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Bridging Continuous and Discrete Optimization

A large part of algorithm design is concerned with problems that optimize or enumerate over discrete structures such as paths, trees, cuts, flows, and matchings in objects such as graphs. Important examples include the following:

(i) Given a graph $G = (V, E)$, a source $s \in V$, a sink $t \in V$, find a **flow** on the edges of G of maximum value from s to t while ensuring that each edge has at most one unit flow going through it.
(ii) Given a graph $G = (V, E)$, find a **matching** of maximum size in G.
(iii) Given a graph $G = (V, E)$, count the number of **spanning trees** in G.

Algorithms for these fundamental problems have been sought for more than a century due to their numerous applications. Traditionally, such algorithms are **discrete** in nature, leverage the rich theory of **duality** and **integrality**, and are studied in the areas of algorithms and combinatorial optimization; see the books by Dasgupta et al. (2006), Kleinberg and Tardos (2005), and Schrijver (2002a). However, classic algorithms for these problems have not always turned out to be fast enough to handle the rapidly increasing input sizes of modern-day problems.

An alternative, **continuous** approach for designing faster algorithms for discrete problems has emerged. At a very high level, the approach is to first formulate the problem as a convex program and then develop continuous algorithms such as gradient descent, the interior point method, or the ellipsoid method to solve it. The innovative use of convex optimization formulations coupled with algorithms that move in geometric spaces and leverage linear solvers has led to faster algorithms for many discrete problems. This pursuit has also significantly improved the state of the art of algorithms for convex optimization. For these improvements to be possible, it is often crucial to abandon an entirely combinatorial viewpoint; simultaneously, fast convergence of continuous algorithms often leverage the underlying combinatorial structure.

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1.1 An Example: The Maximum Flow Problem

We illustrate the interplay between continuous and discrete optimization through the $s-t$-maximum flow problem on undirected graphs.

The maximum flow problem. Given an undirected graph $G=(V,E)$ with $n:=|V|$ and $m:=|E|$, we first define the **vertex-edge incidence matrix** $B\in\mathbb{R}^{n\times m}$ associated to it. Direct each edge $i\in E$ arbitrarily and let i^+ denote the head vertex of i and i^- denote its tail vertex. For every edge i, the matrix B contains a column $b_i := e_{i^+}-e_{i^-}\in\mathbb{R}^n$, where $\{e_j\}_{j\in[n]}$ are the standard basis vectors for $\mathbb{R}^n$.

Given $s\neq t\in V$, an $s-t$-flow in G is an assignment $x:E\to\mathbb{R}$ that satisfies the following **conservation of flow** property: For all vertices $j\in V\setminus\{s,t\}$, we require that the **incoming** flow is equal to the **outgoing** flow, i.e.,

$$\langle e_j, Bx\rangle = 0.$$

An $s-t$-flow is said to be **feasible** if

$$|x_i|\leq 1$$

for all $i\in E$, i.e., the magnitude of the flow in each edge respects its capacity (1 here). The objective of the $s-t$-maximum flow problem is to find a feasible $s-t$-flow in G that maximizes the flow out of s, i.e., the value

$$\langle e_s, Bx\rangle.$$

The $s-t$-maximum flow problem was not only used to encode various real-world routing and scheduling problems; also many fundamental combinatorial problems such as finding a maximum matching in bipartite graph were shown to be its special cases; see Schrijver (2002a,b) for an extensive discussion.

Combinatorial algorithms for the maximum flow problem. An important fact about the $s-t$-maximum flow problem is that there always exists an **integral** flow that maximizes the objective function. As it will be explained later in this book, this is a consequence of the fact that the matrix B is **totally unimodular**: Every square submatrix of B has determinant 0, 1, or -1. Thus, we can restrict

$$x_i\in\{-1,0,1\}$$

for each $i\in E$, making the search space for the optimal $s-t$-maximum flow discrete. Because of this, the problem has been traditionally viewed as a combinatorial optimization problem.

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One of the first combinatorial algorithms for the $s-t$-maximum flow problem was presented in the seminal work by Ford and Fulkerson (1956). Roughly speaking, the **Ford-Fulkerson method** starts by setting $x_i = 0$ for all edges i and checks if there is a path from s to t such that the capacity of each edge on it is 1. If there is such a path, the method adds 1 to the flow value of the edges that point (from head to tail) in the direction of this path and subtracts 1 from the flow values of edges that point in the opposite direction. Given the new flow value on each edge, it constructs a **residual graph** where the capacity of each edge is updated to reflect how much additional flow can still be pushed through it, and the algorithm repeats. If there is no path left between s and t in the residual graph, it stops and outputs the current flow values.

The fact that the algorithm always outputs a maximum $s-t$-flow is nontrivial and a consequence of **duality** – in particular, of the **max-flow min-cut theorem** that states that the maximum amount of flow that can be pushed from s to t is equal to the minimum number of edges in G whose deletion leads to disconnecting s from t. This latter problem is referred to as the $s-t$-minimum cut problem and is the **dual** of the $s-t$-maximum flow problem. Duality gives a way to certify that we are at an optimal solution and, if not, suggests a way to improve the current solution.

It is not hard to see that the Ford-Fulkerson method generalizes to the setting of nonnegative and integral capacities: Now the flow values are

$$x_i \in \{-U, \ldots, -1, 0, 1, \ldots, U\}$$

for some $U \in \mathbb{Z}_{\geq 0}$. However, the running time of the Ford-Fulkerson method in this general capacity case depends linearly on U. As the number of bits required to specify U is roughly $\log U$, this is not a polynomial time algorithm.

Following the work of Ford and Fulkerson (1956), a host of combinatorial algorithms for the $s-t$-maximum flow problem were developed. Roughly, each of them augments the flow in the graph iteratively in an increasingly faster, but combinatorial, manner. The first polynomial time algorithms were by Dinic (1970) and by Edmonds and Karp (1972), who used breadth-first search to augment flows. This line of work culminated in an algorithm by Goldberg and Rao (1998) that runs in $\widetilde{O}\left(m \min\left\{n^{2/3}, m^{1/2}\right\} \log U\right)$ time. Note that unlike the Ford-Fulkerson method, these latter combinatorial algorithms are polynomial time: They find the exact solution to the problem and run in time that depends polynomially on the number of bits required to describe the input. However, since the result of Goldberg and Rao (1998), there was no real progress on improving the running times for algorithms for the $s-t$-maximum flow problem until 2011.

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Convex programming-based algorithms. Starting with the paper by Christiano et al. (2011), the last decade has seen striking progress on the $s-t$-maximum flow problem. One of the keys to this success has been to abandon combinatorial approaches and view the $s-t$-maximum flow problem through the lens of continuous optimization. At a very high level, these approaches still maintain a vector $x \in \mathbb{R}^m$ which is updated in every iteration, but this update is dictated by continuous and geometric quantities associated to the graph and is not constrained to be a feasible $s-t$-flow in the intermediate steps of the algorithm. Here, we outline one such approach for the $s-t$-maximum flow problem from the paper by Lee et al. (2013).

For this discussion, assume that we are also given a value F and that we would like to find a feasible $s-t$-flow of value F.[1] Lee et al. (2013) start with the observation that the problem of checking if there is a feasible $s-t$-flow of value F in G is equivalent to determining if the intersection of the sets

$$\{x \in \mathbb{R}^m : Bx = F(e_s - e_t)\} \cap \{x \in \mathbb{R}^m : |x_i| \leq 1,\ \forall i \in [m]\} \tag{1.1}$$

is nonempty. Moreover, finding a feasible $s-t$-flow of value F is equivalent to finding a point in this intersection. Note that the first set in Equation (1.1) is the set of all $s-t$-flows (a linear space) and the second set is the set of all vectors that satisfy the capacity constraints, in this case the ℓ_∞-ball of radius one, denoted by B_∞, which is a polytope.

Their main idea is to reduce this nonemptiness testing problem to a convex optimization problem. To motivate their idea, suppose that we have convex sets K_1 and K_2 and the goal is to find a point in their intersection (or assert that there is none). One way to formulate this problem as a convex optimization problem is as follows: Find a point $x \in K_1$ that minimizes the distance to K_2. As K_1 is convex, for this formulation to be a convex optimization problem, we need to find a convex function that captures the distance of a point x to K_2. It can be checked that the squared Euclidean distance has this property. Alternatively, one could consider the convex optimization problem where we switch the roles of K_1 and K_2: Find a point $x \in K_2$ that minimizes the distance to K_1. Note here that, while the squared Euclidean distance to a set is a convex function, it is nonlinear. Thus, at this point it may seem like we are heading in the wrong direction. We started off with a combinatorial problem that is a special type of a linear programming problem, and here we are with a nonlinear optimization formulation for it.

[1] Using a solution to this problem, we could solve the $s-t$-maximum flow problem by performing a binary search over F.

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Thus the following questions arise: Which formulation should we choose? And why should this convex optimization approach lead us to faster algorithms?

Lee et al. (2013) considered the following convex optimization formulation for the $s-t$-maximum flow problem:

$$\begin{aligned} \min_{x \in \mathbb{R}^m} \quad & \operatorname{dist}^2(x, B_\infty) \\ \text{such that} \quad & Bx = F(e_s - e_t), \end{aligned} \tag{1.2}$$

where $\operatorname{dist}(x, B_\infty)$ is the Euclidean distance of x to the set $B_\infty \coloneqq \{y \in \mathbb{R}^m : \|y\|_\infty \leq 1\}$. As the optimization problem above minimizes a convex function over a convex set, it is indeed a convex program. The choice of this formulation, however, comes with a foresight that relies on an understanding of algorithms for convex optimization.

A basic method to minimize a convex function is **gradient descent**, which is an iterative algorithm that, in each iteration, takes a step in the direction of the negative gradient of the function it is supposed to minimize. While gradient descent does so in an attempt to optimize the function locally, the convexity of the objective function implies that a local minimum of a convex function is also its global minimum. Gradient descent only requires oracle access to the gradient, or first derivative of the objective function and is, thus, called a **first-order** method. It is really a meta-algorithm and, to instantiate it, one has to fix its parameters such as the step-size and must specify a starting point. These parameters, in turn, depend on various properties of the program including estimates of smoothness of the objective function and those of the closeness of the starting point to the optimal point.

For the convex program in Equation (1.2), the objective function has an easy-to-compute first-order oracle. This follows from the observation that it decomposes into a sum of squared-distances, one for each coordinate, and each of these functions is quadratic. Moreover, the objective function is **smooth**: The change in its gradient is bounded by a constant times the change in its argument; one can visually inspect this in Figure 1.1.

One problem with the application of gradient descent is that the convex program in (1.2) has constraints $\{x \in \mathbb{R}^m : Bx = F(e_s - e_t)\}$ and, hence, the direction gradient descent asks us to move can take us out of this set. A way to get around this is to project the gradient of the objective function onto the subspace $\{x \in \mathbb{R}^m : Bx = 0\}$ at every step and move in the direction of the projected gradient. However, this projection step requires solving a least squares problem that, in turn, reduces to the numerical problem of solving a linear system of equations. While one can appeal to the Gaussian elimination

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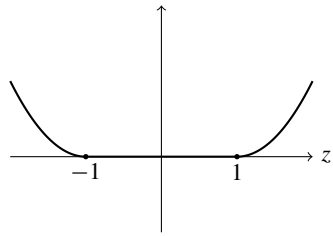

Figure 1.1 The function $\text{dist}^2(z, [-1, 1])$.

method for this latter task, it is not fast enough to warrant improvements over combinatorial algorithms mentioned earlier. Here, a major result discovered by Spielman and Teng (2004) implies that such a projection can, in fact, be computed in time $\widetilde{O}(m)$. This is achieved by noting that the linear system that arises when projecting a vector onto the subspace $\{x \in \mathbb{R}^m : Bx = 0\}$ is the same as solving **Laplacian systems** that are of the form $BB^\top y = a$ (for a given vector a), where B is a vertex-edge incidence matrix of the given graph. Such a result is not known for general linear systems and (implicitly) relies on the combinatorial structure of the graph that gets encoded in the matrix B.

Thus, roughly speaking, in each iteration the projected gradient descent algorithm takes a point x_t in the space of all $s - t$-flows of value F, moves toward the set B_∞ along the negative gradient of the objective function, and then projects the new point back to the linear space; see Figure 1.2 for an illustration. While each iterate is an $s - t$-flow, it is not a feasible flow.

A final issue is that such a method may not lead to an exact solution but only an approximate solution. Moreover, in general, the number of iterations depends inverse polynomially on the quality of the desired approximation. Lee et al. (2013) proved the following result: There is an algorithm that, given an $\varepsilon > 0$, can compute a feasible $s - t$-flow of value $(1 - \varepsilon)F$ in time $\widetilde{O}(mn^{1/3}\varepsilon^{-2/3})$. If we ignore the ε in their bound, this improved the result of Goldberg and Rao (1998) mentioned earlier.

We point out that the combinatorial algorithm of Goldberg and Rao (1998) has the same running time even when the input graph is directed. It is not clear how to generalize the gradient descent–based algorithm for the $s-t$-maximum flow problem presented above to run for directed graphs.

The results of Christiano et al. (2011) and Lee et al. (2013) were further improved using increasingly sophisticated ideas from continuous optimization and finally led to a nearly linear time algorithm for the undirected $s - t$-maximum flow problem in a sequence of work by Sherman (2013), Kelner et al. (2014), and Peng (2016). Remarkably, while these improvements

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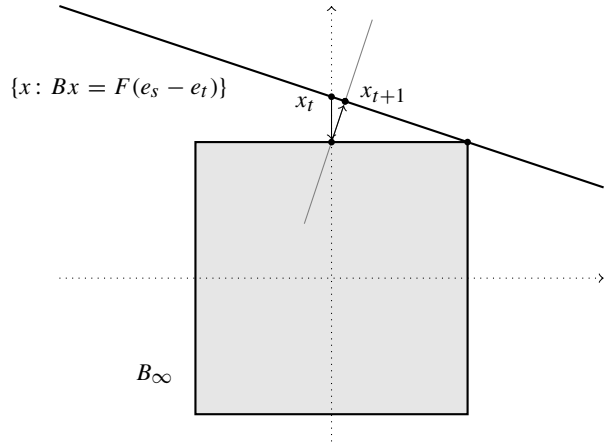

Figure 1.2 An illustration of one step of the projected gradient descent in the algorithm by Lee et al. (2013).

abandoned discrete approaches and used algorithms for convex optimization, beating the running times of combinatorial algorithms leveraged the underlying combinatorial structure of the $s-t$-maximum flow problem.

The goal of this book is to enable a reader to gain an in-depth understanding of algorithms for convex optimization in a manner that allows them to apply these algorithms in domains such as combinatorial optimization, algorithm design, and machine learning. The emphasis is to derive various convex optimization methods in a principled manner and to establish precise running time bounds in terms of the input length (and not just on the number of iterations). The book also contains several examples, such as the one of $s-t$-maximum flow presented earlier, that illustrate the bridge between continuous and discrete optimization. Laplacian solvers are not discussed in this book. The reader is referred to the monograph by Vishnoi (2013) for more on that topic.

The focus of Chapters 3–5 is on basics of convexity, computational models, and duality. Chapters 6–8 present three different first-order methods: gradient descent, mirror descent and multiplicative weights update method, and **accelerated gradient descent.** In particular, the discussion here is presented in detail as an application in Chapter 6. In fact, the fastest version of the method of Lee et al. (2013) uses the accelerated gradient method. Chapter 7 also draws a connection between **mirror descent** and the **multiplicative weights update** method and shows how the latter can be used to design a fast (approximate) algorithm for the bipartite maximum matching problem. We

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remark that the algorithm of Christiano et al. (2011) relies on the multiplicative weights update method.

Beyond approximate algorithms? The combinatorial algorithms for the $s-t$-maximum flow problem, unlike the first-order convex optimization–based algorithms described above, are exact. One can convert the latter approximate algorithms to exact ones, but it may require setting a very small value of ε making the overall running time non-polynomial. The remainder of the book is dedicated to developing algorithms for convex optimization – **interior point** and **ellipsoid** – whose number of iterations depend **poly-logarithmically** on ε^{-1} as opposed to polynomially on ε^{-1}. Thus, if we use such algorithms, we can set ε to be small enough to recover exact algorithms for combinatorial problems at hand. These algorithms use deeper mathematical structures and more sophisticated strategies (as explained later). The advantage in learning these algorithms is that they work more generally – for linear programs and even convex programs in a very general form. Chapters 9–13 develop these methods, their variants, and exhibit applications to a variety of discrete optimization and counting problems.

1.2 Linear Programming

The $s-t$-maximum flow problem on undirected graphs is a type of linear program: a convex optimization problem where the objective function is a linear function and all the constraints are either linear equalities or inequalities. In fact, the objective function is to maximize the flow value $F \geq 0$ constrained to the set of feasible $s-t$-flows of value F; see Equation (1.1).

A linear program can be written in many different ways and we consider its **standard form**, where one is given a matrix $A \in \mathbb{R}^{n \times m}$, a constraint vector $b \in \mathbb{R}^m$, and a cost vector $c \in \mathbb{R}^n$, and the goal is to solve the following optimization problem:

$$\begin{aligned} &\max_{x \in \mathbb{R}^m} \langle c, x \rangle \\ &\text{such that } Ax = b, \\ &x \geq 0. \end{aligned}$$

Typically we assume $n \leq m$ and, hence, the rank of A is at most n. Analogous to the $s-t$-maximum flow problem, linear programming has a rich duality theory, and in particular the following is the **dual** of the above linear program:

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$$\min_{y \in \mathbb{R}^n} \langle b, y \rangle$$

$$\text{such that } A^\top y \geq c.$$

Note that the dual is also a linear program and has n variables.

Linear programming duality asserts that if there is a feasible solution to both the linear program and its dual, then the optimal values of these two linear programs are the same. Moreover, it is often enough to solve the dual if one wants to solve the primal and vice versa. While duality has been known for linear programming for a very long time (see Farkas [1902]), a polynomial time algorithm for linear programming was discovered much later. What was special about the $s-t$-maximum flow problem that led to a polynomial time algorithm for it before linear programming?

As mentioned earlier, one crucial property that underlies the $s-t$-maximum flow problem is integrality. If one encodes the $s-t$-maximum flow problem as a linear program in the standard form, the matrix A turns out to be totally unimodular: Determinants of all square submatrices of A are $0, 1$, or -1. In fact, in the case of the $s-t$-maximum flow problem, A is just the vertex-edge incidence matrix of the graph G (which we denoted by B) that can be shown to be totally unimodular. Because of linearity, one can always assume without loss of generality that the optimal solution is an extreme point, i.e., a **vertex** of the polyhedra of constraints (not to be confused with the vertex of a graph). Every such vertex arises as a solution to a system of linear equations involving a subset of rows of the matrix A. The total unimodularity of A, then, along with Cramer's rule from linear algebra, implies that each vertex of the polyhedra of constraints has integral coordinates.

While duality and integrality do not directly imply a polynomial time algorithm for the $s-t$-maximum flow problem, the mathematical structure that enables these properties is relevant to the design of efficient algorithms for this problem. It is worth mentioning that these ideas were generalized in a major way by Edmonds (1965a,b) who figured out an integral polyhedral representation for the **matching problem** and gave a polynomial time algorithm for optimizing linear functions over this polyhedron.

For general linear programs, however, integrality does not hold. The reason is that for a general matrix A, the determinants of submatrices that show up in the denominator of the vertices of the associated polyhedra may not be 1 or -1. However, for A with integer entries, these determinants cannot be more than $\text{poly}(n, L)$ in magnitude, where L is the number of bits required to encode A, b and c. This is a consequence of the fact that determinant of a matrix with integer entries bounded by 2^L is no more than $n!\, 2^{nL}$. While there was a

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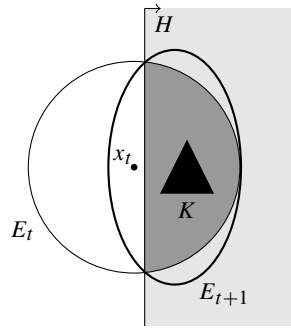

Figure 1.3 Illustration of one step of the ellipsoid method for the polytope K.

combinatorial algorithm for linear programming, e.g., the **simplex method** of Dantzig (1990) that moved from one vertex to another, none was known to run in polynomial time (in the bit complexity) in the worst case.

Ellipsoid method. In the late 1970s, a breakthrough occurred and a polynomial time algorithm for linear programming was discovered by Khachiyan (1979, 1980). The ellipsoid method is a geometric algorithm that checks if a given linear program is feasible or not. As in the case of the $s-t$-maximum flow problem, solving this feasibility problem implies an algorithm to optimize a linear function via a binary search argument.

In iteration t, the ellipsoid method approximates the feasible region of the linear program with an **ellipsoid** E_t and outputs the center (x_t) of this ellipsoid as its guess for a feasible point. If this guess is incorrect, it requires a **certificate** – a hyperplane H that separates the center from the feasible region. It uses this **separating hyperplane** to find a new ellipsoid (E_{t+1}) that encloses the intersection of E_t and the halfspace of H in which the feasible region lies; see Figure 1.3. The key point is that the update ensures that the volume of the ellipsoid reduces at a fast enough rate and only requires solving a linear system of equations to find the new ellipsoid from the previous one. If the **volume** of the ellipsoid becomes so small that it cannot contain any feasible point, we can safely assert the infeasibility of the linear program.

The ellipsoid method belongs to the larger class of **cutting plane methods** as, in each step, the current ellipsoid is cut by an affine halfspace and a new ellipsoid that contains this intersection is determined. The final running time of Khachiyan's ellipsoid method was a polynomial in n, L and, importantly, in $\log \frac{1}{\varepsilon}$: The algorithm output a point $\hat{x}$ in the feasible region such that

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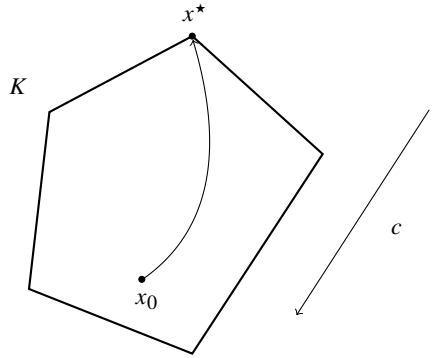

Figure 1.4 Illustration of the interior point method for the polytope K.

$$\langle c, \hat{x} \rangle \leq \langle c, x^\star \rangle + \varepsilon.$$

This implied that one can handle an error as small as $2^{-\text{poly}(n,L)}$, and this is all we need for linear programming to be in polynomial time. While this put linear programming in the complexity class **P** for the first time, the resulting algorithm, when specialized for combinatorial problems such as the $s-t$-maximum flow problem, was far from competitive in terms of running time.

Interior point methods. In 1984, another continuous polynomial time algorithm to solve linear programs was discovered by Karmarkar (1984): This time the idea was to move in the interior of the feasible region until one reaches the optimal solution; see Figure 1.4. Karmarkar's algorithm had its roots in the barrier method from nonlinear optimization. The barrier method is one way to convert a constrained optimization problem to an unconstrained one by choosing a **barrier function** for the constraint set. Roughly speaking, a barrier function for a convex set is a function that is finite only in the interior of it and increases to infinity as one approaches the boundary of the feasible region. Once we have a barrier function for a constraint set, we can add it to the objective function to penalize any violation of the constraints. For such a function to be algorithmically useful, it is desirable that it is a convex a function and also has certain smoothness properties (as explained later in this book).

Renegar (1988) combined the barrier approach with **Newton's method** for root finding to improve upon Karmarkar's method. Unlike gradient descent, which was based on the first-order approximation of the objective function, Renegar's algorithm, following Newton's method, considered a **quadratic**

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approximation to the objective function around the current point and optimized it to find the next point. His method took roughly $\widetilde{O}\left(\sqrt{m}\,L\right)$ iterations to find a solution to the given linear program. Here L is the bit complexity of the input (A, b, c). Further, each iteration just had to solve a linear system of equations of size $m \times m$.

While one way to view Newton's method is as a second-order method, an equivalent way is to view it as performing steepest descent in a geometric space where the inner product and the norm change depending on the current location of the algorithm: At a point x, the inner product between vectors u and v is defined as $u^\top \nabla^2 F(x) v$ for a barrier function F. As F is convex, this gives rise to a **local norm** and is an example of a **Riemannian metric**.

However, unlike the ellipsoid method that just needs a separating hyperplane, interior point methods require the constraints explicitly in order to compute the Hessian of the barrier function for the constraint set. In Chapter 9, we derive Newton's method both as a quadratic approximation and as a steepest descent on a Riemannian manifold and present its analysis using local norms. In Chapter 10, we introduce barrier functions and present Renegar's path-following interior point method for linear programming.

1.3 Fast and Exact Algorithms via Interior Point Methods

Despite the remarkable effort that went into improving interior point methods in the late 1980s, they could still not compete with combinatorial algorithms for problems such as the $s-t$-maximum flow problem. A key obstacle was the fact that solving a linear system of equations (a primitive used at each step of ellipsoid and interior point method) required roughly $O(m^{2.373})$ time.

Vaidya (1990) observed that the combinatorial structure of the problem manifests in the linear systems that arise, and this structure could be used to speed up certain linear programs. For instance, and as mentioned earlier, for the $s-t$-maximum flow problem, the linear systems that arise correspond to **Laplacian systems**. Vaidya presented some initial results for such linear systems that gave hope for improving the per-iteration cost. His program was completed by Spielman and Teng (2004), who gave an $\widetilde{O}(m)$ time algorithm to solve Laplacian systems. And, using this (and a few more ideas), Daitch and Spielman (2008) gave an exact interior point method for the maximum flow problem that runs in time $\widetilde{O}(m^{1.5} \log U)$, matching the performance of the algorithm by Goldberg and Rao (1998) when $m = O(n)$. In fact, their method could also solve the more general $s-t$-minimum cost flow problem

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in $\widetilde{O}(m^{1.5} \log U)$ time and improved upon all prior algorithms by a factor of about $\widetilde{O}(n/m^{1/2})$. This is presented in Chapter 11 and was the first sign that general-purpose convex optimization methods can be specialized to be comparable to or even beat combinatorial algorithms.

Beyond log-barrier functions. Meanwhile, in a sequence of papers, Vaidya (1987, 1989a,b) introduced the **volumetric barrier** as a generalization of Karmarkar's barrier function and obtain modest improvements on the number of iterations while ensuring that each iteration of his interior point method for linear programming still just required multiplying two $m \times m$ matrices.

Nesterov and Nemirovskii (1994) abstracted the essence of barrier functions and introduced the notion of **self-concordance**. They introduced the **universal barrier** function and showed that the number of iterations in an interior point method that uses their universal barrier function is $\sqrt{n}$, where n is the dimension of the feasible region. They also showed that this bound cannot, in general, go below $\sqrt{n}$. Computing the barrier function that achieved this, however, was not easier than solving the linear programming problem itself.

Finally, Lee and Sidford (2014), building upon the ideas of Vaidya, gave a new barrier function for interior point methods that not only came sensationally close to the bound of $O(\sqrt{n})$ of Nesterov and Nemirovskii (1994), but each iteration of their method just solves a small number of linear systems. Using these ideas, Lee and Sidford (2014) gave an exact algorithm for the $s - t$-maximum flow problem that runs in $\widetilde{O}(m\sqrt{n} \log^2 U)$ time, the first improvement since Goldberg and Rao (1998). Their method also gave an algorithm that runs in the same time for the $s - t$-minimum cost flow problem, improving upon the results of Daitch and Spielman (2008) and Goldberg and Tarjan (1987). Chapter 11 outlines the methods of Vaidya, Nesterov-Nemirovskii, and Lee-Sidford.

1.4 Ellipsoid Method beyond Succinct Linear Programs

As mentioned earlier, an advantage of the ellipsoid method over interior point methods was the fact that they just needed a **separation oracle** to the polytope to optimize a linear function over it. A separation oracle for a convex set is an algorithm that, given a point, either asserts that the point is in the convex set or outputs a hyperplane that separates the point from the convex set. This fact was exploited by Grötschel et al. (1981) to show that the ellipsoid method can also be used to perform linear optimization over combinatorial polytopes

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that do not have a succinct linear description. Prominent examples include the matching polytope for general graphs and various matroid polytopes.

Chapter 12 presents the general framework of cutting plane methods, derives the ellipsoid method of Khachiyan, and applies it to the problem of linear optimization over combinatorial **0-1-polytopes** for which we only have a separation oracle. Thus, the ellipsoid method may, sometimes, be the only way one can obtain polynomial time algorithms for combinatorial problems.

Grötschel et al. (1981, 1988) noticed something more about the ellipsoid method: It can be extended to **general convex programs** of the form

$$\min_{x \in K} f(x), \tag{1.3}$$

where both f and K are convex. Their method outputs a point $\hat{x}$ such that

$$f(\hat{x}) \leq \min_{x \in K} f(x) + \varepsilon$$

in time, roughly, $\text{poly}\left(n, \log \frac{R}{\varepsilon}, T_f, T_K\right)$, where R is such that K is contained in a ball of radius R, time T_f is required to compute the gradient of f, and time T_K is required to separate a point from K. Thus, this settled the problem of designing algorithms for convex optimization in its most generality. However, we emphasize that this result does not imply that any convex program of the form (1.3) is in **P**. The reason is that sometimes it may be impossible to get an efficient gradient oracle for f, an efficient separation oracle for K, or a good enough bound on R.

In Chapter 13, we present an algorithm to minimize a convex function over a convex set and prove the guarantee mentioned above. Subsequently, we show how this can be used to give a polynomial time algorithm for another combinatorial problem – **submodular function minimization** – given just an evaluation oracle to the function. A submodular function $f : 2^{[m]} \to \mathbb{R}$ has the **diminishing returns** property: For sets $S \subseteq T \subseteq [m]$, the marginal gain of adding an element not in T to S is at least the marginal gain of adding it to T, i.e., for all $i \notin T$:

$$f(S \cup \{i\}) - f(S) \geq f(T \cup \{i\}) - f(T).$$

The ability to minimize submodular set functions allows us to obtain separation oracles for matroid polytopes. Submodular functions arose in discrete optimization and have recently also appeared as objective functions of machine learning tasks such as data summarization.

Finally, in Chapter 13, we consider convex programs that have been recently used for designing various counting problems over discrete sets, such as spanning trees. Given a graph $G = (V, E)$, let $\mathcal{T}_G$ denote the set of spanning trees

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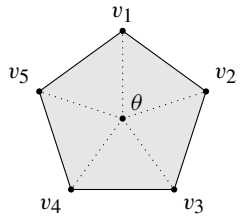

$$\max -\sum_i p_i \log p_i$$

subject to

$$\sum_i p_i v_i = \theta,$$

$$\sum_i p_i = 1,$$

$$\forall i,\ p_i \geq 0.$$

Figure 1.5 The maximum entropy problem and its convex program.

in G and let P_G denote the **spanning tree polytope**, i.e., the convex hull of indicator vectors of all the spanning trees in $\mathcal{T}_G$. Each vertex of P_G corresponds to a spanning tree in G. The problem that we consider is the following: Given a point $\theta \in P_G$, find a way to write θ as a convex combination of the vertices of the polytope P_G so that the probability distribution corresponding to this convex combination maximizes **Shannon entropy**; see Figure 1.5. To see what this problem has to do with counting spanning trees, the reader is encouraged to check that if we let θ be the average of all the vertex vectors of P_G, the value of this optimization problem is exactly $\log |\mathcal{T}_G|$.

As stated, this is an optimization problem where there is a variable corresponding to each vertex of the polytope, the constraints on these variables are linear, and the objective function maximizes a concave function; see Figure 1.5. Thus, this is a convex program. Note, however, that $|\mathcal{T}_G|$ can be exponential in the number of vertices in G; the complete graph on n vertices has n^{n-2} spanning trees. Thus, the number of variables can be exponential in the input size, and it is not clear how to solve this problem. Interestingly, if one considers the dual of this convex optimization problem, the number of variables becomes the number of edges in the graph.

However, there are obstacles to applying the general convex optimization method to this setting, and this is discussed in detail in Chapter 13. In particular, Chapter 13 presents a polynomial time algorithm for the **maximum entropy problem** over polytopes due to Singh and Vishnoi (2014) and Straszak and Vishnoi (2019). Such algorithms have been used to design very general **approximate counting** algorithms for discrete problems by Anari and Oveis Gharan (2017) and Straszak and Vishnoi (2017), and have enabled breakthrough results for the **traveling salesman problem** by Oveis Gharan et al. (2011) and Karlin et al. (2020).

Curiously, nature – via evolution – has developed continuous algorithms to solve discrete problems. An example is the organism *Physarum polycephalum*,

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a slime mold, that uses a continuous-time dynamical system to solve the shortest path problem in a maze; see the papers by Nakagaki et al. (2000) and Bonifaci et al. (2012). Interestingly, the dynamics of slime mold has served as an inspiration to new continuous algorithms for the maximum flow problem and for linear programming; see the papers by Straszak and Vishnoi (2016a,b, 2021) and Exercises 9.11 and 11.8. Another striking example is a work by Chastain et al. (2014), which argues that the mathematical description of sexual evolution is equivalent to the multiplicative weight updates algorithm. Thus, the bridge between continuous and discrete optimization transcends the artificial world.

In summary, the last few years have seen dramatic progress in approximate and exact algorithms for discrete problems. This progress is a result of viewing discrete problems through the powerful lens of continuous methods and has been fueled by major advances in algorithms for convex optimization. The examples presented here hint at how continuous formulations allow algorithms to harness geometric and analytic structures absent in the discrete world. However, much remains to be done: from discovering even faster algorithms, to conceptually simplifying the existing ones, to explaining the effectiveness of continuous methods for discrete problems.