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ORDINARY AND PARTIAL DIFFERENCE EQUATIONS

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Abstract

Ordinary difference equations (OΔE's), mostly of order two and three, are derived for the trigonometric, Jacobian elliptic, and hyperbolic functions. The results are used to derive partial difference equations (PΔE's) for simple solutions of the wave equation and three nonlinear evolutionary partial differential equations.

1. Introduction

In a sequence of recent papers [4]–[9], it has been shown that, in choosing a difference equation (ΔE) approximating a differential equation (DE), theoretical advantage can be obtained by exploiting a wider range of approximations than is customary. If the solution to an ordinary differential equation (ODE) satisfies an addition formula, then this can be used to determine a 'best' ordinary difference equation (OΔE) approximating the ODE. If $f(x)$ is the solution of the ODE then an OΔE is called a best approximating OΔE if it is exactly satisfied by

$$f_m = f(mp) \tag{1.1}$$

where m is an integer, and p a constant stepsize of any magnitude, not necessarily small in any sense. In the limit $p \to 0$, the best OΔE will be expected to converge to the corresponding ODE.

In this paper, OΔE's and their limiting ODE's will be obtained for various simple functions—trigonometric, Jacobian elliptic, and hyperbolic—which satisfy addition formulae. The analysis will then be extended to some partial difference equations (PΔE's) and their associated PDE's. The results encompass some of the canonical nonlinear evolutionary PDE's for which the hyperbolic functions tanh

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and sech play an important role in describing solitary-wave and single soliton solutions.

The determination of PΔE analogues of nonlinear evolutionary PDEs has been considered from a different point of view by Hirota [3].

2. Trigonometric functions

In this section, first and second-order OΔE's will be determined for the trigonometric functions. The details will be given for the sine fuction; for other trig functions, just the results will be presented.

2.1 $f(x) = A\sin kx$

From the addition formula

$$\sin k(x+p) = \sin kx \cos kp + \cos kx \sin kp, \tag{2.1}$$

and with f_m defined by (1.1) as

$$f_m = A \sin kmp, \tag{2.2}$$

follow the first-order OΔE's

$$f_{m+1} = f_m \cos kp + \left(A^2 - f_m^2\right)^{1/2} \sin kp, \tag{2.3}$$

$$f_{m-1} = f_m \cos kp - \left(A^2 - f_m^2\right)^{1/2} \sin kp. \tag{2.4}$$

Equations (2.3) and (2.4) can be subtracted to give the nonlinear second-order OΔE

$$\frac{f_{m+1} - f_{m-1}}{2k^{-1}\sin kp} = k\left(A^2 - f_m^2\right)^{1/2} \tag{2.5}$$

which is seen to converge to the first-order ODE

$$f'(x) = k\left(A^2 - f(x)^2\right)^{1/2} \tag{2.6}$$

in the limit $p \to 0$.

Alternatively (2.3) and (2.4) can be added to give, after some manipulation, the second-order OΔE

$$\frac{f_{m+1} - 2f_m + f_{m-1}}{4k^{-2}\sin^2(kp/2)} + k^2 f_m = 0 \tag{2.7}$$

which converges to the second-order ODE

$$f''(x) + k^2 f(x) = 0 \tag{2.8}$$

in the limit $p \to 0$.

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There are two interesting features of the OΔE (2.7). First it is exactly satisfied by $f(x) = A\sin kx$ at $x = mp$ for any nonzero p, not necessarily small, so that it is a best approximating OΔE to the ODE (2.8). Secondly, the term $4k^{-2}\sin^2(kp/2)$, which is $O(p^2)$, replaces the usual p^2 in the denominator of the quotient approximating the second derivative.

The function $f(x) = A\cos kx$ gives the same second-order OΔE (2.7) and the same ODE (2.8).

2.2 $f(x) = A\tan kx$

The addition formula

$$\tan k(x+p) = (\tan kx + \tan kp)/(1 - \tan kx \tan kp) \tag{2.9}$$

gives the nonlinear second-order OΔE

$$\frac{f_{m+1} - 2f_m + f_{m-1}}{k^{-2}\tan^2 kp} - 2k^2 f_m - k^2A^{-2}f_m^2(f_{m+1} + f_{m-1}) = 0 \tag{2.10}$$

for

$$f_m = A\tan kmp \tag{2.11}$$

with the limiting ODE

$$f''(x) - 2k^2 f(x) - 2k^2A^{-2}f(x)^3 = 0. \tag{2.12}$$

Regarded as an approximation of (2.12), the OΔE (2.10) uses $k^{-2}\tan^2 kp$, which is $O(p^2)$, in the denominator of the quotient approximating the second derivative, and $f(x)^3$ is replaced by $f_m^2(f_{m+1} + f_{m-1})/2$.

The function $f(x) = A\cot kx$ gives the same OΔE (2.10) and the same ODE (2.12).

2.3 $f(x) = A\csc kx$, $f(x) = A\sec kx$

The nonlinear second-order OΔE is

$$\frac{f_{m+1} - 2f_m + f_{m-1}}{k^{-2}\sin^2 kp} + \frac{k^2 f_m}{\cos^2(kp/2)} - k^2A^{-2}f_m^2(f_{m+1} + f_{m-1}) = 0 \tag{2.13}$$

with the limiting ODE

$$f''(x) + k^2 f(x) - 2k^2A^{-2}f(x)^3 = 0. \tag{2.14}$$

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3. Jacobian elliptic functions

The Jacobian elliptic functions satisfy addition formulae which will now be used to derive appropriate OΔE's. The notation used in this section has been altered to conform with [1]. The derivations are straightforward and only the results are presented.

3.1 $g(t) = A\,\mathrm{sn}(\omega t \,|\, m)$

With constant stepsize q for the variable t, define

$$g_n = A\,\mathrm{sn}(\omega n q \,|\, m) \tag{3.1}$$

where n is an integer. The addition formula [1]

$$\mathrm{sn}\,\omega(t+q) = \frac{\mathrm{sn}\,\omega t\,\mathrm{cn}\,\omega q\,\mathrm{dn}\,\omega q + \mathrm{sn}\,\omega q\,\mathrm{cn}\,\omega t\,\mathrm{dn}\,\omega t}{1 - m\,\mathrm{sn}^2\omega t\,\mathrm{sn}^2\omega q} \tag{3.2}$$

leads to the nonlinear second-order OΔE

$$\begin{aligned}\frac{g_{n+1} - 2g_n + g_{n-1}}{\omega^{-2}\,\mathrm{sn}^2\omega q} &+ \frac{2(m+1-m\,\mathrm{sn}^2\omega q)}{1+\mathrm{cn}\,\omega q\,\mathrm{dn}\,\omega q}\omega^2 g_n \\ &- m\omega^2 A^{-2} g_n^2 (g_{n+1} + g_{n-1}) = 0.\end{aligned} \tag{3.3}$$

In the limit $q \to 0$, the OΔE (3.3) converges to the nonlinear ODE

$$\ddot{g}(t) + (m+1)\omega^2 g(t) - 2m\omega^2 A^{-2} g(t)^3 = 0. \tag{3.4}$$

3.2 $g(t) = A\,\mathrm{cn}(\omega t \,|\, m)$

The nonlinear second-order OΔE is

$$\begin{aligned}\frac{g_{n+1} - 2g_n + g_{n-1}}{\omega^{-2}\,\mathrm{sn}^2\omega q} &+ \frac{2\omega^2}{1+\mathrm{cn}\,\omega q} g_n - m\omega^2(g_{n+1} + g_{n-1}) \\ &+ m\omega^2 A^{-2} g_n^2 (g_{n+1} + g_{n-1}) = 0\end{aligned} \tag{3.5}$$

with the limiting nonlinear ODE

$$\ddot{g}(t) + (1-2m)\omega^2 g(t) + 2m\omega^2 A^{-2} g(t)^3 = 0. \tag{3.6}$$

3.3 $g(t) = A\,\mathrm{dn}(\omega t \,|\, m)$

The nonlinear OΔE is

$$\begin{aligned}\frac{g_{n+1} - 2g_n + g_{n-1}}{\omega^{-2}\,\mathrm{sn}^2\omega q} &+ \frac{2m\omega^2}{1+\mathrm{dn}\,\omega q} g_n \\ &- \omega^2(g_{n+1} + g_{n-1}) + \omega^2 A^{-2} g_n^2 (g_{n+1} + g_{n-1}) = 0\end{aligned} \tag{3.7}$$

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with the limiting ODE

$$\ddot{g}(t) + (m-2)\omega^2 g(t) + 2\omega^2 A^{-2} g(t)^3 = 0. \tag{3.8}$$

More-complicated OΔE's can be constructed in the same way for $A\,\mathrm{ns}(\omega t\,|\,m)$ and related functions. The important case of the Weierstrass elliptic function $\mathscr{P}(z)$ has been treated elsewhere [9].

4. Hyperbolic functions

The OΔE's for the hyperbolic functions can be obtained directly from those for the trigonometric functions. Detailed derivations of third-order OΔE's for tanh and sech will be given because of their application to certain nonlinear evolution PΔE's to be discussed later.

4.1 $f(x) = A \sinh kx$, $f(x) = A \cosh kx$

Since the replacements

$$k \to ik, \quad A \to -iA \tag{4.1}$$

transform $A \sin kx$ to $A \sinh kx$, the OΔE and ODE for $A \sinh kx$ are immediately obtained from (2.7) and (2.8) as

$$\frac{f_{m+1} - 2f_m + f_{m-1}}{4k^{-2}\sinh^2(kp/2)} - k^2 f_m = 0 \tag{4.2}$$

and

$$f''(x) - k^2 f(x) = 0. \tag{4.3}$$

These are also obtained for $f(x) = \cosh kx$ by using (2.7) and (2.8) with the single replacement

$$k \to ik. \tag{4.4}$$

4.2 $f(x) = A \,\mathrm{csch}\, kx$

In a similar way, the second-order OΔE and ODE for $A \cosh kx$ are obtained as

$$\frac{f_{m+1} - 2f_m + f_{m-1}}{k^{-2}\sinh^2 kp} - \frac{k^2 f_m}{\cosh^2(kp/2)} - k^2 A^{-2} f_m^2 (f_{m+1} + f_{m-1}) = 0 \tag{4.5}$$

and

$$f''(x) - k^2 f(x) - 2k^2 A^{-2} f(x)^3 = 0. \tag{4.6}$$

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4.3 $f(x) = A \tanh kx$

This function will be considered in more detail because it is to be used later and first, second and third order equations for it will be derived.

For

$$f_m = A \tanh kmp \tag{4.7}$$

the first-order OΔE is obtained from the addition formula for tanh, in the forms

$$(f_{m+1} - f_{m-1})(\cosh^2 kmp + \sinh^2 kp) = A \sinh 2kp, \tag{4.8}$$

$$(f_{m+1} + f_{m-1})(\cosh^2 kmp + \sinh^2 kp) = A \sinh 2kmp \tag{4.9}$$

which combine to give the OΔE

$$\frac{f_{m+1} - f_{m-1}}{k^{-1} \sinh 2kp} = k \frac{f_{m+1} + f_{m-1}}{2 f_m} A\left(1 - A^{-2} f_m^2\right). \tag{4.10}$$

This converges when $p \to 0$ to the ODE

$$f'(x) = kA\left(1 - A^{-2} f(x)^2\right). \tag{4.11}$$

It will be noted that (4.10) is in fact a second-order OΔE because evaluation of f_{m+1} requires knowledge of f_m and f_{m-1}.

The second-order OΔE

$$\frac{f_{m+1} - 2f_m + f_{m-1}}{k^{-2} \tanh^2 kp} + 2k^2 f_m - k^2 A^{-2} f_m^2 (f_{m+1} + f_{m-1}) = 0 \tag{4.12}$$

can be obtained from the corresponding OΔE (2.10) for $A \tan kx$, together with the ODE

$$f''(x) + 2k^2 f(x) - 2k^2 A^{-2} f(x)^3 = 0. \tag{4.13}$$

To obtain third-order equations, (4.10) is modified to

$$\frac{f_{m+3} - f_{m-3}}{\sinh 6kp} = \frac{f_{m+3} + f_{m-3}}{2 f_m} A\left(1 - A^{-2} f_m^2\right) \tag{4.14}$$

which, together with (4.10), leads after some manipulation to the OΔE

$$\begin{aligned} &\frac{f_{m+3} - 3f_{m+1} + 3f_{m-1} - f_{m-3}}{(k^{-1} \sinh 2kp)^3} \\ &\qquad + 6kA^{-1} \cosh 2kp \left[\frac{f_{m+3} + f_{m-3}}{f_{m+1} + f_{m-1}}\right]\left[\frac{f_{m+1} - f_{m-1}}{k^{-1} \sinh 2kp}\right]^2 \\ &\qquad - 4k^2 \left[\frac{f_{m+3} + f_{m-3}}{f_{m+1} + f_{m-1}}\right]\left[\frac{f_{m+1} - f_{m-1}}{k^{-1} \sinh 2kp}\right] = 0. \end{aligned} \tag{4.15}$$

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This rather formidable-looking difference equation is of third order, and allows the calculation of f_{m+3} for given f_{m+1}, f_{m-1}, f_{m-3}. In the limit it converges to the third-order ODE

$$f'''(x) + 6kA^{-1}f'(x)^2 - 4k^2f'(x) = 0. \qquad (4.16)$$

These results will be used in the later discussion of the Korteweg-de Vries equation.

4.4 $f(x) = A \operatorname{sech} kx$

This function also arises in the analysis of nonlinear evolutionary equations and first, second and third order equations will be derived.

For

$$f_m = A \operatorname{sech} kmp \qquad (4.17)$$

the first-order OΔE is obtained from the addition formula for sech in the forms

$$(f_{m+1} - f_{m-1})(\cosh^2 kmp + \sinh^2 kp) = -2A \sinh kmp \sinh kp, \qquad (4.18)$$

$$(f_{m+1} + f_{m-1})(\cosh^2 kmp + \sinh^2 kp) = 2A \cosh kmp \cosh kp \qquad (4.19)$$

which combine to give the second-order OΔE

$$\frac{f_{m+1} - f_{m-1}}{2k^{-1}\tanh kp} = -k\left(1 - A^{-2}f_m^2\right)^{1/2}(f_{m+1} + f_{m-1})/2. \qquad (4.20)$$

This converges to the first-order ODE

$$f'(x) = -k\left(1 - A^{-2}f(x)^2\right)^{1/2}f(x). \qquad (4.21)$$

The second order OΔE which leads to the second-order ODE is similar to that for $A \operatorname{csch} kx$. The OΔE is

$$\frac{f_{m+1} - 2f_m + f_{m-1}}{k^{-2}\sinh^2 kp} - \frac{k^2 f_m}{\cosh^2(kp/2)} + k^2A^{-2}f_m^2(f_{m+1} + f_{m-1}) = 0, \qquad (4.22)$$

and the corresponding ODE

$$f''(x) - k^2f(x) + 2k^2A^{-2}f(x)^3 = 0. \qquad (4.23)$$

To obtain third-order equations, (4.20) is modified to give

$$\frac{f_{m+3} - f_{m-3}}{2\tanh 3kp} = -\left(1 - A^{-2}f_m^2\right)^{1/2}(f_{m+3} + f_{m-3})/2 \qquad (4.24)$$

which, again after some manipulation, leads to the OΔE

$$\frac{f_{m+3}-3f_{m+1}+3f_{m-1}-f_{m-3}}{(2k^{-1}\sinh kp)^3}\cosh 3kp$$
$$+3k^2A^{-2}f_m(f_{m+3}+f_{m-3})\cosh kp\cosh 2kp\frac{f_{m+1}-f_{m-1}}{2k^{-1}\tanh kp} \quad (4.25)$$
$$-k^2\left(\frac{f_{m+3}+f_{m-3}}{f_{m+1}+f_{m-1}}\right)\frac{f_{m+1}-f_{m-1}}{2k^{-1}\tanh kp}=0.$$

The convergence to the third-order ODE

$$f'''(x)+6k^2A^{-2}f(x)^2f'(x)-k^2f'(x)=0 \quad (4.26)$$

is evident. This equation will be used in the later discussion of the modified Korteweg-de Vries equation.

5. Wave equation

As a first example of a partial difference equation (PΔE) and an associated partial differential equation (PDE) consider the one-dimensional wave equation

$$u_{tt}=c^2u_{xx} \quad (5.1)$$

satisfied, for example, by the function

$$u(x,t)=A\sin k(x+ct). \quad (5.2)$$

To obtain the PΔE corresponding to this solution, the discretization used is

$$x=mp,\qquad \Delta x=p, \quad (5.3)$$
$$t=nq,\qquad \Delta t=q, \quad (5.4)$$

with

$$u(x,t)=u(mp,mq)=u_{m,n}. \quad (5.5)$$

From (2.7) it immediately follows that

$$\frac{u_{m+1,n}-2u_{m,n}+u_{m-1,n}}{4k^{-2}\sin^2(kp/2)}+k^2u_{m,n}=0 \quad (5.6)$$

and

$$\frac{u_{m,n+1}-2u_{m,n}+u_{m,n-1}}{4k^{-2}c^{-2}\sin^2(kcq/2)}+k^2c^2u_{m,n}=0 \quad (5.7)$$

whence

$$\frac{u_{m,n+1}-2u_{m,n}+u_{m,n-1}}{4k^{-2}c^{-2}\sin^2(kcq/2)}=c^2\frac{u_{m+1,n}-2u_{m,n}+u_{m-1,n}}{4k^{-2}\sin^2(kp/2)}. \quad (5.8)$$

This linear second-order PΔE converges in the limit $p \to 0$, $q \to 0$ to the PDE (5.1). The function $u(x,t) = A \sin k(x - ct)$ is also a solution of the PΔE (5.8) and the PDE (5.1).

The analysis can be extended to a solution consisting of a sum of two terms, for example

$$u(x,t) = A_1 \sin k_1(x + ct) + A_2 \sin k_2(x + ct). \tag{5.9}$$

From the relation

$$\begin{aligned} &u(x+p,t) - (\cos k_1 p + \cos k_2 p)u(x,t) + u(x-p,t) \\ &\quad = (\cos k_1 p - \cos k_2 p)[A_1 \sin k_1(x+ct) - A_2 \sin k_2(x+ct)] \end{aligned} \tag{5.10}$$

follows the required PΔE

$$\begin{aligned} &[u_{m,n+1} - (\cos k_1 cq + \cos k_2 cq)u_{m,n} + u_{m,n-1}]/(\cos k_1 cq - \cos k_2 cq) \\ &\quad = [u_{m+1,n} - (\cos k_1 p + \cos k_2 p)u_{m,n} + u_{m-1,n}]/(\cos k_1 p - \cos k_2 p). \end{aligned} \tag{5.11}$$

This converges to the wave equation (5.1) as $p, q \to 0$ since

$$\cos k_1 cq + \cos k_2 cq \to 2, \qquad \cos k_1 cq - \cos k_2 cq \to c^2 q^2 (k_2^2 - k_1^2)/2, \tag{5.12}$$

$$\cos k_1 p + \cos k_2 p \to 2, \qquad \cos k_1 p - \cos k_2 p \to p^2 (k_2^2 - k_1^2)/2. \tag{5.13}$$

It is not possible to find a simple PΔE for the general solution $u(x,t) = \Sigma A_j \cos k_j(x + ct)$ although this is trivially still a solution of the wave equation. The appearance of k in the denominator $4k^{-2}\sin^2(kp/2)$ loses the advantage enjoyed by the usual approximation term p^2.

One generalization is possible and that is to the three-dimensional wave equation

$$u_{tt} = c^2 \Delta u. \tag{5.14}$$

The plane-wave solution

$$u = A \sin(kx + \omega t) \tag{5.15}$$

is a solution of an approximating PΔE which is a straightforward generalization of (5.8).

6. FitzHugh-Nagumo equation

As a first example of a nonlinear evolution equation, consider the PDE

$$u_t = u_{xx}/2 + (\omega A - k^2 u)(A^{-2}u^2 - 1) \tag{6.1}$$

which was considered by FitzHugh and Nagumo in modelling the propagation of a nerve pulse [2]. The PΔE has the solitary-wave solution

$$u(x,t) = A\tanh(kx + \omega t). \tag{6.2}$$

The derivation of the first and second-order OΔE's (4.10) and (4.12) for $A\tanh kx$ can now be extended to find the PΔE satisfied by (6.2). From

$$u_{m,n} = \tanh(kmp + \omega nq) \tag{6.3}$$

first follows

$$u_{m,n+1} - u_{m,n} = A\tanh\omega q\left(A^{-2}u_{m,n}u_{m,n+1} - 1\right) \tag{6.4}$$

which, combined with the obvious extension of (4.12), gives the required PΔE

$$\frac{u_{m,n+1} - u_{m,n}}{\omega^{-1}\tanh\omega q} = \frac{u_{m+1,n} - 2u_{m,n} + u_{m-1,n}}{2k^{-2}\tanh^2 kp} - \omega A + \omega A^{-1}u_{m,n}u_{m,n+1}$$
$$+k^2u_{m,n} - k^2A^{-2}u_{m,n}^2(u_{m+1,n} + u_{m-1,n})/2. \tag{6.5}$$

In the limit $p, q \to 0$, this PΔE gives

$$u_t = u_{xx}/2 - \omega A + \omega A^{-1}u^2 + k^2u - k^2A^{-2}u^3 \tag{6.6}$$

which is equivalent to (6.1).

Equation (6.5) is a simple example of a nonlinear evolutionary PΔE. Given, say, the initial values $u_{m,0}$ for all m, (6.5) can be solved for $u_{m,1}$ for all m, and then for $u_{m,2}$ and so on. The evolution of $u_{m,n}$ exactly follows that of the nerve-pulse solution of the PΔE, regardless of the magnitude of the stepsizes p and q. In this sense the PΔE (6.5) is the best approximation to the PDE (6.1).

7. Korteweg-de Vries equation

As a second, and more difficult, example of a nonlinear evolutionary PDE consider the equation

$$u_{xxx} - 3u_x^2 + u_t = 0 \tag{7.1}$$

which, when differentiated with respect to x and the substitution $u_x = v$ made, gives the Korteweg-de Vries equation in the usual form

$$v_{xxx} - 6vv_x + v_t = 0. \tag{7.2}$$

We shall determine a PΔE for the single-soliton solution

$$u(x,t) = -2k\tanh(kx - 4k^3t) \tag{7.3}$$

of the equation (7.1) by considering the third-order equations satisfied by

$$u(x,t) = A\tanh(kx + \omega t) \tag{7.4}$$

and

$$u_{m,n} = A\tanh(kmp + \omega nq). \tag{7.5}$$

From (4.10) follows the relation

$$\left[\frac{u_{m+1,n} - u_{m-1,n}}{u_{m+1,n} + u_{m-1,n}}\right]\frac{1}{\sinh 2kp} = \left[\frac{u_{m,n+1} - u_{m,n-1}}{u_{m,n+1} + u_{m,n-1}}\right]\frac{1}{\sinh 2\omega q} \tag{7.6}$$

which, when used with the OΔE (4.15), gives the PΔE

$$\begin{aligned}&\frac{u_{m+3,n} - 3u_{m+1,n} + 3u_{m-1,n} - u_{m-3,n}}{(k^{-1}\sinh 2kp)^3}\\ &\quad + 6kA^{-1}\cosh 2kp\left[\frac{u_{m+3,n} + u_{m-3,n}}{u_{m+1,n} + u_{m-1,n}}\right]\left[\frac{u_{m+1,n} - u_{m-1,n}}{k^{-1}\sinh 2kp}\right]^2\\ &\quad - \frac{4k^3}{\omega}\left[\frac{u_{m+3,n} + u_{m-3,n}}{u_{m,n+1} + u_{m,n-1}}\right]\left[\frac{u_{m,n+1} - u_{m,n-1}}{\omega^{-1}\sinh 2\omega q}\right] = 0.\end{aligned} \tag{7.7}$$

The limiting ODE is

$$u_{xxx} + 6kA^{-1}u_x^2 - (4k^3/\omega)u_t = 0. \tag{7.8}$$

When $A = -2k$ and $w = -4k^3$, so that $u(x, t)$ becomes (7.3), the ODE (7.8) takes the required form (7.1).

8. Modified KdV equation

As a third example of a nonlinear evolutionary PDE consider the modified Korteweg-de Vries equation [2, 10] in the form

$$u_{xxx} + 6u^2u_x + u_t = 0. \tag{8.1}$$

We shall determine a PΔE for the solitary-wave solution given by

$$u(x,t) = k\,\mathrm{sech}(kx - k^3t). \tag{8.2}$$

Consider first the function

$$u(x,t) = A\,\mathrm{sech}(kx + \omega t) \tag{8.3}$$

and consequently

$$u_{m,n} = A\,\mathrm{sech}(kmp + \omega nq). \tag{8.4}$$

From (4.20) follows the relation

$$\left[\frac{u_{m+1,n} - u_{m-1,n}}{u_{m+1,n} + u_{m-1,n}}\right]\frac{1}{\tanh kp} = \left[\frac{u_{m,n+1} - u_{m,n-1}}{u_{m,n+1} + u_{m,n-1}}\right]\frac{1}{\tanh \omega q} \tag{8.5}$$

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which, used with the OΔE (4.25), gives the PΔE

$$\frac{u_{m+3,n}-3u_{m+1,n}+3u_{m-1,n}-u_{m-3,n}}{(2k^{-1}\sinh kp)^3}\cosh 3kp$$
$$+3k^2A^{-2}u_{m,n}(u_{m+3,n}+u_{m-3,n})\cosh kp\cosh 2kp\left[\frac{u_{m+1,n}-u_{m-1,n}}{2k^{-1}\tanh kp}\right]$$
$$-\frac{k^3}{\omega}\left[\frac{u_{m+3,n}+u_{m-3,n}}{u_{m,n+1}+u_{m,n-1}}\right]\left[\frac{u_{m,n+1}-u_{m,n-1}}{2\omega^{-1}\tanh\omega q}\right]=0. \quad (8.6)$$

The limiting ODE is

$$u_{xxx}+6k^2A^{-2}u^2u_x-(k^3/\omega)u_t=0. \quad (8.7)$$

The special choices $A = k$ and $\omega = -k^3$ lead to the modified KdV equation (8.1).

It will be noted that the PΔE (8.6) is of second order in the time index n, the values of $u_{m,n-1}$ and $u_{m,n}$ being required for $u_{m+1,n}$ to be determined.

9. Discussion

There are four significant points to be made about the special difference equations which have been derived in this paper.

The first point is the importance of generating a class of difference equations for which exact solutions are known. The usual difference equation approximations for differential equations cannot be solved exactly. Their behaviour can be quite bizarre, and their indiscriminant use fraught with danger, especially when nonlinear equations are approximated. For even the simplest nonlinear ordinary differential equation, the logistic equation, it has been shown that a central discretization scheme for any mesh size gives a difference equation which cannot be solved exactly and which exhibits chaotic behaviour! As has been pointed out [6], a more careful discretization along the lines of this paper leads to a difference equation which can be solved exactly and which exhibits precisely the true behaviour of the logistic equation. As Whitham [10] has highlighted, one of the most remarkable developments in recent work on nonlinear waves has been the discovery of explicit exact solutions for some of the simple standard nonlinear evolutionary partial differential equations. The present paper shows how to generate corresponding difference equations for which exact solutions are also available.

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This leads to the second significant point. For nonlinear differential equations for which explicit exact solutions are not known, it is often necessary to resort to a perturbation method based on a simpler equation which can be solved. If the simpler equation can be discretized along the lines developed in this paper, then this can provide a suitable zeroth-order approximation for a perturbation scheme. The Van der Pol and Mathieu differential equations, for which exact solutions are not known, have been investigated in this way. As has been shown [7, 8], difference schemes can be chosen which have the advantage that for the unperturbed problem they are exact. Why choose an inexact zeroth-order approximation?

The third significant point is that the difference equations derived in this paper are exact approximations regardless of the magnitude of the stepsize. It is now a well-known and sobering fact that large three-dimensional nonlinear problems, which can only be analysed numerically with meshes with comparatively large step sizes, may be producing solutions which appear plausible but which are indeed spurious, simply a product of the discretization scheme used. The spurious behaviour is, in general, enhanced by increasing stepsize so that the difference equations valid for any stepsize are of particular interest.

The last point is that, although the approach followed in this paper is a limited one, being confined to functions for which simple addition formulae are available, yet it covers an important class which includes the standard nonlinear evolutionary equations. Had their exact solutions not satisfied addition formulae, then the method of this paper would not be applicable. Is it just a happy coincidence that the functions which figure prominently in the exact solutions, trigonometric, hyperbolic, Jacobian elliptic and Weierstrass elliptic, are just the functions which satisfy addition formulae? It is a point which is being explored, one clue being the relation of these differential equations to the ordinary Euler differential equation with separated variables.

The results presented in this paper extend an approach used for linear and nonlinear ordinary difference and differential equations to standard partial equations. It is proposed to apply the method to other such equations, and to two and more soliton solutions. The numerical behaviour of the difference equations that have been developed is also being investigated.

References

[1] M. Abramowitz and I. A. Stegun (eds.), *Handbook of Mathematical functions* (Dover, New York, 1970).

[2] G. Eilenberger, *Solitons: mathematical methods for physicists* (Springer-Verlag, 1983).

[3] R. Hirota, "Difference analogues of nonlinear evolutionary equations in Hamiltonian form", *Hiroshima University Technical Report* A-12 (1982).
[4] R. B. Potts, "Best difference approximation to Duffing's equation", *J. Austral. Math. Soc. Ser. B* 23 (1981), 64–77.
[5] R. B. Potts, "Differential and difference equations", *Amer. Math. Monthly* 89 (1982), 402–407.
[6] R. B Potts, "Nonlinear difference equations", *Nonlinear Analysis* 6 (1982), 659–665.
[7] R. B. Potts, "Van der Pol difference equation", *Nonlinear Analysis* 7 (1983), 801–812.
[8] R. B. Potts, "Mathieu difference equation", in press.
[9] R. B. Potts, "Weierstrass elliptic difference equations", submitted for publication.
[10] G. B. Whitham, *Linear and nonlinear waves* (Wiley, 1974).