COEFFICIENT CONDITIONS FOR STARLIKE FUNCTIONS

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Let $\{a_k\}$ be a sequence of non-negative real numbers satisfying $a_1 = 1$ and

$$(k+1)a_{k+1} \leqq ka_k \qquad (k \in \mathbb{N}). \tag{1}$$

Brannan [**1**] proved that the function

$$f(z) = \sum_{k=1}^{\infty} a_k z^k \tag{2}$$

is close-to-convex univalent in the unit disc $\mathbb{D}$. The example

$$f(z) = z + \frac{z^2}{2} + \frac{z^3}{3}$$

shows that the conclusion in Brannan's theorem is sharp in that sense that "close-to-convex" cannot be replaced by the stronger one: "starlike". It is therefore of interest to see which additional condition can guarantee this stronger conclusion.

THEOREM. *Let $a_k \geqq 0$, $a_1 = 1$, satisfy* (1) *and*

$$(2k+1)a_{2k+1} \leqq (2k-1)a_{2k} \qquad (k \in \mathbb{N}). \tag{3}$$

Then the function (2) *is starlike univalent in* $\mathbb{D}$.

While Brannan's theorem rests on the plain fact that (1) implies

$$\operatorname{Re}(1-z)f'(z) > 0 \qquad (z \in \mathbb{D})$$

the proof of our theorem seems to require a fairly deep result of Vietoris [**2**]:

LEMMA 1. *Let $b_0 > 0$ and b_k a non-increasing sequence of non-negative real numbers satisfying*

$$(2k)b_{2k} \leqq (2k-1)b_{2k-1} \qquad (k \in \mathbb{N}). \tag{4}$$

Then, for $n \in \mathbb{N}$,

$$\sum_{k=0}^{n} b_k \cos(k\varphi) > 0 \qquad (0 < \varphi < \pi), \tag{5}$$

and

$$\sum_{k=0}^{n} b_k \sin(k\varphi) > 0 \qquad (0 < \varphi < \pi). \tag{6}$$

LEMMA 2. *Let f be analytic in $\mathbb{D}$, $f(0) = 0$, $f'(0) = 1$, and assume, that f' is typically real in $\mathbb{D}$ and satisfies $\operatorname{Re} f'(z) > 0$, $z \in \mathbb{D}$. Then f is starlike univalent in $\mathbb{D}$.*

We recall that a function F is typically real in $\mathbb{D}$ if $\operatorname{Im} F(z) \,.\, \operatorname{Im} z > 0$ for $z \in \mathbb{D} \backslash \mathbb{R}$. To prove Lemma 2 we write

$$\frac{f(z)}{zf'(z)} = \int_0^1 \frac{f'(tz)}{f'(z)}\,dt \qquad (z \in \mathbb{D}).$$

If $\operatorname{Im} z > 0$ (<0) we see that both, $f'(tz)$ and $f'(z)$, are in the upper (lower) halfplane since f' is typically real. But they are also in the right halfplane since $\operatorname{Re} f' > 0$. This shows that

$$\operatorname{Re}\frac{f'(tz)}{f'(z)} > 0, \qquad 0 \leqq t \leqq 1,$$

and hence $\operatorname{Re}[f(z)/(zf'(z))] > 0$ in $\mathbb{D}$, which implies the assertion.

Proof of the Theorem. Since the set of normalized starlike univalent functions is compact it suffices to prove the Theorem for sequences a_k satisfying (1), (3), and $a_k = 0$ for $k \geqq n+1$ for certain $n \in \mathbb{N}$. We then have

$$f'(z) = \sum_{k=0}^{n} (k+1)a_{k+1}z^k.$$

Now we write $b_k = (k+1)a_{k+1}$ and observe that (1), (3) are precisely the conditions on b_k in Lemma 1. (5) and the minimum principle for harmonic functions imply $\operatorname{Re} f'(z) > 0$, $z \in \mathbb{D}$. Similarly, the minimum principle applied to $\operatorname{Im} f'(z)$ and $z \in \mathbb{D}^+ := \mathbb{D} \cap \{z : \operatorname{Im} z > 0\}$, shows that either $\operatorname{Im} f' \equiv 0$ or $\operatorname{Im} f'(z) > 0$ in $\mathbb{D}^+$. In the first case we have $f(z) \equiv z$ and the conclusion is trivial. In the second case, using a reflection at $(-1, 1)$, we deduce that f' is typically real. Hence Lemma 2 applies to f and the assertion follows.

REFERENCES

1. D. A. Brannan, On univalent polynomials, *Glasgow Math. J.*, **11** (1970), 102–107.

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