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A generalization of a theorem of Aquaro

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In this paper we introduce the concept of a $\delta\theta$-cover to generalize Aquaro's Theorem that every point countable open cover of a topological space such that every discrete closed family of sets is countable has a countable subcover. A $\delta\theta$-cover of a space X is defined to be a family of open sets $V = \bigcup_n V_n$ where each V_n covers X and for $x \in X$ there exists n such that V_n is of countable order at x. We replace point countable open cover by a $\delta\theta$-cover in Aquaro's Theorem and also generalize the result of Worrell and Wicke that a θ-refinable countably compact space is compact and Jones' result that an $\aleph_1$-compact Moore space is Lindelöf which was used to prove his classic result that a normal separable Moore space is metrizable, using the continuum hypothesis.

In this paper we introduce the concept of a $\delta\theta$-cover to generalize Aquaro's Theorem [1] that every point countable open cover of a topological space such that every discrete closed family of sets is countable has a countable subcover. We replace point countable open cover by a $\delta\theta$-cover and also generalize the result of Worrell and Wicke [8] that a θ-refinable countably compact space is compact and Jones' [6] result that an $\aleph_1$-compact Moore space is Lindelöf which was used to prove his classic result that a normal separable Moore space is metrizable, using the continuum hypothesis.

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DEFINITION 1. A $\delta\theta$-cover of a space X is a family of open sets $V = \bigcup_n V_n$ where each V_n covers X and for $x \in X$ there exists n such that V_n is of countable order at x. A space X is said to be $\delta\theta$-refinable if every open cover is refined by a $\delta\theta$-cover.

THEOREM 1. *Let every closed discrete family of sets of a space X be countable. Then every $\delta\theta$-cover has a countable subcover.*

We use a series of lemmas to establish the theorem.

DEFINITION 2. A set M is distinguished with respect to an open cover $\mathcal{U}$ of a topological space X if for $x, y \in M$, $x \neq y$, $x \in U \in \mathcal{U} \Rightarrow y \notin U$.

From Definition 2, Lemma 1 follows.

LEMMA 1. *A distinguished set is discrete.*

DEFINITION 3. A set M is maximally distinguished with respect to an open cover of a topological space X on a set H if $M \subset H$, M is distinguished and if P is distinguished and $M \subset P \subset H$ then $P = M$.

LEMMA 2. *If a distinguished set M with respect to an open cover $\mathcal{U}$ is contained in a set H then it is contained in a maximally distinguished set on H with respect to $\mathcal{U}$.*

Proof. The union of any chain of distinguished sets on H with respect to $\mathcal{U}$ is a distinguished set on H. Hence by the maximal principle there exists a maximal distinguished set on H with respect to $\mathcal{U}$ containing M.

LEMMA 3. *Let $\mathcal{W}$ consist of the subfamily of an open cover $\mathcal{U}$ that intersects a maximally distinguished set M on H with respect to $\mathcal{U}$. Then $\mathcal{W}$ covers H.*

Proof. Suppose there exists $z \in H$ such that there does not exist $W \in \mathcal{W}$ such that $z \in W$. Then $M \cup [z]$ is a distinguished set with respect to $\mathcal{U}$ on H, and $M \cup [z]$ properly contains M contrary to M being maximal.

Moore [7] obtained a discrete set on the space with the property of the distinguished set in Lemma 3 in a different manner.

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Proof of Theorem 1. Let H_n consist of the points of countable order with respect to V_n of Definition 1. Let W_n consist of the members of V_n that intersect M_n, a maximal distinguished set on H_n with respect to V_n. By the condition of the theorem and Lemma 1, M_n is countable, so that W_n is also countable. By Lemma 3, $W = \cup W_n$ is a countable cover of X.

COROLLARY 1. *If a T_1 space is $\aleph_1$-compact and satisfies any of the following properties, it is Lindelöf:*

(a) *$\delta\theta$-refinable;*

(b) *θ-refinable;*

(c) *meta-Lindelöf* (Aquaro [1]);

(d) *metacompact or paracompact* (Arens and Dugundji [2]);

(e) *subparacompact* (Christian [5]);

(f) *developable* (Jones [6]).

Proof. In a T_1-space $\aleph_1$-compactness is equivalent to every closed discrete set being countable. Then *(a)*, *(b)*, *(c)* and *(d)* are immediate and *(e)* follows from Burke's [4] result that a subparacompact space is θ-refinable and *(f)* follows from the result of Worrell and Wicke [8] that Moore spaces are θ-refinable.

COROLLARY 2. *A countable compact space satisfying any of the conditions (a) through (f) is compact.*

Corollary 2 *(b)* is due to Worrell and Wicke [8], and Boyte [3] has given a short proof of Corollary 2 *(c)*.

COROLLARY 3. *An $\aleph_1$-compact T_3 space is metrizable iff it has a σ-locally countable base.*

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