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# A NOTE ON CONVEX CONES IN TOPOLOGICAL VECTOR SPACES

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The aim of the present note is an independent study of a class of convex cones, which is the largest possible with regard to existence of cone-maximal points in abstract vector optimization problems.

## 1. Introduction

This paper is motivated by results obtained in [8,9] for abstract vector maximisation problems. Let $X$ be a Hausdorff topological vector space (t.v.s.), let $C$ be a convex cone in $X$, that is a convex nonempty subset which is closed under nonnegative scalar multiplication and let $B$ be a nonempty subset in $X$. We say that $e \in B$ is $i$-maximal (maximal up to indifference with respect to $C$) in $B$, and write $e \in E_C(B)$, if $e - b \in C$ whenever $b - e \in C$ and $b \in B$. The elements of $E_C(B)$ are also called nondominated, efficient or Pareto optimals (see the references in [8]). We shall not use notions of maximality different from the above one, so we shall call an $i$-maximal element simply maximal. See also [1,7] and the references therein for a study of maximal points in connection with multicriteria optimization problems.

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Hartley [3] proved that in a finite dimensional t.v.s. $E_C(B) \neq \emptyset$ for every convex cone $C$ and for every $C$ - compact subset $B$ of $X$ ($C$-compct means $B \cap (x + \overline{C})$ is nonempty and compact for some $x \in X$ ). Here and subsequently the bar means the closure in the topology of $X$. Borwein [2] showed that $E_C(B) = \emptyset$ in the infinite dimensional case if $C$ is closed and $B$ is $C$-compact (see also [4]). The class $\mathcal{C}$ of convex cones defined by the author in [8] is the largest possible with respect to the existence of maximal points. A convex cone $C$ belongs to the class $\mathcal{C}$ if the following condition holds:

for every closed vector subspace $L$ of $X$ $C \cap L$ is a vector subspace whenever $\overline{C \cap L}$ is a vector subspace.

It was observed in [8] that (*) is equivalent to the apparently weaker condition where $L$ is assumed to be a closed subspace of $\overline{C} \cap (-\overline{C})$.

The class $\mathcal{C}$ contains each of the following classes of convex cones $C$ :

(i) $C$ is contained in a finite dimensional space; or more generally

(ii) $\overline{C} \cap (-\overline{C})$ is a finite dimensional space; in particular, if the closure of $C$ is pointed, that is $\overline{C} \cap (-\overline{C}) = \{0\}$ then $C \in \mathcal{C}$ ;

(iii) $C$ is closed;

(iv) $C \{0\}$ is open; or more generally

(v) $C$ admits a continuous $C$ - positive functional $f$, that is $f \in X^*$, the topological dual of $X$, $f(x) \geq 0$ for every $x \in C$ and $f(x) > 0$ if $x \notin C \cap (-C)$.

The above facts (i) - (v) are easy to prove using the definition of the class $\mathcal{C}$ and separation arguments. However, in every normed space, there exist convex cones with nonempty interior which do not belong to $\mathcal{C}$ [8].

The following existence results are known regarding the class $\mathcal{C}$ :

(i) [8] (sufficient condition) If $C \subseteq X$ and $C \in \mathcal{C}$ then $E_C(B) \neq \emptyset$ for every $C$-compact subset $B$ of $X$.

(ii) [8] (necessary condition) Let $C \subseteq X$ be such that $\overline{C} \cap (-\overline{C})$ is a metrisable linear space. If $E_C(B) = \emptyset$ for every nonempty compact subset $B$ of $X$ then $C \in \mathcal{C}$ .

(iii) [9] If $X$ is locally convex then $E_C(B) \neq \emptyset$ for every weakly $C$-compact (that is $C$-compact in the weak topology of $X$ ) subset $B$ of $X$ and every $C \in \mathcal{C}$ .

The only results previously known which covered the existence of maximal points for non-closed cones were the following:

(1) Maximal points always exist in $C$-compact sets if there exists a continuous $C$-positive functional on $X$ (see for example [2] ).

(2) Maximal points always exist in $C$-compact sets if $\overline{C}$ is pointed.

The latter is a straightforward consequence of the existence result for closed cones since in the case when $\overline{C}$ is pointed we have that $E_{\overline{C}}(B) \subseteq E_C(B)$ .

Moreover the following generalisation of the Krein-Milman theorem was proved in [9] :

if $X$ is a locally convex t.v.s. and $B$ is a nonempty convex weakly compact subset of $X$ then $E_C(B)$ contains an extreme point of $B$ whenever $C \subseteq X$ and $C \in \mathcal{C}$ . This result was proved by Borwein [2] for the class of weakly closed convex cones.

Thus an independent study of convex cones satisfying (*) is clearly of interest. The aim of this paper is to describe the geometrical and topological structure for such cones in arbitrary topological vector spaces.

Obviously, a convex cone $C$ which is dense in the vector space, yet is not itself a vector subspace, does not satisfy condition (*). However, cones which fail to satisfy (*) follow a similar pattern. Namely, we shall show (Theorem 2.1) that every convex cone $C$ can be written in the form $C = C_1 + C_2$ (even stronger, $C = C_1 \cup C_2$) where $C_1$ and $C_2$ are convex cones such that $\overline{C}_1$ is a linear subspace, $C_2$ is pointed, $C_2 \in \mathcal{C}$ , and

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$\overline{C}_1 \cap C_2 = \{0\}$ . Moreover, $C \in \mathcal{C}$ if and only if $C_1$ above is a linear subspace.

The paper is organized as follows. The main results, concerning the structure of convex cones in an arbitrary t.v.s., are proved in Section 2. Some additional facts on the existence of maximal elements are given in Section 3.

## 2. On the structure of convex cones

The results of this section hold for an arbitrary t.v.s. $X$ , not necessarily Hausdorff. $C$ denotes any convex cone in $X$ , and by $\ell(C)$ we shall denote the greatest vector subspace of $X$ contained in $C$ ; that is $\ell(C) = C \cap (-C)$ . Let $K$ be the set of all convex cones in $X$ . Define the operation $T : K \to K$ by $T(C) = C \cap \ell(\overline{C})$ . For every ordinal $\alpha$ put $T^{\alpha}(C) \equiv T(T^{\alpha-1}(C))$ if $\alpha-1$ exists and $T^{\alpha}(C) \equiv \bigcap_{\beta<\alpha} T^{\beta}(C)$ otherwise. It is easy to see that $T^{\alpha}(C) = C \cap \ell(\overline{T^{\alpha-1}(C)})$ whenever $\alpha-1$ exists and $T^{\alpha}(C) = C \cap \bigcap_{\beta<\alpha} \ell(\overline{T^{\beta}(C)})$ for $\alpha$ a limit ordinal. Moreover, since the transfinite sequence of vector subspaces $\ell(\overline{T^{\alpha}(C)})$ is non-increasing, it must terminate and so also does the transfinite sequence $\{T^{\alpha}(C)\}$ . Hence there must exist an ordinal $\alpha(C)$ such that $T^{\alpha(C)}(C) = T^{\alpha}(C)$ for every $\alpha \geq \alpha(C)$ .

Observe that always $\ell(C) \subseteq T^{\alpha(C)}(C) \subseteq C$ . Thus $C$ is pointed, that is $\ell(C) = \{0\}$ if and only if $T^{\alpha(C)}(C)$ is pointed. If $C$ is closed then $T^{\alpha(C)}(C) = \ell(C)$ . We shall see (Proposition 2.1) that the latter is a characteristic property of convex cones from the class $\mathcal{C}$ .

The convex cone $T^{\alpha(C)}(C)$ has the following properties.

LEMMA 2.1. *(i)* $\overline{T^{\alpha(C)}(C)}$ *is a vector subspace.*

*(ii)* $T^{\alpha(C)}(C) = C \cap \overline{T^{\alpha(C)}(C)}$.

*(iii) If* $L$ *is a vector subspace such that* $\overline{C \cap L}$ *is a vector subspace then* $C \cap L = T^{\alpha}(C) \cap L$ *for every ordinal* $\alpha$ *and in particular*

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$C \cap L = T^{\alpha(C)}(C) \cap L$.

*(iv) If* $T^{\alpha(C)}(C)$ *is a vector subspace then* $T^{\alpha(C)}(C) = \ell(C)$ .

Proof. Since $T^{\alpha(C)}(C) = T^{\alpha(C)+1}(C) = C \cap \ell(\overline{T^{\alpha(C)}(C)})$ we obtain that $T^{\alpha(C)}(C) \subseteq \ell(\overline{T^{\alpha(C)}(C)}) \subseteq \overline{T^{\alpha(C)}(C)}$, whence $\overline{T^{\alpha(C)}(C)} = \ell(\overline{T^{\alpha(C)}(C)})$, which proves the first two assertions. For (iii) take a vector subspace $L$ of $X$ such that $\overline{C \cap L}$ is a vector space. Then $\overline{C \cap L} \subseteq \ell(\overline{C})$ and $C \cap L \subseteq C \cap \overline{C \cap L} \subseteq C \cap \ell(\overline{C}) = T(C)$ . Using transfinite induction we can actually show that $C \cap L \subseteq T^{\alpha}(C)$ for every ordinal $\alpha$ . Thus $C \cap L \subseteq T^{\alpha}(C) \cap L \subseteq C \cap L$ for every $\alpha$ , which proves (iii). Assume now that $T^{\alpha(C)}(C)$ is a vector space. As was observed, $\ell(C) \subseteq T^{\alpha(C)}(C)$ and, since $\ell(C)$ is the greatest vector subspace contained in $C$ , we must have that $\ell(C) = T^{\alpha(C)}(C)$, which completes the proof of the lemma. □

Note that (ii) in Lemma 2.1 is a particular case of (iii) in view of (i). Moreover, there exists a pointed convex cone $C$ such that $T^{\alpha(C)}(C) = C$ and $\overline{T^{\alpha(C)}(C)} = X$. [5, p.9].

PROPOSITION 2.1. *Let* $C$ *be a convex cone in a t.v.s.* $X$ *. Then the following conditions are equivalent.*

*(i)* $C \in \mathcal{C}$ .

*(ii)* $T^{\alpha(C)}(C) = \ell(C)$.

Proof. Assume that $C \in \mathcal{C}$ Applying Lemma 2.1 (i) and (ii) we obtain that $\overline{C \cap \overline{T^{\alpha(C)}(C)}} = \overline{T^{\alpha(C)}(C)}$ is a vector space. Since $C \in \mathcal{C}$ we must actually have, by (*), that $T^{\alpha(C)}(C) = C \cap \overline{T^{\alpha(C)}(C)}$ is a vector space. Thus, by Lemma 2.1 (iv), $T^{\alpha(C)}(C) = \ell(C)$ .

Conversely, assume that $T^{\alpha(C)}(C) = \ell(C)$ . Take any vector subspace $L$ of $X$ such that $\overline{C \cap L}$ is a vector space. By Lemma 2.1 (iii)

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$C \cap L = \overline{T^{\alpha(C)}(C)} \cap L = \ell(C) \cap L$, and this is a vector space, which means that $C \in \mathcal{C}$. □

As one can see from the above proof the condition (*) implies, unexpectedly, the apparently stronger condition in which we do not impose the assumption of closure on the vector space $L$ of $X$. Let us summarise the equivalent statements to the condition (*) in the following remark.

Remark 2.1. Let $C$ be a convex cone in a t.v.s. $X$. Then the following conditions are equivalent:

(i) $C \in \mathcal{C}$.

(ii) [8] For every closed subspace $L$ of $\ell(\overline{C})$ if $\overline{C \cap L}$ is a linear space then so is $C \cap L$.

(iii) For every subspace $L$ of $X$ if $\overline{C \cap L}$ is a linear space then so is $C \cap L$.

On the other hand, it is not true that if $\overline{C \cap L}$ is a vector space for some vector subspace $L$ in $X$, then $C \cap \overline{L}$ is also a vector subspace, although $C \cap L$ and $\overline{C \cap \overline{L}}$ may both be vector subspaces. A simple example can be provided, if we take a vector subspace $L$ in $X$ which is not closed. Put $C \equiv L + \{t x : t \geq 0\}$ for some $x \in \overline{L} \setminus L$. Then $C$ is a convex cone such that $\overline{C} = \overline{L}$ and $\overline{C \cap L} = \overline{L}$, $C \cap L = L$, $\overline{C \cap \overline{L}} = \overline{L}$ are vector spaces but $C \cap \overline{L} = C$ is not a vector space.

LEMMA 2.2. *Let* $x_1, x_2 \in C$. *If* $x_1 + x_2 \in T^{\alpha(C)}(C)$ *then* $x_1, x_2 \in T^{\alpha(C)}(C)$.

Proof. Since $\overline{T^{\alpha(C)}(C)}$ is a linear space (Lemma 2.1(i)) and $x \equiv x_1 + x_2 \in T^{\alpha(C)}(C)$ we must have that $-x \in \overline{T^{\alpha(C)}(C)}$. Thus for $i \neq j$, $i,j = 1,2$, $-x_i = -x + x_j \in \overline{T^{\alpha(C)}(C)} + C \subseteq \overline{C} + C \subseteq \overline{C}$,

which proves that $x_1, x_2 \in C \cap \ell(\overline{C}) = T(C)$ and again writing

$-x_i = -x + x_j \in \overline{T^{\alpha(C)}(C)} + T(C) \subseteq \overline{T(C)}$ we obtain that $x_1, x_2 \in \ell(\overline{T(C)})$.

Using transfinite induction one can show that $x_1, x_2 \in \ell(\overline{T^{\alpha}(C)})$ for every ordinal $\alpha$ . Hence $x_1, x_2 \in C \cap \overline{T^{\alpha(C)}(C)} = T^{\alpha(C)}(C)$ by Lemma 2.1 (i) and (ii) . □

The proof of the following lemma is straightforward.

LEMMA 2.3. *Let $D$ and $C$ be convex cones such that $D \subseteq C$ . Then $T^{\alpha(D)}(D) \subseteq T^{\alpha(C)}(C)$ .*

On the other hand, we cannot conclude that $D \in \mathcal{C}$ whenever $C \in \mathcal{C}$ , unless $T^{\alpha(C)}(C)$ is finite dimensional (take $C = X$ and $D$ any convex cone with $D \notin \mathcal{C}$) .

The next theorem gives a decomposition of an arbitrary convex cone $C$ as a union and a sum of $T^{\alpha(C)}(C)$ and some pointed convex cone $D \in \mathcal{C}$ , with $\overline{T^{\alpha(C)}(C)} \cap D = \{0\}$ .

THEOREM 2.1. *Let $C$ be a convex cone in a t.v.s. $X$ . Then $D = (C \setminus \overline{T^{\alpha(C)}(C)}) \cup \{0\}$ is a pointed convex cone satisfying (*) and $C = T^{\alpha(C)}(C) \cup D$ is a disjoint union; in particular $T^{\alpha(D)}(D) = \{0\}$. Moreover, $C = T^{\alpha(C)}(C) + D$ , and $\overline{T^{\alpha(C)}(C)} \cap D = \{0\}$ .*

Proof. Since $\overline{T^{\alpha(C)}(C)} \cap C = T^{\alpha(C)}(C)$ (Lemma 2.1 (ii)) we must have that $C \setminus \overline{T^{\alpha(C)}(C)} = C \setminus T^{\alpha(C)}(C)$ . It is clear that $D$ is closed under nonnegative scalar multiplication. If $x_1, x_2 \in D$ and $x_1 + x_2 \notin D$ then $x_1 + x_2 \in T^{\alpha(C)}(C)$ , and by Lemma 2.2 , $x_1, x_2 \in T^{\alpha(C)}(C)$ , which is a contradiction. Hence $D$ is a convex cone. By Lemma 2.3, $T^{\alpha(D)}(D) \subseteq T^{\alpha(C)}(C) \cap D = \{0\}$ . Therefore, $T^{\alpha(D)}(D)$ is a vector subspace, so by Lemma 2.1, (iv) $\ell(D) = \{0\}$ ; thus $D$ is pointed and Proposition 2.1 gives $D \in \mathcal{C}$ . Moreover, it is easy to see that $C = T^{\alpha(C)}(C) + D$ . □

Let us observe that Theorem 2.1 gives a decomposition of the identity map $Id : K \to K$ as a sum of two orthogonal idempotents. Indeed, let

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$K : K \to K$ and $R : K \to K$ be maps defined by $K(C) = T^{\alpha(C)}(C)$ and $R(C) = D$ respectively, for $C \in K$. Then we have $K(T^{\alpha(C)}(C)) = T^{\alpha(C)}(C)$ and $R(D) = D$, which shows that $K$ and $R$ are idempotents. Moreover $K\,R(C) = R\,K(C) = \{0\}$ for every $C \in \mathcal{C}$ and $(K + R)\,(C) = K(C) + R(C) = T^{\alpha(C)}(C) + D = C$ by Theorem 2.1. Hence we have $K^2 = R^2 = Id$, $KR = RK = 0$ and $K + R = Id$.

The next corollary gives a necessary condition for a convex cone to be in the class $\mathcal{C}$.

COROLLARY 2.1. *If $C \in \mathcal{C}$ then there is a pointed cone $C_0 \in \mathcal{C}$ such that $\ell(C) \cap \operatorname{span} C_0 = \{0\}$ and $C = \ell(C) + C_0$, where span $C_0$ denotes the vector subspace generated by $C_0$.*

The proof is straightforward using Theorem 2.1, Proposition 2.1 and verifying that $C_0 = D \cap X_0$, where $X_0$ is an algebraic complement to $\ell(C)$ in $X$ and $D$ is the convex cone defined in Theorem 2.1. □

However, the conditions on $C$ given in Corollary 2.1 are not sufficient to ensure $C \in \mathcal{C}$. Indeed, if $L$ is any dense, proper vector subspace of $X$ and $x \notin L$, then the convex cone $C \equiv L + \{tx : t \geq 0\}$ fulfills the conditions in Corollary 2.1, that is $C_0$ exists with the stated properties, yet $C$ does not have (*). The converse to Corollary 2.1 holds under additional assumptions. The following proposition provides a sufficient condition for a convex cone to belong to $\mathcal{C}$.

PROPOSITION 2.2. *Let $C$ be a convex cone in a t.v.s. $X$, such that $\ell(C)$ is a closed subspace with a topological complement $X_0$ in $X$. If $C = \ell(C) + C_0$, where $C_0 \in \mathcal{C}$ is a pointed convex cone in $X_0$ then $C \in \mathcal{C}$.*

Proof. Using transfinite induction we show that, under the assumptions of Proposition 2.2, $T^{\alpha}(C) = \ell(C) + T^{\alpha}(C_0)$ for every ordinal $\alpha$ (here we use the result that $X$ is a topological sum of $\ell(C)$ and $X_0$). Since $C_0$ is pointed and satisfies (*) we have that $T^{\alpha(C_0)}(C_0)$

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$= \{0\}$ by Proposition 2.1. Thus $T^{\alpha(C)}(C) = \ell(C)$ and again applying Proposition 2.1 one obtains that $C \in \mathcal{C}$ .

It is easy to show that, in the above proposition, $C_0$ must be actually equal to $D \cap X_0$ , where $D$ is the cone defined in Theorem 2.1. Simple examples in the Euclidean plane show that the assumptions of Proposition 2.2 do not imply that $\overline{C}_0$ is pointed (take any pointed convex cone $C \subseteq \mathbb{R}^2$ such that $\overline{C}$ is not pointed).

Corollary 2.1 and Proposition 2.2 delimit the class $\mathcal{C}$ . We know by Lemma 2.1 (i) that $\overline{T^{\alpha(C)}(C)}$ is a linear space for every convex cone $C$ . Hence we are tempted to verify whether $C = T^{\alpha(C)}(C) + D \cap X_0$ if $\overline{T^{\alpha(C)}(C)}$ has a topological commplement $X_0$ in $X$ . However, Example 2.1 shows that the answer is negative.

EXAMPLE 2.1. Let $X$ be a t.v.s. space with nontrivial topological dual and let $X_1$ be a closed vector subspace of codimension 1 in $X$ . Then any algebraic complement to $X_1$ is also a topological complement and we can write $X = X_1 \oplus \mathbb{R} x_0$ for some $x_0 \notin X_1$ . Let $L$ be a dense proper vector subspace of $X_1$ . Define the convex cone $C \equiv L + \{t\, x_0 : t \geq 0\}$ . Then $T(C) = T^{\alpha(C)}(C) = L$ is a vector space, which means that $C \in \mathcal{C}$ . Moreover, $\overline{T^{\alpha(C)}(C)} = X_1$ has a topological complement. Put $X_0 = \mathbb{R}(x_0 + x_1)$ for any $x_1 \in X_1 \setminus L$ . Then we must have that $X = X_1 \oplus X_0 = \overline{T^{\alpha(C)}(C)} \oplus X_0$ is a topological sum. It is easy to check that $C \neq T^{\alpha(C)}(C) + D \cap X_0 = L + D \cap X_0$ since $D \cap X_0 = C \cap X_0 = \{0\}$ . □

This example shows also that if $C \in \mathcal{C}$ and $\overline{\ell(C)}$ ($\equiv \overline{T^{\alpha(C)}(C)}$ in this case) has a topological complement $X_0$ then it is not necessarily true that $C = \ell(C) + D \cap X_0$ . On the other hand, by Corollary 2.1 $C = \ell(C) + D \cap X_0$ for every algebraic complement $X_0$ of $\ell(C)$ in $X$ .

## 3. Remarks on existence of maximal points

Using the notion of the cone $T^{\alpha(C)}(C)$ we can improve some existence results from [8]. In this section we shall assume that $X$ is a Hausdorff t.v.s. We say that a cone is finite dimensional if it is contained in a finite dimensional vector space. Let us recall (see the introduction) that a subset $B$ of $X$ is $C$-compact if $B \cap (x + C)$ is nonempty and compact for some $x \in X$.

PROPOSITION 3.1. *Let $M$ be a vector subspace of $X$ such that the cone $T^{\alpha(C)}(C) \cap M$ is finite dimensional. Then $E_C(B) \neq \emptyset$ for every $C$-compact subset $B \subseteq M$, in particular for every nonempty compact subset $B \subseteq M$.*

Proof. Let $B \subseteq M$ be a $C$-compact subset and let $L$ be a vector subspace of $X$ such that $\overline{(C \cap M)} \cap L$ is a vector subspace. Then by Lemma 2.1(iii) $C \cap (M \cap L) = T^{\alpha(C)}(C) \cap M \cap L \subseteq T^{\alpha(C)}(C) \cap M$. Since the latter cone is finite dimensional so is $C \cap M \cap L$. Therefore its closure cannot be a vector subspace unless $C \cap M \cap L$ is itself a vector subspace. This shows that $C \cap M \in \mathcal{C}$. Using the existence result proved in [8] and recalled in the introduction we have that $E_{C \cap M}(B) \neq \emptyset$. Since $B \subseteq M$ we obtain that $E_{C \cap M}(B) \subseteq E_C(B)$, which completes the proof. □

The above proposition was proved in [8] with $\ell(\overline{C})$ instead of $T^{\alpha(C)}(C)$, which can now be seen to be far too strong a requirement. Observe that even the assumption "$\overline{T^{\alpha(C)}(C)} \cap M$ is finite dimensional" is too strong. Indeed, let $X_0$ be a vector subspace which is dense in $X$ and of infinite codimension (see for instance [6]). Take a convex cone $C \subseteq X$ with $\ell(C) = \{0\}$ such that $X_0$ is the set of linearly accessible points from $C$ (see [5, p.9]). Then $T^{\alpha(C)}(C) = C$ and $\overline{T^{\alpha(C)}(C)} = \overline{X_0} = X$. Let $M$ be any algebraic complement of $X_0$. Then $T^{\alpha(C)}(C) \cap M = \{0\}$ and Proposition 3.1 can be applied. However $M \cap \overline{T^{\alpha(C)}(C)} = M \cap X = M$ is not finite dimensional.

Let us recall that the convex cone defined in Theorem 2.1 is $D \equiv (C \setminus \overline{T^{\alpha(C)}(C)}) \cup \{0\}$ .

THEOREM 3.1. *If* $C$ *is a convex cone then for every subset* $B$ *of* $X$,

$$E_C(B) = E_D(B) \cap E_{T^{\alpha(C)}(C)}(B)$$

Proof. The inclusion

$$E_D(B) \cap E_{T^{\alpha(C)}(C)}(B) \subseteq E_C(B)$$

is straightforward since by Theorem 2.1 $C = D \cup T^{\alpha(C)}(C)$ .

Conversely, let $x \in E_C(B)$ . If $b - x \in D$ for some $b \in B$ then $x - b \in C$ . Hence $x - b \in D$ or else $x - b \neq 0$ and $x - b \in T^{\alpha(C)}(C)$ . In the latter case we apply Lemma 2.1 (i) and (ii) and obtain that

$$b - x \in \overline{T^{\alpha(C)}(C)} \cap C = T^{\alpha(C)}(C)$$

This contradiction shows that $x - b \in D$ . Thus $E_C(B) \subseteq E_D(B)$ . Now let $x \in E_C(B)$ and $b - x \in T^{\alpha(C)}(C)$ for some $b \in B$ .

Hence $x - b \in C$ and since $\overline{T^{\alpha(C)}(C)}$ is a vector space we must actually have that

$$x - b \in \overline{T^{\alpha(C)}(C)} \cap C = T^{\alpha(C)}(C)$$

(again by Lemma 2.1(ii)). This proves that $E_C(B) \subseteq E_{T^{\alpha(C)}(C)}(B)$ too. □

THEOREM 3.2. *(i) If* $C \in \mathcal{C}$ *then* $E_C(B) = E_D(B)$ *for every subset* $B$ *of* $X$ . *(ii) If* $\overline{T^{\alpha(C)}(C)}$ *is metrisable then* $C \in \mathcal{C}$ *if and only if* $E_C(B) = E_D(B)$ *for every (compact) subset* $B$ *of* $X$ .

Proof. By using Theorem 3.1 and Proposition 2.1 the proof of (i) is straightforward since then $T^{\alpha(C)}(C)$ is a vector subspace and

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$E_{T^{\alpha(C)}(C)}(B) = B$ for every subset $B$ of $X$. To prove (ii) suppose that $X_0 \equiv \overline{T^{\alpha(C)}(C)}$ is metrisable and that $T^{\alpha(C)}(C)$ is not a vector subspace. Hence $T^{\alpha(C)}(C)$ is a convex cone which does not satisfy (*). Using Proposition 2.1 in [8] (see the necessary condition recalled in the introduction) one can find a compact subset $B$ of $X_0$ such that $E_{T^{\alpha(C)}(C)}(B) = \emptyset$. Applying Theorem 3.2 we obtain that $E_C(B) = \emptyset$. On the other hand, by Theorem 2.1, $D \in \mathcal{C}$ and we must have that $E_D(B) \neq \emptyset$, which contradicts that $E_C(B) = E_D(B)$. Hence $T^{\alpha(C)}(C)$ is a vector space and using Proposition 2.1 we obtain that $C \in \mathcal{C}$.

Of course, $\overline{T^{\alpha(C)}(C)}$ is metrisable when $\ell(\overline{C})$ is metrisable. Moreover, let us note that the metrisability of $\overline{T^{\alpha(C)}(C)}$ is essential for the existence of a compact set $B$ with $E_{T^{\alpha(C)}(C)}(B) = \emptyset$ in the above proof. An example can be provided with any vector space $X$ endowed with its strongest locally convex topology and applying Proposition 3.1, since then every compact subset of $X$ is finite dimensional (Example 1.1 in [9]).

## References


[1] J.M. Borwein, "The geometry of Pareto efficiency over cones". *Math. Operationsforsch. Statist. Ser. Optim.* 11 (1980) 235-248.

[2] J.M. Borwein, "On the existence of Pareto efficient points", *Math. Oper. Res.* 8 (1983) 64-73.

[3] R. Hartley, "On cone-efficiency, cone-convexity and cone-compactness", *SIAM J. Appl. Math.* 34 (1978) 211-222.

[4] G.B. Hazen, T.L. Morin, "Optimality conditions in non-conical multiple objective programming", *J. Optim. Theory Appl.* 40 (1983) 25 - 60.

[5] R.B. Holmes, *Geometric Functional Analysis and Its Applications* (Springer-Verlag, New York, 1975).

[6] Z. Lipecki, "On some dense subspaces of topological linear spaces", *Studia Math.* 77 (1984) 413 - 421.

[7] J.P. Penot, A. Sterna-Karwat, "Parameterized multicriteria optimization; continuity and closedness of optimal multifunctions", *J. Math. Anal. Appl.* (to appear).

[8] A. Sterna-Karwat, "On existence of cone-maximal points in real topological linear spaces", *Israel J. Math* (to appear).

[9] A. Sterna-Karwat, "A note on cone-maximal and extreme points in topological vector spaces", Analysis Paper 49, Monash University (1985), submitted.

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