value of $r_5 < 1$ if $Q_{13} > Q_{12}$, while $r_5 = r_4 = 1$ if $Q_{13} = Q_{12}$. Hence the second alternative must obtain.

The second alternative leads to a further equation

$$Q_{23}(r_4 r_5 - 1) = Q_{12}(-1 - e_4 r_5) + 1,$$

along with the additional conditions that $e_2 = 1$ and $r_3 = 1$ (Lemma 3 (i)). Since the left side of this equation is non-negative, it follows that $e_4 = -1$, because $Q_{12} = r_1 r_2 + 1 \neq 1$. Hence, if $r_5 = 1$, we have $Q_{23}(r_4 - 1) = 1$. This implies that $r_4 \geqq 2$ and $Q_{23} = 1$. But $Q_{23} = 1$ leads to the statement $r_1 = 0$ or $r_2 = -e_3$, which is nonsense. Therefore $r_5 \geqq 2$, and we obtain

$$r_5 = \frac{1}{Q_{23} r_4 - Q_{12}} + \frac{Q_{23} - Q_{12}}{Q_{23} r_4 - Q_{12}}. \tag{13}$$

Under the present set of conditions, we deduce that $Q_{23} = 2r_1 r_2 + e_3 r_1 + 1 \geqq Q_{12} + 2$ if $e_3 = 1$, and $Q_{23} \geqq Q_{12} + 1$ if $e_3 = -1$. If $r_4 = 1$, then $e_3 = 1$, since $r_3 = 1$. We find from (13) that r_5 has a value less than 2. If $r_4 \geqq 2$, we find that the first and second members in (13) do not exceed $\frac{1}{2}$ and 1, respectively, so that $r_5 < 2$. We therefore conclude that $Q_{25} > Q_{15}$.

There are two further conditions on the components of c that can be obtained by the same kind of argument as we have been using all through the paper. To avoid repetition we state the results only.

LEMMA 6. *$3Q_{13}-Q_{23}+1 \geqq 0$ or $Q_{12} = Q_{22}\pm 1$.*

LEMMA 7. *$2Q_{1m}-Q_{2m}+2 \geqq 0$ for $m \geqq 4$.*

Although the concluding theorem is a contribution to the converse problem, we include it here because it seems to be interesting.

THEOREM 6. *The units $[(1+5^{\frac{1}{2}})/2]^m$ ($m = 0, \pm 1, \pm 2, \ldots$) are finite λ-fractions and consequently parabolic points.*

Proof. We shall prove the theorem for $m \geqq 0$, as the statement is then obvious for $m < 0$. The units, when represented in terms of the basis $(1, \lambda)$, have the form $\lambda^m = U_m\lambda + U_{m-1}$, where U_i is the ith Fibonnaci number in the sequence 1, 1, 2, 3, ..., and $U_{n+1} = U_n + U_{n-1}$. The proof is by induction.

We verify that for $m = 0$, 1, and 2 the units are finite λ-fractions: $\lambda^0 = \lambda - 1/\lambda$, $\lambda^1 = \lambda$, $\lambda^2 = \lambda + 1 = 2\lambda - 1/\lambda$. We assume that λ^{n-2} is a finite λ-fraction and we shall prove that λ^n is also a finite λ-fraction.

We write $\lambda^n = \lambda^{2n-2}/\lambda^{n-2} = (U_{2n-2}\lambda + U_{2n-3})/(U_{n-2}\lambda + U_{n-3})$. We expand the right side by the nearest integer algorithm [5, p. 560] and obtain formally

$$U_{2n-2}\lambda + U_{2n-3} = (U_{n-2}\lambda + U_{n-3})r_0\lambda + e_1(a_1 + b_1\lambda), \tag{14}$$

where $e_i = \pm 1$ and is chosen so that $a_1 + b_1\lambda > 0$. By using (1) and equating rational and λ-components of both sides, we find that $e_1a_1 = U_{2n-2} - U_{n-2}r_0$ and $e_1b_1 = U_{2n-2} - r_0U_{n-1}$. If we choose $r_0 = U_{n-2} + U_n$ and use the two well known formulae [6, p. 10],

$$U_{2n} = U_n(U_{n-1} + U_{n+1})$$

and

$$U_{2n-3} = U_{n-3}U_{n-1} + U_{n-2}U_n$$

judiciously, we find that $b_1 = 0$ and that $e_1a_1 = (-1)^{n-3}$. We can choose e_1 so that $a_1 > 0$.

Since we now have $(a_1 + b_1\lambda)/(U_{n-2}\lambda + U_{n-3}) < \lambda/2$, our choice of r_0 is the " nearest integer ", in the sense of [5], to λ^n/λ. Hence the resulting λ-fraction is unique. Another way of writing (14) is $\lambda^n = r_0\lambda + e_1/\lambda^{n-2}$. The usual inductive argument for odd and even n proves the theorem.

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THE UNIVERSITY
GLASGOW
Permanent Address: Swarthmore College, Swarthmore, Pa, U.S.A.