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Predual of the Multiplier Algebra of $A_p(G)$ and Amenability

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Abstract. For a locally compact group G and $1 < p < \infty$, let $A_p(G)$ be the Herz-Figà-Talamanca algebra and let $PM_p(G)$ be its dual Banach space. For a Banach $A_p(G)$-module X of $PM_p(G)$, we prove that the multiplier space $\mathcal{M}(A_p(G), X^*)$ is the dual Banach space of Q_X, where Q_X is the norm closure of the linear span $A_p(G)X$ of uf for $u \in A_p(G)$ and $f \in X$ in the dual of $\mathcal{M}(A_p(G), X^*)$. If $p = 2$ and $PF_p(G) \subseteq X$, then $A_p(G)X$ is closed in X if and only if G is amenable. In particular, we prove that the multiplier algebra $MA_p(G)$ of $A_p(G)$ is the dual of Q, where Q is the completion of $L^1(G)$ in the $\|\cdot\|_M$-norm. Q is characterized by the following: $f \in Q$ if an only if there are $u_i \in A_p(G)$ and $f_i \in PF_p(G)$ $(i = 1, 2, \dots)$ with $\sum_{i=1}^{\infty} \|u_i\|_{A_p(G)} \|f_i\|_{PF_p(G)} < \infty$ such that $f = \sum_{i=1}^{\infty} u_i f_i$ on $MA_p(G)$. It is also proved that if $A_p(G)$ is dense in $MA_p(G)$ in the associated w^*-topology, then the multiplier norm and $\|\cdot\|_{A_p(G)}$-norm are equivalent on $A_p(G)$ if and only if G is amenable.

1 Introduction and Notation

Let G be a locally compact group equipped with a fixed left Haar measure λ. If G is compact, we assume $\lambda(G) = 1$. Let $L^p(G)$, $1 \le p \le \infty$, be the usual Lebesgue spaces on G with norm $\|\cdot\|_p$.

Suppose that $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. The Herz-Figà-Talamanca algebra $A_p(G)$ is the space of continuous functions u which can be represented as

$$u = \sum_{n=1}^{\infty} f_i * \check{g}_i \quad \text{with } f_i \in L^q(G),\ g_i \in L^p(G), \quad \text{and } \sum_{n=1}^{\infty} \|f_i\|_q \|g_i\|_p < \infty,$$

where $\check{g} \in L^p(G)$ is defined by $\check{g}(x) = g(x^{-1})$, $x \in G$. The norm of u is defined by

$$\|u\|_{A_p(G)} = \inf \sum_{n=1}^{\infty} \|f_i\|_q \|g_i\|_p,$$

where the infimum is taken over all the representations of u above. It is known that $A_p(G)$ is a subspace of $C_0(G)$ and, equipped with the norm $\|\cdot\|_{A_p(G)}$ above and the pointwise multiplication is a regular tauberian algebra whose Gelfand spectrum is G. Furthermore, the algebra $A_p(G)$ has a bounded approximate identity if and only if the group G is amenable (see Herz [8], Theorem 6). For $p = 2$, $A_p(G) = A(G)$, the Fourier algebra of G (see Eymard [3]). The dual of $A_p(G)$ is $PM_p(G)$ and $PF_p(G)^* =$

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$B_p(G)$ is a Banach algebra such that $A_p(G)$ is dense in the associated w^*-topology. For the definitions and properties of $PM_p(G)$ and $PF_p(G)$, see Pier [11].

Let X be a Banach $A_p(G)$-module of $PM_p(G)$. Then the dual Banach space X^* is also a Banach $A_p(G)$-module defined by $\langle uF, f\rangle = \langle F, uf\rangle$ for all $u \in A_p(G)$, $f \in X$ and $F \in X^*$. Suppose $\mathcal{M}(A_p(G), X^*)$ is the multiplier space of $A_p(G)$ into X^*, *i.e.*, all bounded linear operators $\phi\colon A_p(G) \to X^*$ such that $\phi(uv) = u\phi(v)$ for all $u, v \in A_p(G)$. Then $\mathcal{M}(A_p(G), X^*)$ is a Banach space equipped with the multiplier norm $\|\cdot\|_M$. We show in this paper that $\mathcal{M}(A_p(G), X^*)$ is the dual Banach space of Q_X, where Q_X is the norm closure of the linear span $A_p(G)X$ of uf for all $u \in A_p(G)$ and $f \in X$ in the dual of $\mathcal{M}(A_p(G), X^*)$. We will characterize Q_X in terms of the elements in $A_p(G)$ and X (see Theorem 2.3). In Lau and Losert [9], it is proved that for $p = 2$, $A_p(G)PM_p(G)$ is closed if and only if G is amenable. We prove that if $p = 2$ and $PF_p(G) \subseteq X$ or $\overline{\ell^1(G)} \subseteq X$, then $A_p(G)X$ is closed in X if and only if G is amenable.

The special cases of $X = PF_p(G)$ and $\overline{\ell^1(G)}$ will be considered. Let $MA_p(G)$ be the space of pointwise multipliers of $A_p(G)$ equipped with the multiplier norm $\|u\|_M = \sup\{\|uv\|_{A_p(G)} : v \in A_p(G), \|v\|_{A_p(G)} \le 1\}$, *i.e.*, the space of all continuous functions u on G such that the pointwise multiplication uv defines a bounded operator from $A_p(G)$ to $A_p(G)$ for every $v \in A_p(G)$. It is obvious that $A_p(G) \subseteq MA_p(G)$ and $\|u\|_M \le \|u\|_{A_p(G)}$ if $u \in A_p(G)$. It will be proved that $MA_p(G) = \mathcal{M}(A_p(G), PF_p(G)^*)$, which is also equal to the space of multiplier algebra of the Banach algebra $A_p(G)$, *i.e.*, all the bounded linear operators $\phi\colon A_p(G) \to A_p(G)$ with $\phi(uv) = u\phi(v)$ for all u and v in $A_p(G)$. We show that the predual $Q_{PF_p(G)}$ of $MA_p(G)$ is equal to the closure of $L^1(G)$ in $MA_p(G)^*$ under the multiplier norm, where for $f \in L^1(G)$, a continuous linear functional on $MA_p(G)$ is defined by $\langle f, \phi\rangle = \int_G f(x)\phi(x)\,dx$ for all $\phi \in MA_p(G)$. Thus, $MA_p(G)$ is a dual Banach space. This result is proved in De Cannière and Haagerup [2] for $p = 2$ and in Xu [14] for discrete G (see the comments on page 466 of Granirer and Leinert [6]). Also, an element f is in its predual $Q_{PF_p(G)}$ if and only if there are $u_i \in A_p(G)$ and $f_i \in PF_p(G)$ $(i = 1, 2, \dots)$ with $\sum_{i=1}^{\infty} \|u_i\|_{A_p(G)} \|f_i\|_{PF_p(G)} < \infty$ such that $f = \sum_{i=1}^{\infty} u_i f_i$ on $MA_p(G)$. We will investigate that when $A_p(G)$ is w^*-dense in $MA_p(G)$. We prove that the w^*-closure $\overline{A_p}^{w^*}(G)$ of $A_p(G)$ in $MA_p(G)$ is also a dual Banach space of the norm closure of $L^1(G)$ in the dual of $\overline{A_p}^{w^*}(G)$. If $A_p(G)$ is w^*-dense in $MA_p(G)$, then the multiplier norm and the $\|\cdot\|_{A_p(G)}$-norm are equivalent if and only if G is amenable. We do not have an example of G for which $A_p(G)$ is not w^*-dense in $MA_p(G)$ even for $p = 2$. For the case of $X = \overline{\ell^1(G)}$, we have a similar characterization for the predual of $\mathcal{M}(A_p(G), \overline{\ell^1(G)}^*)$. The relation between $MA_p(G)$ and $\mathcal{M}(A_p(G), \overline{\ell^1(G)}^*)$ will be discussed.

For Banach spaces X and Y, let $\mathcal{L}(X, Y)$ denote the space of all bounded linear operators from X to Y and let X^* be the conjugate Banach space of X. For $x \in X$ and $f \in X^*$, the value of f at x, $f(x)$, is sometimes denoted by $\langle f, x\rangle$ or $\langle x, f\rangle$ in duality. The norm of x (respectively, f) is sometimes written as $\|x\|_X$ (respectively, $\|f\|_{X^*}$) or $\|x\|_{X^*}$ (respectively, $\|f\|_X$). The projective tensor product of X and Y is the Banach space $X \hat{\otimes} Y$ such that each tensor t in $X \hat{\otimes} Y$ has a representation of the form $t = \sum_{i=1}^{\infty} x_i \otimes y_i$, where $\sum_{i=1}^{\infty} \|x_i\|\,\|y_i\| \le \infty$ and the norm of t in $X \hat{\otimes} Y$

is the infimum of $\sum_{i=1}^{\infty}\|x_i\|\,\|y_i\|$ over all the representations. The most important property of $X\hat{\otimes}Y$ used in this paper is that the Banach space dual of $X\hat{\otimes}Y$ can be isometrically identified with $\mathcal{L}(X,Y^*)$ by

$$\Big\langle T,\sum_{i=1}^{\infty}x_i\hat{\otimes}y_i\Big\rangle=\sum_{i=1}^{\infty}\langle T(x_i),y_i\rangle$$

(see Wojtaszczyk [13] p. 125).

2 The Multiplier Space $\mathcal{M}(A_p(G),X^*)$

Let X be a Banach $A_p(G)$-module of $PM_p(G)$. We will show in this section that $\mathcal{M}(A_p(G),X^*)$ is a dual Banach space and characterize its predual in terms of elements in $A_p(G)$ and X. We will also investigate when $A_p(G)X$ is closed.

Proposition 2.1 *Let G be a locally compact group and X be a Banach $A_p(G)$-module of $PM_p(G)$. Then*

- (i) *$\mathcal{M}(A_p(G),X^*)$ is a Banach $A_p(G)$-module with respect to the action defined by $(u\phi)(v)=\phi(uv)$ for $u,v\in A_p(G)$ and $\phi\in\mathcal{M}(A_p(G),X^*)$;*
- (ii) *For every $u\in A_p(G)$ and $f\in X$, uf is a bounded linear functional on $\mathcal{M}(A_p(G),X^*)$ defined by $\langle uf,\phi\rangle=\langle f,\phi(u)\rangle$ for $\phi\in\mathcal{M}(A_p(G),X^*)$ with $\|uf\|_M\le\|u\|_{A_p(G)}\|f\|_X$.*

Proof (i) For every $u\in A_p(G)$ and $\phi\in\mathcal{M}(A_p(G),X^*)$, we have $(u\phi)(vw)=\phi(uvw)=v((u\phi))(w)$ for all v and w in $A_p(G)$. So $u\phi\in\mathcal{M}(A_p(G),X^*)$ and $\|u\phi\|_M\le\|u\|_{A_p(G)}\|\phi\|_M$.

(ii) Let $\phi\in\mathcal{M}(A_p(G),X^*)$, then

$$|\langle uf,\phi\rangle|=|\langle f,\phi(u)\rangle|\le\|f\|_X\|\phi(u)\|_X\le\|f\|_X\|\phi\|_M\|u\|_{A_p(G)}.$$

So uf is in $\mathcal{M}(A_p(G),X^*)^*$ and $\|uf\|_M\le\|u\|_{A_p(G)}\|f\|_X$. ■

We denote the linear span of $\{uf : u\in A_p(G), f\in X\}$ by $A_p(G)X$. Then $A_p(G)X\subseteq\mathcal{M}(A_p(G),X^*)^*$ by Proposition 2.1. Let Q_X be the norm closure of $A_p(G)X$ in $\mathcal{M}(A_p(G),X^*)^*$.

Theorem 2.2 *Let G be a locally compact group and let X be a Banach $A_p(G)$-module. Then $\mathcal{M}(A_p(G),X^*)=(Q_X)^*$.*

Proof Define $J\colon A_p(G)\hat{\otimes}X\to Q_X$ as follows. For $\sum_{i=1}^{\infty}u_i\otimes f_i\in A_p(G)\hat{\otimes}X$, where $u_i\in A_p(G)$ and $f_i\in X$, $(i=1,2,\dots)$, we define $J(\sum_{i=1}^{\infty}u_i\otimes f_i)=\sum_{i=1}^{\infty}u_if_i$. Then J is well defined. In fact, it is obvious that $\mathcal{M}(A_p(G),X^*)$ is a closed subspace of $(A_p(G)\hat{\otimes}X)^*=\mathcal{L}(A_p(G),X^*)$. If $\sum_{i=1}^{\infty}u_i\otimes f_i=0$ in $A_p(G)\hat{\otimes}X$, then

$$\Big\langle\sum_{i=1}^{\infty}u_i\otimes f_i,\phi\Big\rangle=0\quad\text{for all }\phi\in\mathcal{L}(A_p(G),X^*).$$

Hence $\sum_{i=1}^{\infty} u_i f_i = 0$ in $\mathcal{M}(A_p(G), X^*)^*$. It follows from Proposition 2.1 that $\|J(\sum_{i=1}^{\infty} u_i \otimes f_i)\|_M \leq \sum_{i=1}^{\infty} \|u_i\|_{A_p(G)} \|f_i\|_X$. So $J(\sum_{i=1}^{\infty} u_i \otimes f_i) \in Q_X$ and $\|J\| \leq 1$.

We have the adjoint operator $J^*\colon (Q_X)^* \to (A_p(G) \hat{\otimes} X)^*$ with $\|J^*\| \leq 1$. For every $\phi \in (Q_X)^*$, since $(A_p(G) \hat{\otimes} X)^* = \mathcal{L}(A_p(G), X^*)$, we have $J^*(\phi)\colon A_p(G) \to X^*$ is a bounded linear operator. We will show that $J^*(\phi) \in \mathcal{M}(A_p(G), X^*)$. In fact, let u and v be in $A_p(G)$. Then $J^*(\phi)(u) \in X^*$ and

$$\begin{aligned} \langle J^*(\phi)(uv), f\rangle &= \langle J^*(\phi), (uv) \otimes f\rangle = \langle \phi, J((uv) \otimes f)\rangle \\ &= \langle \phi, J(u \otimes (vf))\rangle = \langle J^*(\phi)(u), vf\rangle \\ &= \langle vJ^*(\phi)(u), f\rangle \end{aligned}$$

for every $f \in X$. Hence $J^*(\phi)(uv) = vJ^*(\phi)(u)$ for all $u, v \in A_p(G)$. Therefore $J^*(\phi) \in \mathcal{M}(A_p(G), X^*)$.

For every $\phi \in \mathcal{M}(A_p(G), X^*)$, it is obvious that $\phi \in (Q_X)^*$ by duality and it is routine to check that $J^*(\phi) = \phi$. Therefore, J^* is a surjective isometry. ∎

Theorem 2.3 *Let $f \in \mathcal{M}(A_p(G), X^*)^*$. Then $f \in Q_X$ if and only if there are $u_i \in A_p(G)$ and $f_i \in X$ $(i = 1, 2, \dots)$ with $\sum_{i=1}^{\infty} \|u_i\|_{A_p(G)} \|f_i\|_X < \infty$ such that*

$$f = \sum_{i=1}^{\infty} u_i f_i \text{ and } \|f\|_M = \inf \sum_{i=1}^{\infty} \|u_i\|_{A_p(G)} \|f_i\|_X,$$

where the infimum is taken over all the representations of f above.

Proof By definition, each element of the form $\sum_{i=1}^{\infty} u_i f_i$ as in the theorem above is in Q_X.

Conversely, let $\mathcal{S}$ be the subspace of $A_p(G) \hat{\otimes} X$ generated by $(uv) \otimes f - u \otimes (vf)$ for $u, v \in A_p(G)$ and $f \in X$. Then an element $\phi \in \mathcal{L}(A_p(G), X^*)$ is in $\mathcal{M}(A_p(G), X^*)$ if and only if $\phi = 0$ on $\mathcal{S}$. Let $I\colon A_p(G) \hat{\otimes} X/\mathcal{S} \to Q_X$ be defined by

$$I\Big(\sum_{i=1}^{\infty} u_i \otimes f_i + \mathcal{S}\Big) = \sum_{i=1}^{\infty} u_i f_i,$$

where $\sum_{i=1}^{\infty} u_i \otimes f_i \in A_p(G) \hat{\otimes} X$. Then it is clear that I is well defined and $\|I\| \leq 1$. Also, that $(A_p(G) \hat{\otimes} X/\mathcal{S})^* = \mathcal{M}(A_p(G), X^*)$ and $\mathcal{M}(A_p(G), X^*) = Q_X^*$ implies that $I^*\colon Q_X^* \to (A_p(G) \hat{\otimes} X/\mathcal{S})^*$ is one-to-one and onto. So I is surjective (see Rudin [12], Theorem 4.15). This proves the first part of the theorem.

For $f \in Q_X$ with $\|f\| = 1$ and $\epsilon > 0$, there are $u_i \in A_p(G)$ and $f_i \in X$ $(i = 1, 2, \dots)$ such that $\sum_{i=1}^{\infty} \|u_i\|_{A_p(G)} \|f_i\|_X < \infty$ and $f = \sum_{i=1}^{\infty} u_i f_i$. Let $\eta = \sum_{i=1}^{\infty} u_i \otimes f_i + \mathcal{S}$ be in $A_p(G) \hat{\otimes} X/\mathcal{S}$. Since $\phi(\eta) = \phi(f)$ for all $\phi \in \mathcal{M}(A_p(G), X^*)$, we have $\|\eta\| \leq 1$. Thus, there exists $v_i \in A_p(G)$ and $h_i \in X$ $(i = 1, 2, \dots)$ such that $\sum_{i=1}^{\infty} \|v_i\|_{A_p(G)} \|h_i\|_X < 1 + \epsilon$ and $\eta = \sum_{i=1}^{\infty} v_i \otimes h_i + \mathcal{S}$ by the definition of the quotient norm. Thus, $f = \sum_{i=1}^{\infty} v_i h_i$ on $\mathcal{M}(A_p(G), X^*)$. This proves the second part of the theorem. ∎

Proposition 2.4 *If $F \in X^*$, then F defines an element of $\mathcal{M}(A_p(G), X^*)$ by $F(u) = uF$ for $u \in A_p(G)$ and $\|F\|_M \leq \|F\|_X$.*

Proof Since X^* is a Banach $A_p(G)$-module, $F \in \mathcal{M}(A_p(G), X^*)$ is well defined and $|\langle F(u), f\rangle| = |\langle F, uf\rangle| \leq \|F\|_X \|uf\|_X \leq \|F\|_X \|u\|_{A_p(G)} \|f\|_X$ for all $u \in A_p(G)$ and $f \in X$. Thus, $\|F\|_M \leq \|F\|_X$. ■

Proposition 2.5 *Let X be a Banach $A_p(G)$-module and $PF_p(G) \subseteq X$. Then $B_p(G)$ is a subalgebra of $\mathcal{M}(A_p(G), X^*)$ with $\|b\|_M \leq \|b\|_{B_p(G)}$ for $b \in B_p(G)$.*

Proof Let $u \in A_p(G)$ and $b \in B_p(G)$, then $bu \in A_p(G) \subseteq PM_p(G)^*$. So $bu \in X^*$. Since $PF_p(G) \subseteq X$, we have $\|bu\|_{PF_p(G)} = \|bu\|_X \leq \|b\|_{PF_p(G)} \|u\|_{A_p(G)}$. Hence $\|b\|_M \leq \|b\|_{B_p(G)}$. ■

We shall investigate the relationship between Q_X and X. In Proposition 2.4, we have $X^* \subseteq \mathcal{M}(A_p(G), X^*)$ with $\|\cdot\|_M \leq \|\cdot\|_X$. Let $R\colon (Q_X)^{**} \to X^{**}$ be the restriction map. Then $\|R\| \leq 1$. If $u \in A_p(G)$ and $f \in X$, then it is routine to check that $R(uf) = uf \in X$. So $R(\eta) \in X$ for every $\eta \in Q_X$.

Theorem 2.6 *Let G be a locally compact group. Then the following statements are equivalent:*

(i) *the restriction map $R\colon Q_X \to X$ is onto;*
(ii) *the norms $\|\cdot\|_X$ and $\|\cdot\|_M$ are equivalent on X^*.*

Proof (i) $\Rightarrow$ (ii) Let R be onto. Then X^* is $\|\cdot\|_M$-closed in $(Q_X)^*$ (see Rudin [12], page 103). So the norms $\|\cdot\|_X$ and $\|\cdot\|_M$ are equivalent on X^*.

(ii) $\Rightarrow$ (i) Since the norms $\|\cdot\|_X$ and $\|\cdot\|_M$ are equivalent on X^*, R^* is one-to-one. That the norms $\|\cdot\|_X$ and $\|\cdot\|_M$ are equivalent on X^* implies that X^* is closed in $\mathcal{M}(A_p(G), X^*)$. Thus, R is onto (see Rudin [12] page 103). ■

Corollary 2.7 *Let X be a Banach $A_p(G)$-module of $PM_p(G)$ such that $\|\cdot\|_X$ and $\|\cdot\|_{A_p(G)}$ are equivalent on $A_p(G)$. If $A_p(G)X$ is closed in X, then the norms $\|\cdot\|_{A_p(G)}$ and $\|\cdot\|_M$ are equivalent on $A_p(G)$. In particular, if $p = 2$, $A(G)X$ is closed in X if and only if G is amenable.*

Proof If $A_p(G)X$ is closed in X, then $\|\cdot\|_{A_p(G)X}$ and $\|\cdot\|_M$ are equivalent on $(A_p(G)X)^*$ by Theorem 2.6 if X is replaced by $A_p(G)X$. Since $\|\cdot\|_{A_p(G)X}$ and $\|\cdot\|_X$ are equivalent on $A_p(G)$, $\|\cdot\|_{A_p(G)}$ and $\|\cdot\|_M$ are equivalent on $A_p(G)$ by the condition. If G is amenable, then $A_p(G)X$ is closed by the Cohen facterization theorem. If $p = 2$ and $A(G)X$ is closed, then that the norms $\|\cdot\|_{A(G)}$ and $\|\cdot\|_M$ are equivalent on $A(G)$ implies that G is amenable (see Losert [10] and Remark 1 below). ■

Remarks 1. It is proved in Losert [10] that $\|\cdot\|_{A(G)}$ and $\|\cdot\|_{MA_2(G)}$ are equivalent on $A(G)$ if and only if G is amenable. Let X satisfy the condition in Corollary 2.7. Since $MA_2(G) \subseteq \mathcal{M}(A(G), X^*)$ and the norms $\|\cdot\|_{MA_2(G)}$ and $\|\cdot\|_M$ are equivalent

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on $A(G)$, we have that $\|\cdot\|_{A(G)}$ and $\|\cdot\|_M$ are equivalent on $A(G)$ if and only if G is amenable.

We do not know whether this result is true for $p \neq 2$. So if $A(G)VN(G)$ is closed, then G is amenable. This result due to Lau and Losert [9].

2. If $p = 2$, let $C^*_\delta(G)$ denote the C^* algebra generated by the point measures δ_x, $x \in G$, and $C^*_\rho(G) = PF_p(G)$. Then both $C^*_\delta(G)$ and $C^*_\rho(G)$ are Banach $A(G)$-modules and satisfy the condition in Corollary 2.7.

Corollary 2.8 *Let G be a locally compact group. Then*

(i) *if $A_p(G)PM_p(G)$ is norm closed, then $\|\cdot\|_{A_p(G)}$ and $\|\cdot\|_M$ are equivalent on $A_p(G)$;*
(ii) *$A(G)C^*_\delta(G)$ is norm closed if and only if G is amenable;*
(iii) *$A(G)C^*_\rho(G)$ is norm closed if and only if G is amenable.*

Proof (i) follows immediately from Corollary 2.7.

(ii) and (iii) are direct consequences of Corollary 2.7 and Losert's theorem mentioned above (see [10] and Remark 1 above). ■

Remark In Xu [14], there is a gap in the proof of the theorem that $A_p(G)PM_p(G)$ is norm closed if and only if G is amenable for discrete groups. In fact, the inclusion $Q_p \subseteq PF_p(G)$ in the proof (see Xu [14], page 3427) is ambiguous since $Q_p \subseteq MA_p(G)^*$ while $PF_p(G) \subseteq B_p(G)^*$. It may happen that an element $f \in Q_p$ is nonzero on $MA_p(G)$, but its restriction f on $B_p(G)$ is zero. This is related to the approximation property, *i.e.*, whether $A_p(G)$ is w^*-dense in $MA_p(G)$ (see Section 3 for $MA_p(G)$). We will discuss this in Proposition 3.7

Theorem 2.9 *Let G be a locally compact group. Then*

(i) *if the restriction map $R\colon Q_X \to X$ is one-to-one, then X^* is w^*-dense in $\mathcal{M}(A_p(G), X^*)$, and if R is onto then X^* is w^*-closed in $\mathcal{M}(A_p(G), X^*)$.*
(ii) *if $PF_p(G) \subseteq X$, then R is one-to-one and onto if and only if G is amenable.*

Proof (i) follows directly from the Corollary of Theorem 4.12 and Theorem 4.14 in Rudin [12].

(ii) If R is a bijection, then $X^* = \mathcal{M}(A_p(G), X^*)$. Since $1 \in \mathcal{M}(A_p(G), X^*)$, 1 is also in X^*. Since $PF_p(G) \subseteq X$, we have 1 is in $PF_p(G)^*$ as well. Thus, $1 \in B_p(G)$ implies that G is amenable.

Conversely, let G be amenable. Then $A_p(G)$ has a bounded approximate identity $\{u_\alpha\}$. For every $\phi \in \mathcal{M}(A_p(G), X^*)$ and $u \in A_p(G)$, we have $\phi(uu_\alpha) = u_\alpha\phi(u)$ converges to $\phi(u)$ in the norm topology in X^* since $\|uu_\alpha - u\|_{A_p(G)} \to 0$. On the other hand, since $\phi(u_\alpha)$ is bounded in X^*, let F be its w^*-limit. So $F \in X^*$. By Proposition 2.4, $F \in \mathcal{M}(A_p(G), X^*)$ and $F(u) = uF = \lim u\phi(u_\alpha) = \lim u_\alpha\phi(u) = \phi(u)$ in the w^* topology in X^*. Thus, $\phi = F$ is in X^*. Hence $X^* = \mathcal{M}(A_p(G), X^*)$, which implies that R is one-to-one (see Rudin [12], page 99). Also, it is easy to see that T^* is one-to-one. Thus, R is onto by Theorem 4.15 in Rudin [12]. ■

3 The Case of $X = PF_p(G)$ and $\overline{\ell^1(G)}$

In this section, we will consider the multiplier spaces for the cases of $X = PF_p(G)$ and $\overline{\ell^1(G)}$. Let $\mathcal{M}(A_p(G))$ be the multiplier algebra of $A_p(G)$ equipped with the multiplier norm, *i.e.*, the algebra of all bounded linear operators ϕ from $A_p(G)$ to $A_p(G)$ with the property $\phi(uv) = u\phi(v)$ equipped with the operator norm. At first we show that the multipliers defined in three different ways are the same.

Proposition 3.1 *Let G be a locally compact group. Then*

$$MA_p(G) = \mathcal{M}(A_p(G)) = \mathcal{M}(A_p(G), PF_p(G)^*).$$

Proof Let $I\colon MA_p(G) \to \mathcal{M}(A_p(G))$ be defined by $I(u)(v) = uv$ for all $u \in MA_p(G)$ and $v \in A_p(G)$. Then it follows from definition that I is an isometry. So $MA_p(G) \subseteq \mathcal{M}(A_p(G))$. Since $A_p(G)$ is a subalgebra of $PF_p(G)^*$, we have $\mathcal{M}(A_p(G)) \subseteq \mathcal{M}(A_p(G), PF_p(G)^*)$. Next we show that for every ϕ in $\mathcal{M}(A_p(G), PF_p(G)^*)$, there exists an $u \in MA_p(G)$ such that $\phi = I(u)$, that is, $\mathcal{M}(A_p(G), PF_p(G)^*) \subseteq MA_p(G)$.

We will define a function $\tilde{\phi}$ on G as follows. For $x \in G$, there is an element $u \in A_p(G)$ with compact support and $u(x) = 1$. Define $\tilde{\phi}(x) = \phi(u)(x)$. Then $\tilde{\phi}(x)$ is independent of u. In fact, let $v \in A_p(G)$ be with compact support and $v(x) = 1$. Then there is an $w \in A_p(G)$ such that $w = 1$ on the supports of u and v. So $\phi(u)(x) = \phi(uw)(x) = u(x)\phi(w)(x) = \phi(w)(x)$. Similarly, $\phi(v)(x) = \phi(w)(x)$. Thus, $\phi(u)(x) = \phi(v)(x)$. Let $x_0 \in G$. There exists a open neighborhood U of x_0 with compact closure. So there exists $u \in A_p(G)$ with compact support such that $u = 1$ on U. Then $\tilde{\phi}(x) = \phi(u)(x)$ for all $x \in U$. Since $\phi(u) \in A_p(G)$ is continuous at x_0, $\tilde{\phi}$ is continuous at x_0. Let $u \in A_p(G)$ and $x \in G$. There is an $v \in A_p(G)$ with compact support such that $v(x) = 1$. Then $\phi(u)(x) = v(x)\phi(u)(x) = \phi(vu)(x) = u(x)\phi(v)(x) = u(x)\tilde{\phi}(x)$. Thus, $\phi(u) = \tilde{\phi}u$ in $A_p(G)$. By definition, $\tilde{\phi} \in MA_p(G)$. Therefore, $\phi = I(\tilde{\phi})$. ■

Let $f \in L^1(G)$. Define a linear functional on $MA_p(G)$ by

$$\langle f, \phi\rangle = \int f(x)\phi(x)\,dx \quad \text{for } \phi \in MA_p(G).$$

Then $|\langle f, \phi\rangle| \leq \|f\|_1\|\phi\|_\infty \leq \|f\|_1\|\phi\|_M$ for every $\phi \in MA_p(G)$. So f is in $MA_p(G)^*$ and its norm, denoted by $\|f\|_M$, is less than or equal to $\|f\|_1$. Define

$$Q = \text{the completion of } L^1(G) \quad \text{with respect to the norm } \|\cdot\|_M.$$

Theorem 3.2 *Let G be a locally compact group. Then $Q_{PF_p(G)} = Q$ and so $MA_p(G) = Q^*$.*

Proof Let $f \in L^1(G)$ be with compact support. Then there exists $u \in A_p(G)$ such that $u = 1$ on the support of u. So $f = uf$ is in $Q_{PF_p(G)}$ and $\langle uf, \phi\rangle = \int_G f(x)\phi(x)\,dx$ for every $\phi \in MA_p(G)$. Thus, there is an isometry between the dense subspace of $Q_{PF_p(G)}$ and a dense subspace of $(L^1(G), \|\cdot\|_M)$. Therefore $Q_{PF_p(G)}$ is the completion of $L^1(G)$ in the $\|\cdot\|_M$ norm. ■

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Corollary 3.3 *Let G be a locally compact group.*

(i) *Let $f \in MA_p(G)^*$. Then $f \in Q$ if and only if there are $u_i \in A_p(G)$ and $f_i \in PF_p(G)$ $(i = 1, 2, \dots)$ with $\sum_{i=1}^{\infty} \|u_i\|_{A_p(G)} \|f_i\|_{PF_p(G)} < \infty$ such that*

$$f = \sum_{i=1}^{\infty} u_i f_i \text{ and } \|f\|_M = \inf \sum_{i=1}^{\infty} \|u_i\|_{A_p(G)} \|f_i\|_{PF_p(G)},$$

where the infimum is taken over all the representations of f above.

(ii) *G is amenable if and only if for any $f \in PF_p(G)$ and $\epsilon > 0$, there are $u_i \in A_p(G)$ and $f_i \in PF_p(G)$ $(i = 1, 2, \dots)$ with $\sum_{i=1}^{\infty} \|u_i\|_{A_p(G)} \|f_i\|_{PF_p(G)} < \|f\| + \epsilon$ such that $f = \sum_{i=1}^{\infty} u_i f_i$ on $B_p(G)$.*

Proof (i) follows immediately from Theorem 2.3. The condition of (ii) is equivalent to that $PF_p(G) = Q$ by (i), *i.e.*, $B_p(G) = MA_p(G)$, which is equivalent to that G is amenable since $1 \in B_p(G)$.

Remarks (1) $Q_{PF_p(G)}$ may be considered as an analogue of the group C^*-algebra $C^*(G)$. But, in general, it is not an algebra under convolution even for $p = 2$ (see Cowling and Haagerup [1] page 512).

(2) This result is proved in De Cannière and Haagerup [2] for $p = 2$ and in Xu [14] for discrete groups.

As is well known, $A_p(G)$ is always w^*-dense in $B_p(G)$ (*i.e.*, in $\sigma(B_p(G), PF_p(G))$-topology), and $A(G)$ is dense in $B(G)$, the Fourier-Stieltjes algebra, in the w^*-topology if and only if G is amenable. It is natural to consider whether $A_p(G)$ is dense in $MA_p(G)$ with respect the w^*-topology. To this end, let

$$\overline{A_p}^{w^*}(G) = \text{the } w^*\text{-closure of } A_p(G) \text{ in } MA_p(G).$$

A locally compact group G is said to have the approximation property if there is a net $\{u_\alpha\}$ of functions in $A_p(G)$ such that $u_\alpha \to 1$ in the associated w^*-topology in $MA_p(G)$. We will see that these two concepts are the same in the following.

Proposition 3.4 *For every locally compact group G, then $A_p(G)$ is w^*-dense in $MA_p(G)$ if and only of G has the approximation property.*

Proof If $A_p(G)$ is w^*-dense in $MA_p(G)$, then $1 \in \overline{A_p}^{w^*}(G)$ since $1 \in MA_p(G)$. So G has the approximation property.

Conversely, suppose there is a net $\{u_\alpha\}$ of functions in $A_p(G)$ such that $u_\alpha \to 1$ in the w^*-topology. For every $\phi \in MA_p(G)$, it is easy to see that $\phi f \in Q$ for all $f \in Q$ by the density of $L^1(G)$ in Q. So for every $f \in Q$, we have $\langle u_\alpha \phi, f \rangle = \langle u_\alpha, \phi f \rangle \to \langle 1, \phi f \rangle = \langle \phi, f \rangle$. Hence the net $\{\phi u_\alpha\}$ of functions in $A_p(G)$ converges to ϕ in the w^*-topology. Therefore, $A_p(G)$ is w^*-dense in $MA_p(G)$. ■

Proposition 3.5 *Let G be a locally compact group. Then $\overline{A_p}^{w^*}(G)$ is an ideal of $MA_p(G)$ and is the dual of Q^L, where Q^L is the Banach space of the restrictions of elements in Q to $\overline{A_p}^{w^*}(G)$. Furthermore, Q^L can be identified with the completion of $L^1(G)$ with the norm, for $f \in L^1(G)$,*

$$\|f\|_L = \sup\Big\{ \Big| \int_G f(x)\phi(x)\,dx \Big| : \phi \in \overline{A_p}^{w^*}(G) \text{ with } \|\phi\|_M \le 1 \Big\}.$$

Proof Let $\phi \in MA_p(G)$ and $\psi \in \overline{A_p}^{w^*}(G)$. Then there is a net $\{u_\alpha\}$ of functions in $A_p(G)$ such that $u_\alpha \to \psi$ in the w^*-topology. The similar argument as in the proof of Proposition 3.4 shows that $u_\alpha\phi \to \psi\phi$ in the w^*-topology. Since each $u_\alpha\phi$ is in $A_p(G)$, we have that $\psi\phi \in \overline{A_p}^{w^*}(G)$ and $\overline{A_p}^{w^*}(G)$ is an ideal of $MA_p(G)$.

It is obvious that the identity map $I\colon \overline{A_p}^{w^*}(G) \to (Q^L)^*$ is an isometry. Let $\phi \in (Q^L)^*$ be with norm 1. Since Q^L is a subspace of $\overline{A_p}^{w^*}(G)^*$, we extend ϕ to $\overline{A_p}^{w^*}(G)^*$ with the same norm. By the Goldstine's theorem, there is a net $\{u_\alpha\}$ in the unit ball of $\overline{A_p}^{w^*}(G)$ such that $u_\alpha \to \phi$ in $\sigma(\overline{A_p}^{w^*}(G)^{**}, \overline{A_p}^{w^*}(G)^*)$ topology. Since $Q \subseteq \overline{A_p}^{w^*}(G)^*$, we have $\langle u_\alpha, f\rangle \to \langle \phi, f\rangle$ for every $f \in Q$. Hence $\phi \in \overline{A_p}^{w^*}(G)$. Therefore, I is a surjective isometry. The proof of the last statement is similar to the proof of Theorem 3.2. ■

Definition A locally compact group G is said to be *p-weakly amenable* if there exists a net $\{u_\alpha\}$ in $A_p(G)$ such that $\{\|u_\alpha\|_M\}$ is bounded and

$$\|u_\alpha a - a\|_{A_p} \to 0 \quad \text{for every } a \in A_p(G).$$

Remark If G is an amenable locally compact group, then $A_p(G)$ has a bounded approximate identity. So G is necessarily p-weakly amenable. Conversely, as shown in Furuta [4], for the noncommutative free group F_r with r generators, there exists a net $\{u_\alpha\}$ in $A_p(F_r)$ such that it is bounded in the multiplier norm and $\|u_\alpha a - a\|_{A_p} \to 0$ for every $a \in A_p(F_r)$. Hence, F_r is p-weakly amenable, but not amenable.

Proposition 3.6 *Let G be a locally compact group.*

(i) *If G is p-weakly amenable, then $A_p(G)$ is dense in $MA_p(G)$ in the weak* topology;*
(ii) *If β is a positive number and $\{u \in A_p(G) : \|u\|_M \le \beta\}$ is w^*-dense in the unit ball of $MA_p(G)$, then G is p-weakly amenable.*

Proof (i) Let u_α be a $\|\cdot\|_M$-bounded net in $A_p(G)$ such that $\|u_\alpha a - a\|_{A_p} \to 0$ for every $a \in A_p(G)$. For every $\phi \in MA_p(G)$, then $u_\alpha\phi$ is a $\|\cdot\|_M$-bounded net in $A_p(G)$ as well. We assume, without loss of generality, $u_\alpha\phi$ converges in the *weak** topology by taking subnet if necessary. It is obvious that $u_\alpha \to 1$ pointwisely. So $u_\alpha\phi \to \phi$ pointwisely. Hence $u_\alpha\phi \to \phi$ in the *weak**-topology.

(ii) Since $\{u \in A_p(G) : \|u\|_M \le \beta\}$ is w^*-dense in the unit ball of $MA_p(G)$ and 1 is in the unit ball, there exists a net u_α such that $u_\alpha \to 1$ in the *weak** topology and $\|u_\alpha\|_M \le \beta$ for all α. Choose a continuous function h on G such that $h(x) \ge 0$ for all $x \in G$, $\int_G h(x)\,dx = 1$ and the support of h is compact. Define $u'_\alpha = h * u_\alpha$. Then $\|u'_\alpha\|_M \le \|h\|_1 \|u_\alpha\|_M \le \beta$ for all α. We will show that $\|uu'_\alpha - u\|_{A_p(G)} \to 0$ for all $u \in A_p(G)$. By the boundedness of the net, we can assume, without loss of generality, that u has a compact support. Let $S = \operatorname{supp}(h)^{-1} \operatorname{supp}(u)$. Then for $x \in \operatorname{supp}(u)$ we have

$$u'_\alpha(x) = \int_G h(t) u_\alpha(t^{-1}x)\,dt = \int_G h(t)(1_S u_\alpha)(t^{-1}x)\,dt.$$

So $uu'_\alpha = u(h * (1_S u_\alpha))$. Similarly, $u = u(h * 1_S)$. Next, we show that $1_S u_\alpha \to 1_S$ in $L^q(G)$. Let $a \in A_p(G)$ such that $a = 1$ on S. For every $f \in PF_p(G)$, we have

$$\langle au_\alpha, f\rangle = \langle u_\alpha, af\rangle \to \langle 1, af\rangle = \langle a, f\rangle$$

since $uf \in Q$. Thus $au_\alpha \to a$ in the $\sigma(B_p(G), PF_p(G))$-topology. So $au_\alpha \to a$ in measure with respect to the left Haar measure. Also, that au_α is bounded in $A_p(G)$-norm implies that au_α is bounded in $\|\cdot\|_\infty$-norm. So $1_S u_\alpha \to 1_S$ in $L^q(G)$. Therefore, uu'_α converges to u in the $A_p(G)$-norm. ■

Remark There are examples of locally compact groups such that they are not p-weakly amenable, but $A_p(G)$ is w^*-dense in $MA_p(G)$ for $p = 2$ (see Haagerup and Kraus [7], page 670).

Proposition 3.7 *Let G be a locally compact group. Then*

(i) *the restriction map $R\colon Q \to PF_p(G)$ is one-to-one if and only if $A_p(G)$ is w^*-dense in $MA_p(G)$;*
(ii) *the restriction map $R\colon Q \to PF_p(G)$ is onto if and only if the norms $\|\cdot\|_{A_p(G)}$ and $\|\cdot\|_M$ are equivalent on $A_p(G)$;*
(iii) *if $A_p(G)$ is w^*-dense in $MA_p(G)$, then the norms $\|\cdot\|_{A_p(G)}$ and $\|\cdot\|_M$ are equivalent on $A_p(G)$ if and only if G is amenable.*

Proof (i) If R is one-to-one, it follows from Theorem 2.9 that $PF_p(G)^*$ is w^*-dense in $MA_p(G)$. Since $A_p(G)$ and $PF_p(G)^*$ have the same w^* closure in $MA_p(G)$ (see the proof of Proposition 3.5), $A_p(G)$ is w^*-dense in $MA_p(G)$. Conversely, if $A_p(G)$ is w^*-dense in $MA_p(G)$, let $R(f) = 0$ for some $f \in Q$. Then $\langle u, f\rangle = 0$ for all $u \in A_p(G)$. So $\langle \phi, f\rangle = 0$ for all $\phi \in MA_p(G)$ by the density of $A_p(G)$. Thus, $f = 0$. So R is one-to-one.

(ii) If R is onto, then $PF_p(G)^*$ is norm-closed in $MA_p(G)$ by Theorem 4.15 in Rudin [12]. So the multiplier norm and the $A_p(G)$-norm are equivalent. Conversely, let the multiplier norm and the $A_p(G)$-norm be equivalent on $A_p(G)$. We will show that the $B_p(G)$-norm and the multiplier norm are equivalent on $B_p(G)$. In fact, the inclusion map $i\colon A_p(G) \to MA_p(G)$ is bounded and $A_p(G)$ is $\|\cdot\|_M$-closed in $MA_p(G)$. By Theorem 4.14 in Rudin [12], $i^*(MA_p(G)^*)$ is w^*-closed in $A_p(G)^*$. By Theorem 4.14 in Rudin [12] again, $i^{**}(A_p(G)^{**})$ is norm-closed in $MA_p(G)^{**}$. Since $B_p(G)$ is a norm-closed subspace of $A_p(G)^{**}$, the norm on $B_p(G)$ and the multiplier norm are equivalent. Thus, $B_p(G)$ is norm-closed in $MA_p(G)$. It is clear that R^* is one-to-one. Therefore, R is onto by Theorem 4.15 in Rudin [12].

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(iii) If G is amenable, then $MA_p(G) = B_p(G)$ and the multiplier norm and the $A_p(G)$-norm are equal. Conversely, let the norms $\|\cdot\|_{A_p(G)}$ and $\|\cdot\|_M$ are equivalent on $A_p(G)$. If $A_p(G)$ is w^*-dense in $MA_p(G)$, then by (ii) and (iii), the restriction map is one-to-one and onto. So Q is isometrically isomorphic to $PF_p(G)$. Thus, $1 \in MA_p(G) = B_p(G)$. Hence G is amenable. ■

Remark We do not have an example of locally compact group for which $A_p(G)$ is not w^*- dense in $MA_p(G)$.

If $x \in G$, then the point measure $\delta_x \in PM_p(G)$. So $\ell^1(G) \subseteq PM_p(G)$. We denote the norm closure of $\ell^1(G)$ in $PM_p(G)$ by $\overline{\ell^1(G)}$. Then $\overline{\ell^1(G)}$ is a Banach $A_p(G)$-module of $PM_p(G)$. Also, every bounded linear functional on $\overline{\ell^1(G)}$ is in $\ell^\infty(G)$.

Theorem 3.8 *$\mathcal{M}(A_p(G), \overline{\ell^1(G)}^*)$ consists of functions ϕ in $\ell^\infty(G)$ such that the pointwise multiplication ϕa defines a bounded operator from $A_p(G)$ to $\overline{\ell^1(G)}^*$. The predual $Q_{\overline{\ell^1(G)}}$ as in Theorem 2.2 is equal to the completion of $\ell^1(G)$ with respect the norm*

$$\|f\| = \sup\Big\{ \sum f(x)\phi(x) : \phi \in \mathcal{M}(A_p(G), \overline{\ell^1(G)}^*) \text{ with } \|\phi\| \le 1 \Big\}.$$

Furthermore, $MA_p(G) \subseteq \mathcal{M}(A_p(G), \overline{\ell^1(G)}^*)$, and the inclusion map from $MA_p(G)$ to $\mathcal{M}(A_p(G), \overline{\ell^1(G)}^*)$ is norm decreasing.

Proof If ϕ is in $\ell^\infty(G)$ such that the pointwise multiplication ϕa defines a bounded linear operator from $A_p(G)$ to $\overline{\ell^1(G)}^*$, then ϕ is also a multiplier. So it is in $\mathcal{M}(A_p(G), \overline{\ell^1(G)}^*)$. Conversely, if $\phi \in \mathcal{M}(A_p(G), \overline{\ell^1(G)}^*)$, we define $\tilde{\phi}$ as follows. For $x \in G$, take an $u \in A_p(G)$ such that $u(x) = 1$ and $\mathrm{supp}(u)$ is compact. Define $\tilde{\phi}(x) = \phi(u)(x)$. Then it is well defined since if there is an $v \in A_p(G)$ with compact support such that $v(x) = 1$. Let $a_K \in A_p(G)$ satisfy that $a_K = 1$ on the support of u and of v. The $\phi(u-v)(x) = \phi((u-v)a_K)(x) = (u-v)(x)\phi(a_K)(x) = 0$. If $u \in A_p(G)$ and $x \in G$, let $v \in A_p(G)$ be with compact support and $v(x) = 1$. Then

$$\phi(u)(x) = v(x)\phi(u)(x) = \phi(vu)(x) = u(x)\phi(v)(x) = u(x)\tilde{\phi}(x).$$

Thus, $\phi = \tilde{\phi}$ as an element of $\mathcal{M}(A_p(G), \overline{\ell^1(G)}^*)$.

It follows from Theorem 2.2 that $\mathcal{M}(A_p(G), \overline{\ell^1(G)}^*)$ is the dual of $Q_{\overline{\ell^1(G)}}$. If $f \in \ell^1(G)$ with finite support, then $\langle \phi, f\rangle = \langle \phi, af\rangle = \langle a\phi, f\rangle = \sum \phi(x)f(x)$, where $a \in A_p(G)$ with $a = 1$ on the support of f. By the density of the set of finite support elements in $\overline{\ell^1(G)}$, $Q_{\overline{\ell^1(G)}}$ is equal to the completion of $\ell^1(G)$ with respect to the multiplier norm.

Since $A_p(G)$ is a subspace of $PM_p(G)^*$ and $\overline{\ell^1(G)}$ is a subspace of $PM_p(G)$, we have $\phi a \in \overline{\ell^1(G)}^*$ and $\|\phi a\|_{\overline{\ell^1(G)}} \le \|\phi a\|_{A_p(G)} \le \|\phi\|_M \|a\|_{A_p(G)}$ for all $\phi \in MA_p(G)$ and $a \in A_p(G)$. Hence the last statement of the theorem is true. ■

Remark When $p = 2$, then the norms of an element of $A_p(G)$ on $PF_p(G)$ and on $\overline{\ell^1(G)}$ are equal (see Eymard [3]). Hence the inclusion map from $A_p(G)$ to $\mathcal{M}(A_p(G), \overline{\ell^1(G)}^*)$ is an isometry.

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