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LIFTING TORSION GALOIS REPRESENTATIONS

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Abstract

Let $p \geqslant 5$ be a prime, and let $\mathcal{O}$ be the ring of integers of a finite extension K of $\mathbb{Q}_p$ with uniformizer π. Let $\rho_n : G_{\mathbb{Q}} \to GL_2(\mathcal{O}/(\pi^n))$ have modular mod-π reduction $\bar{\rho}$, be ordinary at p, and satisfy some mild technical conditions. We show that ρ_n can be lifted to an $\mathcal{O}$-valued characteristic-zero geometric representation which arises from a newform. This is new in the case when K is a ramified extension of $\mathbb{Q}_p$. We also show that a prescribed ramified complete discrete valuation ring $\mathcal{O}$ is the weight-2 deformation ring for $\bar{\rho}$ for a suitable choice of auxiliary level. This implies that the field of Fourier coefficients of newforms of weight 2, square-free level, and trivial nebentype that give rise to semistable $\bar{\rho}$ of weight 2 can have arbitrarily large ramification index at p.

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1. Introduction

Let $p \geqslant 5$ be a prime, and let $\bar{\rho} : G_{\mathbb{Q}} \to GL_2(\mathbb{F}_{p^f})$ be a continuous, odd, irreducible representation of the absolute Galois group $G_{\mathbb{Q}}$ of $\mathbb{Q}$. We say that such a representation is of S-type.

Let $W(\mathbb{F}_{p^f})$ be the ring of Witt vectors of $\mathbb{F}_{p^f}$. A method due to the second author [13] lifts representations $\rho_n : G_{\mathbb{Q}} \to GL_2(W(\mathbb{F}_{p^f})/(p^n))$, with reduction $\bar{\rho}$ of S-type, to geometric characteristic-zero representations if $p \geqslant 5$, provided that ρ_n is balanced in a sense made precise below. For finite sets of places T disjoint from S we have various deformation rings associated to $\bar{\rho}$, for example $R^{ord,T\text{-}new}$, the universal ordinary ring associated to $\bar{\rho}$ (no weight is fixed) whose universal deformation is ramified only at $S \cup T$, minimal at $v \in S$ and Steinberg at $v \in T$. We denote its weight-2 quotient by $R_2^{ord,T\text{-}new}$. The technical conditions of Theorem 1 (cf. Theorem 17 below) and Theorem 2 (cf. Theorem 25 below) below will be explained in the body of the paper (for example, see Definition 4 for the definition of being full, and ordinary of weight 2, and see Definition 7 for the definition of being balanced).

THEOREM 1. *Let $\mathcal{O}$ be the ring of integers of a finite extension K of $\mathbb{Q}_p$ with uniformizer π. Suppose that $\rho_n : G_{\mathbb{Q}} \to GL_2(\mathcal{O}/(\pi^n))$ is odd, ordinary, weight 2, modular, has full image and determinant ϵ, and is balanced. Then there exists a finite set of primes $T \supseteq S$ such that the universal ordinary 'new at T' ring $R^{ord,T\text{-}new} \simeq W(\mathbb{F}_{p^f})[[U]]$, and there are surjections*

$$R^{ord,T\text{-}new} \twoheadrightarrow R_2^{ord,T\text{-}new} \twoheadrightarrow \mathcal{O}/(\pi^n)$$

from this ring to its weight-2 quotient and then to $\mathcal{O}/(\pi^n)$ inducing ρ_n.

The innovation here compared to [13] is that $K/\mathbb{Q}_p$ need not be unramified.

While Theorem 1 gives, via the smoothness of $R^{ord,T\text{-}new}$, many $\mathcal{O}$-valued ordinary lifts of ρ_n, it leaves open the possibility that none of these are arithmetic. To remedy this we also prove the following.

THEOREM 2. *Let ρ_n be as in Theorem 1. There exists a finite set of primes $T \supseteq S$ such that*

$$R_2^{ord,T\text{-}new} \simeq \mathcal{O} \twoheadrightarrow \mathcal{O}/(\pi^n)$$

inducing ρ_n.

By similar means, we next prove Theorem 3 (cf. Corollary 32), which may be regarded to be in the direction of a converse to Wiles' results that Hecke algebras are usually finite flat complete intersections over $\mathbb{Z}_p$. Our theorem proves that a monogenic finite flat complete intersection over $\mathbb{Z}_p$ occurs as a Hecke algebra (or equivalently deformation ring) for a given $\bar{\rho}$.

THEOREM 3. *Let $\bar{\rho} : G_{\mathbb{Q}} \to GL_2(\mathbb{F}_{p^f})$ be odd, full, ordinary, weight 2, and have determinant ϵ. Let $g(U) \in W(\mathbb{F}_{p^f})[U]$ be a distinguished polynomial. Then there exists a set $T \supset S_0$ such that*

$$W(\mathbb{F}_{p^f})[[U]] \simeq R^{ord,T\text{-}new} \twoheadrightarrow R_2^{ord,T\text{-}new} \simeq W(\mathbb{F}_{p^f})[[U]]/(g(U)).$$

We sketch the proofs. We first study the ordinary deformation rings $R^{ord,T\text{-}new}$ and construct sets of primes T such that $R^{ord,T\text{-}new}$ has one-dimensional tangent space and thus is a quotient of $W(\mathbb{F}_{p^f})[[U]]$. We choose the primes of T carefully so that ρ_n arises as a specialization of the universal representation associated to $R^{ord,T\text{-}new}$. As a certain dual Selmer group vanishes, we in fact deduce that $R^{ord,T\text{-}new} \simeq W(\mathbb{F}_{p^f})[[U]]$. This smoothness already ensures that ρ_n lifts to an ordinary characteristic-zero representation which is ramified at finitely many primes (cf. Theorem 1). We also deduce (see Corollary 35) that $R^{ord,T\text{-}new}$ is isomorphic to the related Hida Hecke algebra $\mathbb{T}^{ord,T\text{-}new}$, as the latter is known to be nonzero and flat over the Iwasawa algebra $\Lambda = \mathbb{Z}_p[[X]]$ by level-raising results of [5] and Hida's results on the structure of ordinary Hecke algebras, respectively.

To prove Theorem 2 (and Theorem 3), a more careful choice of T is required to guarantee that the weight-2 quotient of $R^{ord,T\text{-}new}$ can be controlled to be $\mathcal{O}$ and that ρ_n arises from this weight-2 quotient. First, we choose T to simultaneously make $R^{ord,T\text{-}new} \simeq W(\mathbb{F}_{p^f})[[U]]$ and $R_2^{ord,T\text{-}new} \simeq W(\mathbb{F}_{p^f})[[U]]/I$, where $I \subset (p, U)^N$ for a large number N. We then add another prime to the level to introduce the minimal polynomial $g_\pi(U)$ of π (the uniformizer of $\mathcal{O}$) as an obstruction in the weight-2 ring. Essential use is made of the techniques of [10].

It is worth remarking that in the proof of Theorem 2 we lift the representation ρ_n to a characteristic-zero geometric lift (of weight 2), in spite of it being impossible to kill the minimal weight-2 dual Selmer (and Selmer) group of $\bar{\rho}$ using primes which are nice for ρ_n (not just $\bar{\rho}$!) when $\mathcal{O}/\mathbb{Z}_p$ is ramified.

In the appendix, we apply the isomorphisms $W(\mathbb{F}_{p^f})[[U]] \simeq R^{ord,T\text{-}new} \simeq \mathbb{T}^{ord,T\text{-}new}$ above to extend the approach of [9] to modularity of geometric lifts ρ of $\bar{\rho}$ by p-adic approximation, without imposing the condition that ρ is defined over the Witt vectors. (This application was the initial impetus for the work done here.) This condition was essential in [9] again because $GL_2(\mathcal{O}/(\pi^2)) \to GL_2(\mathcal{O}/(\pi))$ is split if $\mathcal{O}/\mathbb{Z}_p$ is ramified. The appendix uses only Theorem 1 of this paper (and not the more involved Theorem 2), together with level-lowering arguments. (In the recent paper [18] the strategy of [9] is combined with the Taylor–Wiles patching argument to prove modularity-lifting theorems in new cases.)

2. The setup

Let $p \geqslant 5$ be a prime. We consider representations $\bar{\rho} : G_\mathbb{Q} \to GL_2(\mathbb{F}_{p^f})$ of S-type.

Let $\mathcal{O}$ be the ring of integers of a finite extension K of $\mathbb{Q}_p$ with uniformizer π and residue field $\mathbb{F}_{p^f}$. Suppose that $[K : \mathbb{Q}_p] = d = ef$, with e, f the ramification and inertial degrees. Let ϵ be the p-adic cyclotomic character, and let $\bar{\epsilon}$ be its mod-p reduction. We consider lifts $\rho_n : G_\mathbb{Q} \to GL_2(\mathcal{O}/(\pi^n))$ and $\rho : G_\mathbb{Q} \to GL_2(\mathcal{O})$ of $\bar{\rho}$.

DEFINITION 4. – We say that ρ_n (respectively, ρ) is full if $\rho_n(G_{\mathbb{Q}})$ contains $SL_2(\mathcal{O}/(\pi^n))$ (respectively, $\rho(G_{\mathbb{Q}})$ contains $SL_2(\mathcal{O})$).

– We say that ρ_n (respectively, ρ) is ordinary of weight 2 if its restriction to an inertia group I_p at p is conjugate to $\left(\begin{smallmatrix}\epsilon & * \\ 0 & 1\end{smallmatrix}\right)$, where ϵ is the cyclotomic character and the $*$ may or may not be trivial. (The arguments in this paper work also for representations whose restriction to I_p is of the form $\left(\begin{smallmatrix}\epsilon^{k-1} & * \\ 0 & 1\end{smallmatrix}\right)$ with $k \geqslant 2$ and $\bar{\rho}$ is distinguished at p, with the further conditions on ρ_n below when $k = 2$ and $\bar{\rho}$ is split at p.)

We assume that the representations ρ, ρ_n, and ρ are odd, have full image and determinant ϵ, and that $\bar{\rho}$ is modular and of weight 2.

We will consider both

– the weight-2 ordinary deformation theory of $\bar{\rho}$; and

– the arbitrary-weight ordinary deformation theory of $\bar{\rho}$.

We always impose the ordinarity condition $\rho_n|_{I_p} \simeq \left(\begin{smallmatrix}\epsilon & * \\ 0 & 1\end{smallmatrix}\right)$, and furthermore the following.

- If $\bar{\rho}|_{G_p} = \left(\begin{smallmatrix}\bar{\epsilon} & * \\ 0 & 1\end{smallmatrix}\right)$ is nonsplit and flat, then $\rho_n|_{G_p}$ is either flat or semistable; that is, if $\rho_n|_{G_p}$ is not flat, $\rho_n|_{G_p} = \left(\begin{smallmatrix}\epsilon & * \\ 0 & 1\end{smallmatrix}\right)$, and the $*$ arises from taking a p-power root of a nonunit of $\mathbb{Z}_p$.
- If $\bar{\rho}|_{G_p} = \left(\begin{smallmatrix}\bar{\epsilon} & 0 \\ 0 & 1\end{smallmatrix}\right)$, then $\rho_n|_{G_p}$ is flat.

The first condition on $\rho_n|_{G_p}$ is necessary for local at p characteristic-zero lifts to exist. The second condition is more restrictive. We do not know how to deal with the second case above if $\rho_n|_{G_p}$ is not flat.

Let S_0 and S be the sets of ramified primes of $\bar{\rho}$ and ρ_n respectively (these include p and ∞). By our assumptions it follows that $S \supset S_0 \supset \{p, \infty\}$. We also assume that ρ_n is *balanced*, a condition which we now explain (cf. Definition 7).

For each $v \in S_0$, a smooth quotient of the versal weight-2 deformation ring of $\bar{\rho}|_{G_v}$ has been defined on pages 120–124 of [13] and in [17]. The points of (the *Spec* of) this smooth quotient are our allowable deformations and are denoted $\mathcal{C}_v$. Corresponding to the tangent space of this smooth quotient is a subspace $\mathcal{L}_v \subset H^1(G_v, Ad^0\bar{\rho})$. Fact 5 follows from the discussion in [13] referred to above.

FACT 5. *For all $v \in S_0$ there exist $\mathcal{C}_v$ and $\mathcal{L}_v$ as above satisfying* $\dim \mathcal{L}_v = \dim H^0(G_v, Ad^0\bar{\rho}) + \delta_{vp}$, *where $\delta_{vp} = 0$ or 1 depending on $v \neq p$ and $v = p$.*

Let M be an $\mathbb{F}_{p^f}[G_{\mathbb{Q}}]$-module with $\mathbb{G}_m$-dual M^*, and let R be the union of the places whose inertial action on M is nontrivial and $\{p, \infty\}$. Let $\mathcal{M}_v \subset H^1(G_v,$

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M) with annihilator $\mathcal{M}_v^\perp \subset H^1(G_v, M^*)$ under the perfect local pairing

$$H^1(G_v, M) \times H^1(G_v, M^*) \to H^2(G_v, \mathbb{F}_{p^f}(1)) \simeq \mathbb{F}_{p^f}.$$

Set the Selmer group for the subspaces $\mathcal{M}_v \subset H^1(G_v, M)$ to be

$$H^1_{\mathcal{M}}(G_R, M) := \operatorname{Ker}\left(H^1(G_R, M) \to \bigoplus_{v\in R} \frac{H^1(G_v, M)}{\mathcal{M}_v}\right),$$

and the dual Selmer group to be

$$H^1_{\mathcal{M}^\perp}(G_R, M^*) := \operatorname{Ker}\left(H^1(G_R, M^*) \to \bigoplus_{v\in R} \frac{H^1(G_v, M^*)}{\mathcal{M}_v^\perp}\right).$$

Recall Proposition 1.6 of [19].

PROPOSITION 6.

$$\begin{aligned}&\dim H^1_{\mathcal{M}}(G_R, M) - \dim H^1_{\mathcal{M}^\perp}(G_R, M^*)\\ &\quad = \dim H^0(G_{\mathbb{Q}}, M) - \dim H^0(G_{\mathbb{Q}}, M^*) + \sum_{v\in R}\left(\dim \mathcal{M}_v - \dim H^0(G_v, M)\right).\end{aligned}$$

Fact 5, Proposition 6, and the fact that $\bar\rho$ is odd with full image together imply that the ordinary weight-2 Selmer group $H^1_{\mathcal{L}}(G_{S_0}, Ad^0\bar\rho)$ and its dual Selmer group $H^1_{\mathcal{L}^\perp}(G_{S_0}, Ad^0\bar\rho^*)$ have the same dimension. We need the following *balancedness assumption*.

DEFINITION 7. Let $v \in S$. We assume that a smooth quotient of the versal deformation ring for $\bar\rho|_{G_v}$ exists with points $\mathcal{C}_v$ and induced subspace $\mathcal{L}_v \subset H^1(G_v, Ad^0\bar\rho)$ such that $\dim H^0(G_v, Ad^0\bar\rho) = \dim \mathcal{L}_v$. We also require that $\rho_n|_{G_v}$ be of type $\mathcal{C}_v$.

We define local conditions $\tilde{\mathcal{L}}_v$ for the full adjoint.

DEFINITION 8. (1) For $v \neq p$, set

$$\tilde{\mathcal{L}}_v := \mathcal{L}_v \oplus H^1_{nr}(G_v, \mathbb{F}_{p^f}) \subset H^1(G_v, Ad^0\bar\rho) \oplus H^1(G_v, \mathbb{F}_{p^f}) = H^1(G_v, Ad\bar\rho),$$

namely, the direct sum of $\mathcal{L}_v$, and the $\mathbb{F}_{p^f}$-valued unramified twists in the dual numbers. Set $\tilde{\mathcal{C}}_v$ to be all unramified twists of the points $\mathcal{C}_v$.

(2) For $v = p$, set $W = \left(\begin{smallmatrix} * & * \\ 0 & 0\end{smallmatrix}\right)$ and $\tilde{\mathcal{L}}_p = \operatorname{Ker}(H^1(G_p, Ad\bar\rho) \to H^1(I_p, Ad\bar\rho/W))$. Then $\tilde{\mathcal{L}}_p \supset \mathcal{L}_p$ and $\tilde{\mathcal{L}}_p \supset H^1_{nr}(G_p, \mathbb{F}_{p^f})$, the unramified twists, though $\tilde{\mathcal{L}}_p$ is larger than the direct sum of these subspaces. Set $\tilde{\mathcal{C}}_p$ to be the ordinary deformations of $\bar\rho$ of any weight.

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3. Local deformation rings

3.1. Ordinary deformation rings at p. We need Lemmas 9 and 10 for (5) of Proposition 11.

LEMMA 9. *Let L be the composite of $\mathbb{Q}_p(\mu_{p^\infty})$ and the $\mathbb{Z}_p$-unramified extension of $\mathbb{Q}_p$. Let $\psi : G_p \to \mathbb{Z}_p^*$ be a character with $\psi \equiv \epsilon \mod p$. Then there exists a nonsplit representation $\rho_\psi : G_p \to GL_2(\mathbb{Z}_p)$ with $\rho_\psi = \left(\begin{smallmatrix} \psi & * \\ 0 & 1 \end{smallmatrix}\right)$, where $* \equiv 0$ mod p but $* \not\equiv 0$ mod p^2. Furthermore, after base change to L, the $*$ arises, via Kummer theory, as the composite of finite extensions of L obtained by taking p-power roots of units of elements of L.*

Proof. That ρ_ψ exists follows from the well-known fact that

$$\dim H^1(G_p, \mathbb{Q}_p(\psi)) = \begin{cases} 2 & \psi = \epsilon \\ 1 & \psi \neq \epsilon. \end{cases}$$

Simply take $g \in H^1(G_p, \mathbb{Q}_p(\psi))$, consider $\rho_g : G_p \to GL_2(\mathbb{Q}_p)$ given by $\rho(\tau) = \left(\begin{smallmatrix} \psi(\tau) & g(\tau) \\ 0 & 1 \end{smallmatrix}\right)$, and conjugate by an appropriate power of $\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)$ to get the desired integral representation ρ_ψ. Rather than deal with ρ_ψ, we deal with its mod-p^m reduction, $\rho_{\psi,m}$. Let L_m be the composite of $\mathbb{Q}_p(\mu_{p^m})$ and the $\mathbb{Z}/(p^{m-1})$-unramified extension of $\mathbb{Q}_p$. Note that the $*$ in $\rho_{\psi,m}$ gives rise to a cyclic extension of order p^{m-1}, *not* order p^m.

When $\psi = \epsilon$ we just take $\rho_{\epsilon,m}$ to arise from the splitting field of $x^{p^m} - (1 + p)^{p^{m-1}}$.

For $\psi \neq \epsilon$, let d be the unique integer satisfying $\psi \equiv \epsilon \mod p^d$ and $\psi \not\equiv \epsilon \mod p^{d+1}$. Let D_m be the maximal abelian extension of L_m whose Galois group is killed by p^{m-1}. Let K_m be the composite of L_m and the field fixed by $\mathrm{Kernel}(\rho_{\psi,m})$.

$$\begin{tikzcd}
D_m \arrow[dd, dash] \arrow[dr, dash] & \\
 & K_m \arrow[dl, dash] \\
L_m \arrow[d, dash] & \\
\mathbb{Q}_p &
\end{tikzcd}$$

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The Kummer pairing $Gal(D_m/L_m) \times L_m^*/L_m^{*\,p^{m-1}} \to \mu_{p^{m-1}}$ is perfect and $Gal(L_m/\mathbb{Q}_p)$-equivariant, so $Gal(D_m/L_m)$ is isomorphic to the $\mathbb{G}_m$-dual of $L_m^*/L_m^{*\,p^{m-1}}$ as a $Gal(L_m/\mathbb{Q}_p)$-module. As K_m/L_m is a cyclic extension of order p^{m-1} and $K_m/\mathbb{Q}_p$ is Galois with $Gal(L_m/\mathbb{Q}_p)$ acting on $Gal(K_m/L_m)$ by the character ψ, we see that K_m arises over L_m by adding p^{m-1}th roots of an element $\alpha \in L_m^*/(L_m^*)^{p^{m-1}}$ which, by the above $\mathbb{G}_m$-duality, generates a ϵ/ψ-eigenspace in this group. So we need to prove such an eigenspace exists in the *unit* part of $L_m^*/(L_m^*)^{p^{m-1}}$.

Recall that $L_m^* \simeq \langle \pi_m \rangle \times U_{L_m}$, where π_m is a uniformizer of L_m and U_{L_m} is the group of units. Write $\alpha = \pi_m^{p^k a} u$, where $p \nmid a$ and $u \in U_{L_m}$, so $K_m = L_m(\alpha^{1/p^{m-1}})$. Let $\sigma \in Gal(L_m/\mathbb{Q}_p)$, and set $\sigma(\pi_m) = \pi_m w_\sigma$ and $\sigma(u) = u_\sigma$, where $w_\sigma, u_\sigma \in U_{L_m}$. We have

$$\sigma(\alpha) = \sigma(\pi_m^{p^k a} u) = \sigma(\pi_m)^{p^k a} \sigma(u) = \pi_m^{p^k a} w_\sigma^{p^k a} u_\sigma.$$

But we also have

$$\sigma(\alpha) \equiv (\alpha)^{\epsilon/\psi(\sigma)} \equiv (\pi_m^{p^k a} u)^{\epsilon/\psi(\sigma)} \bmod (L_m^*)^{p^{m-1}},$$

so we get

$$(\pi_m^{p^k a})^{\epsilon/\psi(\sigma)-1} \equiv \text{a unit} \bmod (L_m^*)^{p^{m-1}}.$$

This can only happen if the left-hand side is trivial; that is, if the exponent of π_m is a multiple of p^{m-1}. Thus

$$p^k \left(\frac{\epsilon}{\psi}(\sigma) - 1 \right) \equiv 0 \bmod p^{m-1}.$$

Since σ is arbitrary, the definition of d implies that $k+d \geqslant m-1$, so $k \geqslant m-d-1$. Thus, when we take the p^{m-d-1}th root of α and adjoin this to L_m, we are taking the root of a unit. So $\rho_{\psi,m-1-d}$ arises as desired, by taking the p-power root of a unit. Now simply let $m \to \infty$. $\square$

It is possible to build mod-p^m representations that are extensions of 1 by ψ that do arise by taking p^{m-1}th roots of nonunits of L_m. The proof of Lemma 9 shows that such extensions, however, do *not* lift to characteristic zero when $\psi \neq \epsilon$.

LEMMA 10. *Let the hypotheses be as in Lemma 9, and let $\psi = \epsilon$. Then Kernel$(\rho_{\epsilon,m})$ fixes the splitting field of $x^{p^m} - a$ for some $a \in \mathbb{Z}_p$. This Galois representation corresponds to a finite flat group scheme over $\mathbb{Z}_p$ if and only if, up to p^mth powers, $a \in \mathbb{Z}_p^*$.*

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Proof. We only sketch the proof.

It is (again) well known that $H^1(G_p, \mathbb{Z}_p/(p^m)(\epsilon)) \simeq (\mathbb{Z}_p/(p^m))^2$. The representations associated to the splitting fields of $x^{p^m} - p$ and $x^{p^m} - (1+p)$ correspond to cohomology classes that form a basis for this module.

Since $* \equiv 0 \bmod p$ and $* \not\equiv 0 \bmod p^2$, we have that a is a pth power in $\mathbb{Z}_p$ but not a p^2th power.

If a is a unit, it is clear that the G_p-module corresponding to $\rho_{\epsilon,m}$ comes from a finite flat group scheme over $\mathbb{Z}_p$.

Using Fontaine–Lafaille theory, [7], one can count how many extensions of $\mathbb{Z}/(p^m)(\epsilon)$ by $\mathbb{Z}/(p^m)$ there are in the category of finite flat group schemes over $\mathbb{Z}_p$ up to isomorphism. One finds $m-1$ of them where the $*$ is as above. These correspond to $a = (1+p)^p, (1+p)^{p^2}, \ldots, (1+p)^{p^{m-1}}$, all of which are units. □

We need to determine the arbitrary-weight ordinary deformation rings for the various possibilities of $\bar{\rho}|_{G_p}$ and check that they are smooth of the correct dimension so that Proposition 12 holds. We also need to ensure that the sets of weight-2 points of each these ordinary rings contain the points of the corresponding weight-2 deformation rings. For $\bar{\rho}$ flat this leads to some minor technicalities in cases (3) and (5) below. In case (5) of Proposition 11 we see that the arbitrary-weight ordinary tangent space is too large and the corresponding ring is not smooth. We construct a specific smooth quotient of this ring in the correct number of variables. Its weight-2 points are flat. This is where the second bullet point at the beginning of Section 2 arises.

PROPOSITION 11. *Let $\bar{\eta} : G_p \to \mathbb{F}^*_{p^f}$ be a nontrivial unramified character. Set $h^0 = \dim H^0(G_p, Ad\,\bar{\rho})$. Up to twist we have the following possibilities for $\bar{\rho}|_{G_p}$ and its local deformation ring.*

(1) $\bar{\rho}|_{G_p} = \begin{pmatrix} \bar{\eta}\bar{\epsilon} & 0 \\ 0 & 1 \end{pmatrix}$. *Here,* $\dim \tilde{\mathcal{L}}_p = 4$, $h^0 = 2$, *and the ordinary deformation ring is smooth in four variables.*

(2) $\bar{\rho}|_{G_p} = \begin{pmatrix} \bar{\eta}\bar{\epsilon} & * \\ 0 & 1 \end{pmatrix}$. *Here,* $\dim \tilde{\mathcal{L}}_p = 3$, $h^0 = 1$, *and the ordinary deformation ring is smooth in three variables.*

(3) $\bar{\rho}|_{G_p} = \begin{pmatrix} \bar{\epsilon} & * \\ 0 & 1 \end{pmatrix}$ *is flat. Here,* $\dim \tilde{\mathcal{L}}_p = 3$, $h^0 = 1$, *and the ordinary deformation ring is smooth in three variables.*

(4) $\bar{\rho}|_{G_p} = \begin{pmatrix} \bar{\epsilon} & * \\ 0 & 1 \end{pmatrix}$ *is not flat. Here,* $\dim \tilde{\mathcal{L}}_p = 3$, $h^0 = 1$, *and the ordinary deformation ring is smooth in three variables.*

(5) $\bar{\rho}|_{G_p} = \begin{pmatrix} \bar{\epsilon} & 0 \\ 0 & 1 \end{pmatrix}$. *Here,* $\dim \tilde{\mathcal{L}}_p = 5$, *and the ordinary deformation ring is not smooth, but it has a smooth quotient in four variables whose characteristic-zero points include all points of weight $k > 2$ and all flat points of weight*

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$k = 2$. So we redefine $\tilde{\mathcal{L}}_p$ to be the four-dimensional subspace induced by this quotient and note that $h^0 = 2$.

Proof. Let $U \subset Ad\bar{\rho}$ be the upper triangular matrices. In each case we will compare the unrestricted (local) upper triangular deformation theory of $\bar{\rho}|_{G_p}$ to its (local) ordinary deformation theory.

(1) $\tilde{\mathcal{L}}_p$ includes the two unramified twists, the one ramified twist of $\bar{\eta}\bar{\epsilon}$, and the nontrivial extension of 1 by $\bar{\eta}\bar{\epsilon}$, so $\dim \tilde{\mathcal{L}}_p = 4$. One easily sees that $\dim H^2(G_p, U) = 0$ and $\dim H^1(G_p, U) = 5$, so the upper triangular deformation ring is smooth in five variables. Its ordinary quotient is formed by forcing the lower right entry to be unramified. This involves the one relation that comes from setting the ramified part of the lower right entry, when evaluated at a topological generator of the Galois group over $\mathbb{Q}_p$ of the cyclotomic extension, to be trivial. Since this relation necessarily cuts the tangent space down by one variable, we can take it to be a variable of the five-dimensional upper triangular ring, so the ordinary ring is smooth in four variables.

(2) One computes $\dim H^2(G_p, U) = 0$ and $\dim H^1(G_p, U) = 4$, so the upper triangular deformation ring is smooth in four variables. Let U^1 be the matrices of the form $\left(\begin{smallmatrix} a & b \\ 0 & -a \end{smallmatrix}\right)$. One computes $\dim H^1(G_p, U^1) = 2$, and from Table 3 of [13] we have $\dim \mathcal{L}_p = 1$. As $\mathcal{L}_p \subset H^1(G_p, U^1)$, any element of $H^1(G_p, U^1)$ not in $\mathcal{L}_p$ is ramified on both diagonals. A linear combination of this class and the ramified twist will be trivial on the lower right entry, and thus ordinary. Of course the unramified twist is in $\tilde{\mathcal{L}}_p$, so $\dim \tilde{\mathcal{L}}_p = 3$. That the ordinary ring is smooth in three variables follows from the argument in the proof of (1).

(3) That $\dim \tilde{\mathcal{L}}_p = 3$ follows from Proposition 13 of [15]. One easily sees that $\dim H^2(G_p, U) = 0$ and $\dim H^1(G_p, U) = 4$, so the upper triangular deformation ring is smooth in four variables. As before, its ordinary quotient involves one relation that forces the lower right entry to be unramified, which again implies that the ordinary ring is smooth in three variables.

(4) Then the short exact sequence $0 \to U^1 \to U \to U/U^1 \to 0$ and routine Galois cohomology computations give that $H^1(G_p, U^1) \to H^1(G_p, U)$ is an injection of a two-dimensional space into a four-dimensional space. The cokernel is spanned by the images of the ramified and unramified twists. There are two independent extensions of 1 by $\bar{\epsilon}$, so at least one dimension of $H^1(G_p, U^1)$ is ordinary.

If all of $H^1(G_p, U^1) \subset H^1(G_p, U)$ is ordinary then, taking into account the unramified twist, $\dim \tilde{\mathcal{L}}_p \geqslant 3$. The only way that $\dim \tilde{\mathcal{L}}_p = 4 = \dim H^1(G_p, U)$ is if the ramified twist belongs to $\tilde{\mathcal{L}}_p$, which we know does not happen. Thus $\dim \tilde{\mathcal{L}}_p = 3$ in this case.

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If only one dimension of $H^1(G_p, U^1)$ is ordinary (this is what actually happens, but proving it is messier than the weaker argument used here) then the same proof as in (2) implies that $\dim \tilde{\mathcal{L}}_p = 3$.

Since $\dim \tilde{\mathcal{L}}_p = 3$ in all cases and the upper triangular ring is smooth in four variables, the ordinary ring is smooth in three variables.

(5) This case is a bit more involved, as the ordinary ring is not smooth. First note that $\tilde{\mathcal{L}}_p$ contains the two unramified twists, the ramified twist of $\bar{\epsilon}$, and the *two* extensions of 1 by $\bar{\epsilon}$, and so $\dim \tilde{\mathcal{L}}_p = 5$. We will replace it by a four-dimensional subspace.

Let D be the maximal pro-p abelian extension of L, the composite of $\mathbb{Z}_p$-unramified extension of $\mathbb{Q}_p$ and $\mathbb{Q}_p(\mu_{p^\infty})$. Then, in this case, any ordinary deformation of $\bar{\rho}$ has meta-abelian image and factors through $Gal(D/\mathbb{Q}_p)$. By Kummer theory D is generated by p-power roots of elements of L. Let $K \subset D$ be the subfield generated by p-power roots of *units* of L.

$$\begin{tikzcd}
& D \arrow[d, dash] & \\
& K \arrow[d, dash] & \\
& L \arrow[dl, dash] \arrow[dr, dash] & \\
\mathbb{Q}_p^{nr} \arrow[dr, dash] & & \mathbb{Q}_p(\mu_{p^\infty}) \arrow[dl, dash] \\
& \mathbb{Q}_p &
\end{tikzcd}$$

Let $\psi_1, \psi_2 : G_p \to \mathbb{Z}_p^*$ be unramified characters, each congruent to 1 mod p.

We will consider a series of ring homomorphisms

$$R^{ord} \twoheadrightarrow R^{ord,unit} \twoheadrightarrow R_k^{ord,unit} \twoheadrightarrow R_k^{ord,unit,\psi_2} \twoheadrightarrow R_k^{ord,unit,\psi_1,\psi_2},$$

where the superscript 'unit' indicates the quotient of the ordinary ring whose deformation factors through $Gal(K/\mathbb{Q}_p)$, and the presence of the unramified character ψ_i as a superscript indicates that we are fixing ψ_i in the ii spot on the

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diagonal. The subscript k indicates the weight. So $R_k^{ord,unit,\psi_1,\psi_2}$ is the deformation ring parameterizing deformations of $\bar{\rho}|_{G_p}$ that factor through $Gal(K/\mathbb{Q}_p)$ and are of the form $\begin{pmatrix} \epsilon^{k-1}\tilde{\epsilon}^{2-k}\psi_1 & * \\ 0 & \psi_2 \end{pmatrix}$. For instance, the ring $R_k^{ord,unit}$ puts no restrictions on the unramified diagonal characters.

Consider $R_k^{ord,unit,\psi_1,\psi_2}$. The tangent space for this ring is one dimensional, as follows. No twists by characters on the diagonal are allowed, and the très ramifiée extension of 1 by $\bar{\epsilon}$ is not allowed either. Only the peu ramifiée extension of 1 by $\bar{\epsilon}$ is allowed. Thus the corresponding deformation ring is $\mathbb{Z}_p[[U]]/I_1$. If I_1 contains a nonzero element $g(U)$, then by the Weierstrass preparation theorem we can assume that $g(U) = p^r h(U)$, where $h(U)$ is a distinguished polynomial, or $h(U) \equiv 1$ or 0. But Lemma 9 gives the existence of nonsplit characteristic-zero deformations. Conjugating these by $\begin{pmatrix} p^m & 0 \\ 0 & 1 \end{pmatrix}$ gives *different deformations* of $\bar{\rho}$ for each m (though of course these representations are all isomorphic), so our ring has infinitely many characteristic-zero points, and $h(U)$ would have infinitely many roots, a contradiction. Thus $h(U) \equiv 0$, I_1 is trivial, and $R_k^{ord,unit,\psi_1,\psi_2} \simeq \mathbb{Z}_p[[U]]$.

The ring $R_k^{ord,unit,\psi_2}$ has two-dimensional tangent space (the unramified twist of $\bar{\epsilon}$ is now allowed), and so $R_k^{ord,unit,\psi_2} \simeq \mathbb{Z}_p[[U_1, U_2]]/I_2$. But, for each choice of ψ_1, we see that this ring has a different quotient isomorphic to $\mathbb{Z}_p[[U]]$. If $I_2 \neq (0)$ Krull's principal ideal theorem (see Corollary 11.18 of [1]) implies that $R_k^{ord,unit,\psi_2}$ has Krull dimension at most 2. Then a Noetherian ring of Krull dimension at most 2 has infinitely many components of Krull dimension 2, a contradiction. Thus $I_2 = 0$. Similarly, $R_k^{ord,unit,}$ has three-dimensional tangent space, and so $R_k^{ord,unit} \simeq \mathbb{Z}_p[[U_1, U_2, U_3]]/I_3$. But, for each choice of ψ_2, we see that this ring has a different quotient isomorphic to $\mathbb{Z}_p[[U_1, U_2]]$. If $I_3 \neq (0)$, the same Krull dimension argument as above gives a contradiction. Thus $I_3 = 0$. Finally, $R^{ord,unit}$ has four-dimensional tangent space, as only the très ramifiée extension of 1 by $\bar{\epsilon}$ is not allowed. We have, for each $k \geqslant 2$,

$$R^{ord,unit} \twoheadrightarrow R_k^{ord,unit} \simeq \mathbb{Z}_p[[U_1, U_2, U_3]],$$

so, arguing as before, $R^{ord,unit} \simeq \mathbb{Z}_p[[U_1, U_2, U_3, U_4]]$.

Using Lemma 9, we see that it remains to show that weight-2 flat deformations of $\bar{\rho}$ factor through $R^{ord,unit}$ in the $\psi_1 = \psi_2 = \psi$ case. This follows immediately from Lemma 10. $\square$

PROPOSITION 12. *For* $v \neq p$, $\dim \tilde{\mathcal{L}}_v = \dim H^0(G_v, Ad\bar{\rho})$ *and* $\dim \tilde{\mathcal{L}}_p = \dim H^0(G_p, Ad\bar{\rho}) + 2$.

Proof. For $v \neq p$, it is known that $\dim \mathcal{L}_v = \dim H^0(G_v, Ad^0\bar{\rho})$. As we switch from $Ad^0\bar{\rho}$ to $Ad\bar{\rho}$,

$$\dim H^0(G_v, Ad\bar{\rho}) - \dim H^0(G_v, Ad^0\bar{\rho}) = 1 = \dim \tilde{\mathcal{L}}_v - \dim \mathcal{L}_v.$$

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The $v = p$ result follows from Proposition 11. $\square$

Propositions 6 and 12 give, taking into account $v = \infty$,

COROLLARY 13. $\dim H^1_{\tilde{\mathcal{L}}}(G_S, Ad\bar{\rho}) - \dim H^1_{\tilde{\mathcal{L}}^\perp}(G_S, Ad\bar{\rho}^*) = 1$.

3.2. Local deformation rings at nice primes. Finally, we recall the notion of *nice* primes for a representation. The definition given below is a blend of that given in [14] (see Section 2 and Proposition 2.2) and that of [17] that is suited for our purposes.

DEFINITION 14. Let $\bar{\rho} : G_{\mathbb{Q}} \to GL_2(\mathbb{F}_{p^f})$ odd, ordinary, full, and weight 2, with trivial nebentype be given. Let R be a local Artin ring with residue field $\mathbb{F}_{p^f}$, and let $\rho_R : G_{\mathbb{Q}} \to GL_2(R)$ lift $\bar{\rho}$. The prime q is called ρ_R-*nice* if q is not ± 1 mod p, ρ_R is unramified at q, and $\rho_R(Fr_q)$ has eigenvalues q and 1 and order prime to p. We simply call q *nice* if it is $\bar{\rho}$-nice.

For nice primes q, the local at q deformation ring has a smooth quotient whose points $\mathcal{C}_q$ consist of Steinberg deformations. There is an induced subspace $\mathcal{L}_q \subset H^1(G_q, Ad^0\bar{\rho})$.

PROPOSITION 15. *Let ρ_R be odd, ordinary, weight 2, with full reduction $\bar{\rho}$. Then ρ_R-nice primes exist, and, for any nice prime q,* $\dim \mathcal{L}_q = \dim H^0(G_q, Ad^0\bar{\rho}) = 1$, *and a smooth quotient of the deformation ring exists with points $\mathcal{C}_q$. Proposition 12 applies for nice primes, so* $\dim \tilde{\mathcal{L}}_q = 2 = \dim H^0(G_q, Ad\bar{\rho})$, *and $\tilde{\mathcal{C}}_q$ consists of all unramified twists of points of $\mathcal{C}_q$.*

Proof. We are given that $\bar{\rho}$ is full and $\det\bar{\rho} = \bar{\epsilon}$, so, for $a \in \mathbb{F}_p$, $a \neq \pm 1$, choose $\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right) \in$ image$(\bar{\rho})$. Any prime q with Frobenius in the conjugacy class of this matrix will be nice. After lifting this matrix to an element of image(ρ_R) and raising to a large power of p (say p^r), the new matrix will have tame order and eigenvalues $\{a^{p^r}, 1\}$ which are distinct. Any prime with Frobenius in the conjugacy class of this element will be ρ_R-nice.

The cohomological results are standard, and we do not give their proofs. $\square$

4. Ordinary smooth deformation rings and ρ_n

For any finite set of primes $T \supset S$ with $T \backslash S$ consisting of only nice primes, we have an ordinary arbitrary-weight deformation ring denoted $R^{ord, T\text{-}new}$. This parameterizes global deformations that restricted to $v \in T$ lie in $\tilde{\mathcal{C}}_v$.

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DEFINITION 16. The weight-2 quotient $R_2^{ord,T\text{-}new}$ of $R^{ord,T\text{-}new}$ parameterizes deformation that when restricted to I_p are of the form $\left(\begin{smallmatrix} \epsilon & * \\ 0 & 1 \end{smallmatrix}\right)$, except in the case when $\bar{\rho}|_{G_p}$ itself is of the form $\left(\begin{smallmatrix} \epsilon & * \\ 0 & 1 \end{smallmatrix}\right)$ and finite flat. We then either consider (i) deformations that are finite flat, or (ii) when $\bar{\rho}|_{G_p}$ is not split, also semistable deformations of weight 2, that is of the form $\left(\begin{smallmatrix} \epsilon & * \\ 0 & 1 \end{smallmatrix}\right)$.

We remark that in previous papers we used the notation $T\backslash S_0$-new to indicate that all nice primes were in the level of the modular form. Since it is less cumbersome, we use T-new here instead. Results toward Theorem 17 (cf. Theorem 1 of introduction) below are proved in [3].

THEOREM 17. *Suppose that $\rho_n : G_{\mathbb{Q}} \to GL_2(\mathcal{O}/(\pi^n))$ is odd, ordinary, weight 2, modular, has full image and determinant ϵ, and is balanced. Then there exists a finite set of primes $T \supseteq S$ such that the universal ordinary 'new at T' ring $R^{ord,T\text{-}new} \simeq W(\mathbb{F}_{p^f})[[U]]$, and there are surjections*

$$R^{ord,T\text{-}new} \twoheadrightarrow R_2^{ord,T\text{-}new} \twoheadrightarrow \mathcal{O}/(\pi^n)$$

from this ring to its weight-2 quotient and then to $\mathcal{O}/(\pi^n)$ inducing ρ_n.

It is not a consequence of Theorem 1 that ρ_n lifts to a *T-new* weight-2 characteristic-zero representation. For instance, if $n = 3$ and $\mathcal{O} = \mathbb{Z}_p[\sqrt{p}]$, it is possible that

$$R_2^{ord,T\text{-}new} \simeq \mathbb{Z}_p[[U]]/\left((U-p)(U-2p)(U-3p)\right),$$

and ρ_3 arises from $U \mapsto \sqrt{p}$. The smoothness of $R^{ord,T\text{-}new}$ immediately implies the existence of characteristic-zero lifts, but these lifts may not have classical weight, let alone weight 2. Theorem 2 addresses this. Indeed, Theorem 3 and Lemma 33 ensure that the above example occurs!

4.1. Group theoretic lemmas.

We need the following lemma of Boston (see [2]) and Lemma 19 for Lemma 20.

LEMMA 18. *(Boston) Let $p \geqslant 3$. Let R be a complete local Noetherian ring with residue characteristic p. Let $\rho : G \to GL_2(R)$ be a representation, and assume that the image of the projection*

$$\rho_2 : G \to GL_2(R) \to GL_2\left(R/m_R^2\right)$$

is full, that is, it contains $SL_2\left(R/m_R^2\right)$. Then the image of ρ contains $SL_2(R)$.

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LEMMA 19. *Let $p \geqslant 5$ and $G \subset GL_2(\mathbb{F}_{p^f})$ be a full subgroup; that is, $SL_2(\mathbb{F}_{p^f}) \subset G$. Assume also when $\mathbb{F}_{p^f} = \mathbb{F}_5$ that $G = GL_2(\mathbb{F}_5)$. Then $H^1(G, Ad^0\bar{\rho}) = 0$.*

Proof. This is Lemma 2.48 of [4], except in the case when $\mathbb{F}_{p^f} = \mathbb{F}_5$ and $G = GL_2(\mathbb{F}_5)$. The latter case is covered in Lemma 1.2 of [6]. □

LEMMA 20. *Let $G \subset GL_2(\mathcal{O}/(\pi^r))$ be a subgroup. Suppose that the hypotheses of Lemma 19 are satisfied for the image of $G \to GL_2(\mathbb{F}_{p^f})$, and that the image of the projection $p_2 : G \to GL_2(\mathcal{O}/(\pi^2))$ is full. Then* $\dim H^1(G, Ad^0\bar{\rho}) = 1$.

Proof. Since the image of p_2 is full, the hypothesis of Lemma 18 is satisfied, so $G \supset SL_2(\mathcal{O}/(\pi^r))$.

Let Γ be the kernel of the projection $G \to GL_2(\mathbb{F}_{p^f})$. We have the exact inflation-restriction sequence

$$0 \to H^1(G/\Gamma, Ad^0\bar{\rho}^{\Gamma}) \to H^1(G, Ad^0\bar{\rho}) \to H^1(\Gamma, Ad^0\bar{\rho})^{G/\Gamma}.$$

As G/Γ is the image of the projection $p_1 : G \to GL_2(\mathbb{F}_{p^f})$, Lemma 19 implies that the first term is trivial.

Also, as Γ acts trivially on $Ad^0\bar{\rho}$,

$$H^1(\Gamma, Ad^0\bar{\rho})^{G/\Gamma} = Hom_{G/\Gamma}(\Gamma, Ad^0\bar{\rho}).$$

For any $\gamma \in Hom_{G/\Gamma}(\Gamma, Ad^0\bar{\rho})$, Kernel$(\gamma) \supset \Gamma'$, the commutator subgroup of Γ.

Set $a = 1 + \pi$ and $r = \pi x/(2 + \pi)$, and use fullness to see that

$$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}^{-1} \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & \pi^2 x \\ 0 & 1 \end{pmatrix} \in \Gamma' \subset \text{Kernel}(\gamma).$$

Similarly,

$$\begin{pmatrix} 1 & 0 \\ \pi^2 y & 1 \end{pmatrix} \in \Gamma' \subset \text{Kernel}(\gamma).$$

As Kernel(γ) is stable under the action of G/Γ,

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & \pi^2 z \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 - \pi^2 z & \pi^2 z \\ -\pi^2 z & 1 + \pi^2 z \end{pmatrix} \subset \text{Kernel}(\gamma).$$

Multiplying on the left and right by suitable matrices

$$\begin{pmatrix} 1 & 0 \\ \pi^2 y & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & \pi^2 x \\ 0 & 1 \end{pmatrix},$$

we have

$$\begin{pmatrix} 1+\pi^2 z & 0 \\ 0 & (1+\pi^2 z)^{-1} \end{pmatrix} \in \mathrm{Kernel}(\gamma).$$

As every element of

$$\Gamma_2 := \{A \in SL_2(\mathcal{O}/(\pi^r)) \mid A \equiv I \bmod (\pi^2)\}$$

can be written as a product

$$A = \begin{pmatrix} 1 & 0 \\ \pi^2 y & 1 \end{pmatrix} \begin{pmatrix} 1+\pi^2 z & 0 \\ 0 & (1+\pi^2 z)^{-1} \end{pmatrix} \begin{pmatrix} 1 & \pi^2 x \\ 0 & 1 \end{pmatrix},$$

we have $\mathrm{Kernel}(\gamma) \supset \Gamma_2$. Since $\Gamma/\Gamma_2 \simeq Ad^0\bar{\rho}$,

$$\begin{aligned} \dim Hom_{G/\Gamma}(\Gamma, Ad^0\bar{\rho}) &= \dim Hom_{G/\Gamma}(\Gamma/\Gamma_2, Ad^0\bar{\rho}) \\ &= \dim Hom_{G/\Gamma}(Ad^0\bar{\rho}, Ad^0\bar{\rho}) = 1, \end{aligned}$$

so $\dim H^1(G, Ad^0\bar{\rho}) \leqslant 1$. As $\mathcal{O}/\mathbb{Z}_p$ is ramified, $GL_2\left(\mathcal{O}/(\pi^2)\right) \simeq GL_2\left(\mathbb{F}_{p^f}[U]/(U^2)\right)$ is nontrivial as we are given full image, so $\dim H^1(G, Ad^0\bar{\rho}) = 1$. □

4.2. Selmer groups and cohomological results. We gather the necessary cohomological results we need to prove Theorems 1 and 2. For Theorem 1 we need Lemma 24 (which depends on Proposition 21) and Proposition 23. Proposition 22 is needed for Theorem 2. All sets of primes Z below will be finite, and contain S, and $Z\backslash S$ will consist of nice primes. Recall the local conditions $\mathcal{L}_v$ and $\tilde{\mathcal{L}}_v$ from Definition 8.

PROPOSITION 21. *Let $h \in H^1_{\tilde{\mathcal{L}}}(G_Z, Ad\bar{\rho})$ and $\phi \in H^1_{\tilde{\mathcal{L}}^\perp}(G_S, Ad\bar{\rho}^*)$. If $h \in H^1(G_Z, \mathbb{F}_{p^f}) \subset H^1(G_Z, Ad\bar{\rho})$ then $h = 0$. If $\phi \in H^1(G_Z, \mathbb{F}_{p^f}(1)) \subset H^1(G_Z, Ad\bar{\rho}^*)$ then $\phi = 0$.*

Proof. If $h \in H^1(G_Z, \mathbb{F}_{p^f})$, it corresponds, when viewed as a lift to the dual numbers, to a twist by a character which gives rise to a $\mathbb{Z}/(p)$-extension of $\mathbb{Q}$. Definition 8 implies that $\tilde{\mathcal{L}}_v \cap H^1(G_v, \mathbb{F}_{p^f})$ is spanned, for all v, by the $\mathbb{F}_{p^f}$-valued unramified twists, so the corresponding global extension is unramified everywhere over $\mathbb{Q}$, so $h = 0$.

Set $M = \mathbb{F}_{p^f}$, and for all v set

$$\mathcal{M}_v = \tilde{\mathcal{L}}_v \cap H^1(G_v, \mathbb{F}_{p^f}) = H^1_{nr}(G_v, \mathbb{F}_{p^f}).$$

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We just showed that $H^1_{\mathcal{M}}(G_Z, M) = 0$. As $\dim \mathcal{M}_v = \dim H^0(G_v, M) = 1$ for $v \neq \infty$, Proposition 6 gives $\dim H^1_{\mathcal{M}^\perp}(G_Z, M^*) = 0$ as well. It suffices to show that $\phi \in H^1_{\mathcal{M}^\perp}(G_Z, M^*)$.

Any $\phi \in H^1(G_Z, \mathbb{F}_{p^f}(1)) \cap H^1_{\tilde{\mathcal{L}}^\perp}(G_Z, Ad\bar{\rho}^*)$ cuts out an extension $L/\mathbb{Q}(\mu_p)$ that is Galois over $\mathbb{Q}$, and $Gal(\mathbb{Q}(\mu_p)/\mathbb{Q})$ acts on $Gal(L/\mathbb{Q}(\mu_p))$ by $\bar{\epsilon}$. At $v \neq p$, unramified cohomologies are exact annihilators under the local duality pairing, so $\tilde{\mathcal{L}}_v^\perp \cap H^1(G_v, \mathbb{F}_{p^f}(1)) = H^1_{nr}(G_v, \mathbb{F}_{p^f}(1))$. (This last group is trivial unless $v \equiv 1 \mod p$.) So, for $v \neq p$, $L/\mathbb{Q}(\mu_p)$ is unramified at v.

For $v = p$, choose a subspace $V \subset H^1(G_p, Ad\bar{\rho})$ such that

$$\tilde{\mathcal{L}}_p = (\mathcal{L}_p \oplus \mathcal{M}_p) + V, \quad V \cap (\mathcal{L}_p \oplus \mathcal{M}_p) = 0,$$

so

$$\tilde{\mathcal{L}}_p^\perp = (\mathcal{L}_p \oplus \mathcal{M}_p)^\perp \cap V^\perp = (\mathcal{L}_p^\perp \oplus \mathcal{M}_p^\perp) \cap V^\perp,$$

and thus

$$\tilde{\mathcal{L}}_p^\perp \cap H^1(G_p, \mathbb{F}_{p^f}(1)) \subset \mathcal{M}_p^\perp.$$

Thus $\phi|_{G_v} \in \mathcal{M}_v^\perp$ for all v; that is, $\phi \in H^1_{\mathcal{M}^\perp}(G_Z, M^*)$, which we already proved is trivial, so $\phi = 0$. $\square$

Note that, for $h \in H^1(G_Z, Ad^0\bar{\rho})$ and $q \notin Z$ nice, $h|_{G_q} \neq 0$ is equivalent to $h|_{G_q} \notin \mathcal{L}_q$. Similarly, for $h \in H^1(G_Z, Ad\bar{\rho})$, write $h = h_{Ad^0\bar{\rho}} + h_{sc}$ with $h_{Ad^0\bar{\rho}} \in H^1(G_Z, Ad^0\bar{\rho})$ and $h_{sc} \in H^1(G_Z, \mathbb{F}_{p^f})$. For $q \notin Z$ nice, $h|_{G_q} \notin \tilde{\mathcal{L}}_q$ is equivalent to $h_{Ad^0\bar{\rho}}|_{G_q} \notin \mathcal{L}_q$, which we just saw is equivalent to $h_{Ad^0\bar{\rho}}|_{G_q} \neq 0$.

Recall that $Ш^1_Z(M)$ is the kernel of the restriction map $H^1(G_Z, M) \to \oplus_{v \in Z} H^1(G_v, M)$.

Proposition 22. *Let $h \in H^1_{\mathcal{L}}(G_Z, Ad^0\bar{\rho})$, $\phi \in H^1_{\mathcal{L}^\perp}(G_Z, Ad^0\bar{\rho}^*)$, and let $q \notin Z$ be nice.*

(1) *The injective inflation map*

$$H^1(G_Z, Ad^0\bar{\rho}) \to H^1(G_{Z\cup\{q\}}, Ad^0\bar{\rho})$$

has codimension 0 or 1. If $Ш^1_Z(Ad^0\bar{\rho}^*)|_{G_q} = 0$, then the codimension is 1.

(2) *If $\phi|_{G_q} \neq 0$, then the maps*

$$H^1(G_{Z\cup\{q\}}, Ad^0\bar{\rho}) \to \bigoplus_{v\in Z} \frac{H^1(G_v, Ad^0\bar{\rho})}{\mathcal{L}_v} \quad \text{and} \quad H^1(G_Z, Ad^0\bar{\rho}) \to \bigoplus_{v\in Z} \frac{H^1(G_v, Ad^0\bar{\rho})}{\mathcal{L}_v}$$

have the same kernel.

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(3) *If* $h, \phi|_{G_q} \neq 0$ *then*

$$\dim H^1_{\mathcal{L}}(G_{Z\cup\{q\}}, Ad^0\bar{\rho}) = H^1_{\mathcal{L}}(G_Z, Ad^0\bar{\rho}) - 1,$$
$$\dim H^1_{\mathcal{L}^\perp}(G_{Z\cup\{q\}}, Ad^0\bar{\rho}^*) = \dim H^1_{\mathcal{L}^\perp}(G_Z, Ad^0\bar{\rho}^*) - 1.$$

(4) *If* $H^1_{\mathcal{L}}(G_Z, Ad^0\bar{\rho})|_{G_q} = 0$, $\phi|_{G_q} \neq 0$, *then*

$$H^1_{\mathcal{L}}(G_{Z\cup\{q\}}, Ad^0\bar{\rho}) = H^1_{\mathcal{L}}(G_Z, Ad^0\bar{\rho}),$$
$$\dim H^1_{\mathcal{L}^\perp}(G_{Z\cup\{q\}}, Ad^0\bar{\rho}^*) = \dim H^1_{\mathcal{L}^\perp}(G_Z, Ad^0\bar{\rho}^*).$$

(5) *If* $H^1(G_Z, Ad^0\bar{\rho}^*)|_{G_q} = 0$, *then*

$$H^1(G_{Z\cup\{q\}}, Ad^0\bar{\rho}) \to \bigoplus_{v\in Z} \frac{H^1(G_v, Ad^0\bar{\rho})}{\mathcal{L}_v}$$

and

$$H^1(G_Z, Ad^0\bar{\rho}) \to \bigoplus_{v\in Z} \frac{H^1(G_v, Ad^0\bar{\rho})}{\mathcal{L}_v}$$

have the same image.

Proof. As the proofs of all parts are similar, we only prove part (2). We use the normal local Selmer condition for $v \in Z$, but just for this proof we set $\mathcal{L}_q = H^1(G_q, Ad^0\bar{\rho})$, so $\mathcal{L}_q^\perp = 0$. We apply Proposition 6 with the sets Z and $Z \cup \{q\}$. Then

$$\dim H^1_{\mathcal{L}}(G_{Z\cup\{q\}}, Ad^0\bar{\rho}) - \dim H^1_{\mathcal{L}^\perp}(G_{Z\cup\{q\}}, Ad^0\bar{\rho}^*)$$
$$= \dim H^1_{\mathcal{L}}(G_Z, Ad^0\bar{\rho}) - \dim H^1_{\mathcal{L}^\perp}(G_Z, Ad^0\bar{\rho}^*) + 1.$$

As $\mathcal{L}_q^\perp = 0$, we have $H^1_{\mathcal{L}^\perp}(G_{Z\cup\{q\}}, Ad^0\bar{\rho}^*) \subset H^1_{\mathcal{L}^\perp}(G_Z, Ad^0\bar{\rho}^*)$, and since $\phi|_{G_q} \neq 0$ this containment is proper. As the dual Selmer goes down by 1 in dimension as we switch from Z to $Z \cup \{q\}$, the above equation implies that the dimension of the Selmer does not change as we switch from Z to $Z \cup \{q\}$. Since $\mathcal{L}_q = H^1(G_q, Ad^0\bar{\rho})$ we have $H^1_{\mathcal{L}}(G_Z, Ad^0\bar{\rho}) \subset H^1_{\mathcal{L}}(G_{Z\cup\{q\}}, Ad^0\bar{\rho})$, and the result follows. □

PROPOSITION 23. *Let* $h \in H^1_{\tilde{\mathcal{L}}}(G_Z, Ad\bar{\rho})$, $\phi \in H^1_{\tilde{\mathcal{L}}^\perp}(G_Z, Ad\bar{\rho}^*)$, *and let* q *be nice.*

(1) *The injective inflation map*

$$H^1(G_Z, Ad\bar{\rho}) \to H^1(G_{Z\cup\{q\}}, Ad\bar{\rho})$$

has codimension 0 *or* 1. *If* $Ш^1(Ad\bar{\rho}^*)|_{G_q} = 0$, *then the codimension is* 1.

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(2) *If* $h|_{G_q} \notin \tilde{\mathcal{L}}_q$ *and* $\phi|_{G_q} \neq 0$, *then*

$$\dim H^1_{\tilde{\mathcal{L}}}(G_{Z\cup\{q\}}, Ad\bar{\rho}) = H^1_{\tilde{\mathcal{L}}}(G_Z, Ad\bar{\rho}) - 1,$$
$$\dim H^1_{\tilde{\mathcal{L}}^\perp}(G_{Z\cup\{q\}}, Ad\bar{\rho}^*) = \dim H^1_{\tilde{\mathcal{L}}^\perp}(G_Z, Ad\bar{\rho}^*) - 1.$$

The proof of Proposition 23 is similar to that of Proposition 22, and is not included. See Section 1, particularly Lemma 1.2, of [17] for the proof of (2).

Consider the deformation to the dual numbers given by

$$G_{\mathbb{Q}} \xrightarrow{\rho_n} GL_2\left(\mathcal{O}/(\pi^n)\right) \to GL_2\left(\mathcal{O}/(\pi^2)\right) \simeq GL_2(\mathbb{F}_{p^f}[U]/(U^2)).$$

Then $f \in H^1(G_{\mathbb{Q}}, Ad^0\bar{\rho})$ corresponds to this composite representation, and the fullness assumption implies that $f \neq 0$. As the determinant of the above composite representation is $\bar{\epsilon}$, $f \in H^1_{\mathcal{L}}(G_S, Ad^0\bar{\rho}) \subset H^1_{\tilde{\mathcal{L}}}(G_S, Ad\bar{\rho})$; that is, f lives in the trace-zero cohomology. This is important, as in the end f will span the tangent space of our arbitrary-weight ordinary ring *and* the tangent space of its weight-2 quotient. By Corollary 13 we may take $\{\phi_1, \ldots, \phi_s\}$ and $\{h_1, \ldots, h_s, f\}$ as bases of $H^1_{\tilde{\mathcal{L}}^\perp}(G_S, Ad\bar{\rho}^*)$ and $H^1_{\tilde{\mathcal{L}}}(G_S, Ad\bar{\rho})$.

LEMMA 24. *For* $i \in \{1, 2, \ldots, s\}$, *let* Q_i *be the set of nice primes such that, for* $q_i \in Q_i$,

- $\phi_i|_{G_{q_i}} \neq 0$ *(equivalently,* $\phi_{i,Ad^0\bar{\rho}^*}|_{G_{q_i}} \neq 0$*),*
- $h_{i,Ad^0\bar{\rho}}|_{G_{q_i}} \neq 0$ *and* $h_{i,sc}|_{G_{q_i}} = 0$,
- *for* $j \neq i$, $\phi_j, h_j|_{G_{q_i}} = 0$, *and*
- q_i *is* ρ_n*-nice; that is,* $\rho_n(Fr_{q_i}) = \left(\begin{smallmatrix} q_i & 0 \\ 0 & 1 \end{smallmatrix}\right)$, *where this element has order prime to* p.

Then Q_i *is nonempty.*

Proof. It suffices to show that the conditions above are independent Chebotarev conditions; that is, they determine linearly disjoint extensions over $K := \mathbb{Q}(\bar{\rho})$, and thus can be simultaneously satisfied.

Each of the cohomology classes above, when restricted to the absolute Galois group of the field K, becomes an element of $Hom_{Gal(K/\mathbb{Q})}(G_K, M)$ for $M = Ad\bar{\rho}$ or $Ad\bar{\rho}^*$. For $M = Ad^0\bar{\rho}$ or $Ad^0\bar{\rho}^*$, the independence of the first three conditions has been established in [13] and [17]. The case of full adjoint cohomology results from these works and Proposition 21 as follows. Write $\phi_i = \phi_{i,Ad^0\bar{\rho}^*} + \phi_{i,\mathbb{F}_{p^f}(1)}$, where $\phi_{i,Ad^0\bar{\rho}^*} \in H^1(G_S, Ad^0\bar{\rho}^*)$ and $\phi_{i,\mathbb{F}_{p^f}(1)} \in H^1(G_S, \mathbb{F}_{p^f}(1))$. We

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claim that the set $\{\phi_{1,Ad^0\bar{\rho}^*}, \ldots, \phi_{s,Ad^0\bar{\rho}^*}\}$ is independent. Indeed, suppose that $\sum_{j=1}^{s} a_j\phi_{j,Ad^0\bar{\rho}^*} = 0$ is a dependence relation. Then

$$\begin{aligned}\sum_{j=1}^{s} a_j\phi_j &= \sum_{j=1}^{s} a_j(\phi_{j,Ad^0\bar{\rho}^*} + \phi_{j,\mathbb{F}_{p^f}(1)})\\ &= \sum_{j=1}^{s} a_j\phi_{j,\mathbb{F}_{p^f}(1)} \in H^1_{\mathcal{L}^\perp}(G_S, Ad^0\bar{\rho}^*) \cap H^1(G_S, \mu_p),\end{aligned}$$

which is 0 by Proposition 21, a contradiction. Let L be the composite of the fields fixed by the kernels of $\phi_i|_{G_K}$. Then $Gal(L/K)$ contains, when viewed as a $\mathbb{F}_{p^f}[Gal(K/\mathbb{Q})]$-module, s copies of $Ad^0\bar{\rho}^*$ by [12], [17]. A similar argument gives that the composite of the fields fixed by the kernels of $h_i|_{G_K}$ and $f|_{G_K}$ contains $s+1$ copies of $Ad^0\bar{\rho}$. This reduces the independence of the first three conditions to the same question with $Ad^0\bar{\rho}$ and $Ad^0\bar{\rho}^*$ cohomology where it is known.

The fourth condition is a complete splitting condition from K to L_n, the field fixed by the kernel of ρ_n. The Jordan–Hölder components of $Gal(L_n/K)$ are $\mathbb{F}_{p^f}[Gal(K/\mathbb{Q})]$-submodules of $Ad\bar{\rho}$ that are either $Ad\bar{\rho}$ or $Ad^0\bar{\rho}$. As the fields fixed by the kernels of the $\phi_i|_{G_K}$ give $Ad\bar{\rho}^*$ (or $Ad^0\bar{\rho}^*$) extensions, these are linearly disjoint over K from L_n. The fields fixed by the kernels of the $h_i|_{G_K}$ give rise to $Ad\bar{\rho}$ (or $Ad^0\bar{\rho}$) extensions of K. If the composite of these fields intersects L_n nontrivially, then, as this intersection is abelian over K, the proof of Lemma 20 applied to ρ_n implies that this composite contains Kernel$(f|_{G_K})$ and f is in the span of the trace zero parts of $\{h_{1,Ad^0\bar{\rho}}, \ldots, h_{s,Ad^0\bar{\rho}}\}$. Proposition 21 then implies that f is in the span of $\{h_1, \ldots, h_s\}$, a contradiction. Thus the composite of the fields fixed by the h_i is linearly disjoint over K from L_n. □

4.3. Proof of Theorem 1.

We prove Theorem 1 in this section. In the case when $n = 1$ so $\rho_1 = \bar{\rho}$, this was proved by Lundell in [11]. Dealing with $n > 1$ is what requires the more complicated parts (3) and (5) of Proposition 11.

Proof of Theorem 1. Choose $q_i \in Q_i$ of Lemma 24, and set $Q = \{q_1, \ldots, q_s\}$ and $T = S \cup Q$. Part (2) of Proposition 23 implies that the Selmer and dual Selmer groups decrease in dimension by 1 for each q_i at which we allow ramification. Thus $H^1_{\tilde{\mathcal{L}}^\perp}(G_T, Ad\bar{\rho}^*) = 0$, and $H^1_{\tilde{\mathcal{L}}}(G_T, Ad\bar{\rho})$ is spanned by f, so $R^{ord,T\text{-}new} \simeq W(\mathbb{F}_{p^f})_p[[U]]$. Since we assume that $\bar{\rho}$ is modular and absolutely irreducible, [5] implies that $R^{ord,T\text{-}new}$ has characteristic-zero points in all classical weights. Thus it is in fact a Hida family. By the fourth condition on the $q_i \in Q_i$ we see that ρ_n arises as a point of $R_2^{ord,T\text{-}new}$, the weight-2 quotient of $R^{ord,T\text{-}new}$. □

5. ρ_n lifts to an $\mathcal{O}$-valued weight 2 point

Let $\rho_n : G_{\mathbb{Q}} \to GL_2(\mathcal{O}/(\pi^n))$ be as in Theorem 1. Suppose also that $\rho_n|_{G_v} \in \mathcal{C}_v$ for all $v \in S$. Recall that for $v = p$ we require that, if $\bar{\rho} = \left(\begin{smallmatrix} \bar{\epsilon} & 0 \\ 0 & 1 \end{smallmatrix}\right)$, then $\mathcal{C}_p$ is taken to be the flat deformations, so we *must* assume that $\rho_n|_{G_p}$ is flat.

If $\bar{\rho}|_{G_p} = \left(\begin{smallmatrix} \bar{\epsilon} & * \\ 0 & 1 \end{smallmatrix}\right)$ is flat and ρ_n is flat (but not semistable), we take $\mathcal{C}_p$ to be the flat deformations. If ρ_n is not flat but semistable, we take $\mathcal{C}_p$ to be the semistable deformations. The universal representations corresponding to the deformation rings considered in Theorem 2 are locally at p of type $\mathcal{C}_p$. We consider accordingly the appropriate ring $R_2^{ord,T\text{-}new}$ as in Definition 16.

The goal of this section is to prove Theorem 25 (cf. Theorem 2 of the introduction).

THEOREM 25. *Let ρ_n be as in Theorem 1. There exists a finite set of primes $T \supseteq S$ such that*

$$R_2^{ord,T\text{-}new} \simeq \mathcal{O} \twoheadrightarrow \mathcal{O}/(\pi^n)$$

inducing ρ_n.

Let $j(U)$ be the determinant of the universal ordinary representation (arising from $R^{ord,X\text{-}new}$ for a given X) evaluated at a topological generator of the Galois group of the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$ so that, at a weight t point $U_t \in \mathfrak{m}_{\mathbb{Z}_p}$, $j(U_t) = (1+p)^{t-1}$. (Note that $j(U)$ is independent of X.) Write $j(U) = 1 + p + g(U)$.

LEMMA 26. *The modularity of $\bar{\rho}$ implies that we may assume that $g(U) \in W(\mathbb{F}_{p^f})[[U]]$ is a distinguished polynomial.*

Proof. The Weierstrass preparation theorem implies that $g(U) = p^t v_2(U)u(U)$, where v_2 is a distinguished polynomial and $u(U)$ is a unit. We need to prove that $t = 0$, so suppose that $t \geqslant 1$.

By [5], $R^{ord,X\text{-}new} \simeq W(\mathbb{F}_{p^f})[[U]]$ has a point of each classical weight $k \geqslant 2$. The weight-3 quotient is formed imposing the relation $j(U) = (1+p)^2$, that is, quotienting out by

$$\begin{aligned} j(U) - (1+p)^2 = g(U) - p - p^2 &= p^t v_2(U)u(U) - p - p^2 \\ &= p\left[-1 - p + p^{t-1}v_2(U)u(U)\right]. \end{aligned}$$

As there is at least one weight-2 point in this Hida family, $v_2(U)$ has positive degree, so the rightmost quantity above is p times a unit, and is never 0 for any choice of U. Thus if $t \geqslant 1$ there are no weight-3 points, a contradiction. □

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Consider the weight-2 quotient $R_2^{ord,X\text{-}new}$ of $R^{ord,X\text{-}new}$. In all cases except (3) and (5) of Proposition 11, we will be considering the quotient by $g(U)$, and in the remaining cases by distinguished polynomial that divides $g(U)$ whose roots correspond to lifts that are either semistable or flat of weight 2 as per the case being considered. For uniformity of notation, we denote by $w_2(U)$ a distinguished polynomial that generates the kernel of the map $R^{ord,X\text{-}new} \to R_2^{ord,X\text{-}new}$. (*A priori* the roots of the $w_2(U)$ in the flat and semistable case could share a root, but geometricity and the Weil bounds imply that this is not the case, though we do not need this here.) Henceforth we will write $R^{ord,X\text{-}new}/(w_2(U))$ to indicate the weight-2 quotient with which we are dealing (the full weight-2 quotient, except in cases (3) and (5) of Proposition 11) and will control to be $\mathcal{O}$.

5.1. Recollection of earlier work in [14]. Let T be as in Theorem 1. A key technical ingredient in this section is the main lifting result of [14], which in turn builds on [10]. The point of [14] was to build a pathological Galois representation by removing all obstructions to deformation problems. Here we repeat this procedure for a *finite number of steps*, but then we *introduce* an obstruction later to force $R_2^{ord,T_2\text{-}new}$ to be 'close to' a specified ring. This closeness will allow us to choose $R_2^{ord,T_2\text{-}new}$ to be isomorphic to a given totally ramified extension of $W(\mathbb{F}_{p^f})$.

We recall some of the key ingredients of [14]. Again, we consider $\mathbb{F}_{p^f}$ with $q = p^f$ and $p \geqslant 5$. First consider the hypotheses of Section 4 of [14].

- Fullness of the image of $\bar{\rho}$, which we assume here.
- Triviality of

$$\text{III}^1_T(Ad^0\bar{\rho}^*) := \text{Kernel}\left(H^1(G_T, Ad^0\bar{\rho}^*) \to \oplus_{v\in T} H^1(G_v, Ad^0\bar{\rho}^*)\right)$$

  can be realized as follows. Note that, for $T \subset Z$, $\text{III}^1_Z(Ad^0\bar{\rho}^*) \subset \text{III}^1_T(Ad^0\bar{\rho}^*)$. Then let $\{\theta_1, \ldots, \theta_t\}$ be a basis for $\text{III}^1_T(Ad^0\bar{\rho}^*)$ and choose nice primes q_i such that the following hold.

  - q_i is ρ_n-nice. Recall that this is a complete splitting condition on q_i in $Gal(L_n/\mathbb{Q}(\bar{\rho}, \mu_p))$.
  - $H^1(G_T, Ad\bar{\rho})|_{G_{q_i}} = 0$.
  - $\theta_i|_{G_{q_i}} \neq 0$ and $j \neq i \implies \theta_j|_{G_{q_i}} = 0$.

  Replacing T by $T \cup \{q_1, \ldots, q_t\}$ (which we rename T) gives $\text{III}^1_T(Ad^0\bar{\rho}^*) = 0$.

- The third hypothesis of Section 4 of [14] was that the local deformations be specified uniquely. This was equivalent to specifying $W(\mathbb{F}_{p^f})$ as our smooth

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quotient of each local deformation ring. We simply ignore that here, and use $\mathcal{C}_v$ as our (weight-2) set of local points as usual.

Let $T \subset Z$, and suppose that we have an ordinary weight-2 deformation of $\bar{\rho}$, $\rho_R : G_Z \to GL_2(R)$, where R is a finite complete local Noetherian ring with residue field $\mathbb{F}_{p^f}$ and $\rho_R|_{G_v} \in \mathcal{C}_v$ for all $v \in Z$. Let S and S' be such rings with

$$S \twoheadrightarrow S' \overset{\delta}{\twoheadrightarrow} R$$

small surjections; that is, the kernels are principal ideals killed by the maximal ideal of the source ring and thus isomorphic to $\mathbb{F}_{p^f}$. It is natural to ask whether ρ_R deforms to a $\rho_{S'} : G_Z \to GL_2(S')$ of weight 2. The obstruction lies in $H^2(G_Z, Ad^0\bar{\rho})$. As $\text{Ш}^1_Z(Ad^0\bar{\rho}^*) \subset \text{Ш}^1_T(Ad^0\bar{\rho}^*) = 0$ and $\text{Ш}^1_Z(Ad^0\bar{\rho}^*)$ is dual to $\text{Ш}^2_Z(Ad^0\bar{\rho})$, this obstruction is realized locally. But as $\rho_R|_{G_v} \in \mathcal{C}_v$ for all $v \in Z$ and the $\mathcal{C}_v$ represent the points of a smooth ring, there are no local obstructions and $\rho_{S'}$ exists. It may be, however, that there are $v_0 \in Z$ with $\rho_{S'}|_{G_{v_0}} \notin \mathcal{C}_{v_0}$. In this case deforming to S may not be possible. The smoothness of the local deformation rings implies that $\rho_R|_{G_v}$ has a deformation to $GL_2(S')$ arising from $\mathcal{C}_v$ for all $v \in Z$. The obstruction to deforming $\rho_{S'}|_{G_{v_0}}$ to S can be removed by a cohomology class $z_{v_0} \in H^1(G_{v_0}, Ad^0\bar{\rho})$. We call the collection $(z_v)_{v\in Z}$ the *local condition problem* for $\rho_{S'}$. Proposition 3.4 of [14] shows that there exists a $\rho_{S'}$-nice prime q, and $h \in H^1(G_{Z\cup\{q\}}, Ad^0\bar{\rho})$, which solves the local condition problem above; that is, $(I + h)\rho_{S'}|_{G_v} \in \mathcal{C}_v$ for all $v \in Z$. The difficulty is that we cannot guarantee that $(I + h)\rho_{S'}|_{G_q} \in \mathcal{C}_q$. If this fails for all $\rho_{S'}$-nice primes, Proposition 3.6 of [14] shows how to add two nice primes q_1 and q_2 to Z and find a cohomology class $h \in H^1(G_{Z\cup\{q_1,q_2\}}, Ad^0\bar{\rho})$ such that $(I + h)\rho_{S'}|_{G_v} \in \mathcal{C}_v$ for all $v \in Z \cup \{q_1, q_2\}$.

5.2. Strategy of the proof of Theorem 2. We use the integer N throughout this section to denote a large natural number. This largeness will depend only on $\rho_n : G_{\mathbb{Q}} \to GL_2(\mathcal{O}/(\pi^n))$. We explain the strategy, which gives an indication of how N is chosen. The first step is to construct a deformation problem where the arbitrary-weight ordinary deformation ring will be $W(\mathbb{F}_{p^f})[[U]]$ and its weight-2 quotient will surject onto $W(\mathbb{F}_{p^f})[[U]]/(p^N, U^{Ne})$, which in turn will surject onto $\mathcal{O}/(\pi^n)$ and give rise to ρ_n. So

$$\begin{aligned} W(\mathbb{F}_{p^f})[[U]] \simeq R^{ord} &\twoheadrightarrow R_2^{ord} \\ &= W(\mathbb{F}_{p^f})[[U]]/(w_{2,N}(U)) \\ &\twoheadrightarrow W(\mathbb{F}_{p^f})[[U]]/(p^N, U^{Ne}) \twoheadrightarrow \mathcal{O}/(\pi^n). \end{aligned}$$

If in this composite U maps to an element of $\mathcal{O}/(\pi^n)$ whose various lifts to $\mathcal{O}$ have valuation greater than $1/e$, then the deformation to $\mathcal{O}/(\pi^2)$ would be trivial,

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contradicting the fullness of ρ_n. Thus U maps to an element whose lifts to $\mathcal{O}$ are uniformizers. After multiplying by a unit, we may assume that $U \mapsto \pi$. Our strategy is to then alter the problem by allowing more ramification so that $w_{2,N}(U)$ is exactly of degree e and 'close to' $g_\pi(U)$, the minimal polynomial of π over $W(\mathbb{F}_{p^f})$. The choice of N will depend on $|\pi^n|$ and the Krasner bound on the distances between roots of $g_\pi(U)$. Furthermore, $w_{2,N}$ has a root $y_{N,1}$ such that the deformation given by $U \mapsto y_{N,1}$ gives rise to ρ_n. Thus ρ_n will have a weight-2 characteristic-zero lifting.

5.3. Weight-2 deformation rings that are large. In this section we will construct large weight-2 deformation rings that give rise to ρ_n. The technical hypotheses on $\rho_n|_{G_p}$ in the introduction arise here.

PROPOSITION 27. *For any integer $N \geqslant n$, there exists a set $X_N \supseteq T$ such that*

$$R_2^{ord, X_N\text{-}new} \twoheadrightarrow W(\mathbb{F}_{p^f})[[U]]/(p^N, U^{Ne}) \twoheadrightarrow \mathcal{O}/(\pi^n).$$

Proof. Theorem 1.1 of [14], based on techniques of [10], gives examples of weight-2 deformation rings that are arbitrarily large and ramified at infinitely many primes. It is proved by taking an inverse limit of certain quotients of deformation rings that are local Artin rings and satisfy a specified property at $v \in S_0$, namely the local representation at G_v is (the reduction of) a specific deformation of $\bar{\rho}|_{G_v}$ to $\mathbb{Z}_p$. While the ring $W(\mathbb{F}_{p^f})[[U]]/(p^N, U^{Ne})$ is not explicitly included there, the techniques apply. The deformation of Theorem 1 to $\mathcal{O}/(\pi^n)$ factors

$$W(\mathbb{F}_{p^f})[[U]] = R^{ord, T\text{-}new} \twoheadrightarrow R_2^{ord, T\text{-}new} = W(\mathbb{F}_{p^f})[[U]]/(w_2(U)) \twoheadrightarrow \mathcal{O}/(\pi^n) \tag{5.1}$$

and $U \mapsto \pi$ in the composite. Let $g_\pi(U)$ be the minimal polynomial of π over $W(\mathbb{F}_{p^f})$. As $U \mapsto \pi$ gives an isomorphism $W(\mathbb{F}_{p^f})[[U]]/(g_\pi(U), U^n) \simeq \mathcal{O}/(\pi^n)$, the kernel of (5.1) is $(g_\pi(U), U^n)$, so $w_2(U) \in (g_\pi(U), U^n)$. For $N \geqslant n$, note that $p^N, U^{Ne} \in (g_\pi(U), U^n)$, as they both map to 0 in (5.1).

We will construct a ring $R = W(\mathbb{F}_{p^f})[[U]]/I$ (not yet a deformation ring!) that surjects onto $W(\mathbb{F}_{p^f})[[U]]/(p^N, U^{Ne})$ and from there onto $W(\mathbb{F}_{p^f})[[U]]/(g_\pi(U), U^n) \simeq \mathcal{O}/(\pi^n)$. We will build R as a series of small extensions of $W(\mathbb{F}_{p^f})[[U]]/(g_\pi(U), U^n)$ and then invoke Proposition 3.6 of [14] to realize all of these rings as quotients of weight-2 deformation rings. Consider the map

$$W(\mathbb{F}_{p^f})[[U]]/(pg_\pi(U), Ug_\pi(U), U^n) \twoheadrightarrow W(\mathbb{F}_{p^f})[[U]]/(g_\pi(U), U^n).$$

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The kernel is just the ideal $(g_\pi(U))$, and this is killed by (p, U), the maximal ideal of $W(\mathbb{F}_{p^f})[[U]]$, so the extension is small. Similarly, the map

$$W(\mathbb{F}_{p^f})[[U]]/(pg_\pi(U), Ug_\pi(U), pU^n, U^{n+1}) \\ \twoheadrightarrow W(\mathbb{F}_{p^f})[[U]]/(pg_\pi(U), Ug_\pi(U), U^n)$$

has kernel (U^n), and this is also killed by (p, U). Repeat this process (with more and more elements in our ideal) until all the generators are of the form $p^r U^s g_\pi(U)$ or $p^r U^s U^n$, where $r+s = N+Ne$. Let I be this ideal of relations. In each relation either $r \geqslant N$ or $s \geqslant Ne$, so $I \subset (p^N, U^{Ne}) \subset (g_\pi(U), U^n)$. Then

$$W(\mathbb{F}_{p^f})[[U]]/I \twoheadrightarrow \cdots \twoheadrightarrow W(\mathbb{F}_{p^f})[[U]]/(g_\pi(U), U^n) \simeq \mathcal{O}/(\pi^n)$$

is a series of small extensions.

At this point we use the technique of [14] to deform ρ_n to each small extension, perhaps allowing ramification at one or two nice primes at each step. If we can deform ρ_n all the way to $W(\mathbb{F}_{p^f})[[U]]/I$ without allowing more ramification, then we are done. If not, there is a first place at which the small deformation problem is obstructed. This is not at ρ_n, as $\text{III}^2_T(Ad^0\bar{\rho}) = 0$ (being dual to $\text{III}^1_T(Ad^0\bar{\rho}^*)$), and the local deformation problems ρ_n are assumed unobstructed. The smoothness of the chosen quotients of the local deformation rings implies that there are local cohomology classes $(h_v)_{v\in T}$ that 'unobstruct' each of the given local deformation problems. Proposition 3.4 of [14] implies that with one nice prime q the local deformation problems at $v \in T$ can be 'unobstructed' by a global class in $H^1(G_{T\cup\{q\}}, Ad^0\bar{\rho})$, but possibly this class introduces an obstruction at q. If all nice primes introduce such an obstruction, the rest of Section 3 of [14] shows how to allow ramification at two nice primes $\{q_1, q_2\}$ so that the obstruction introduced at these primes cancel one another. Then we deform and move on to the next small extension. Set X_N to be the final set of nice primes. □

Recall the cohomology class f that gives rise to $\rho_n \mod (\pi^2)$. The primes q used in Proposition 27 were ρ_n-nice, so $f|_{G_q} = 0$ and $f \in H^1_{\mathcal{L}}(G_{X_N}, Ad^0\bar{\rho}) \subset H^1_{\tilde{\mathcal{L}}}(G_{X_N}, Ad\bar{\rho})$, but this last space could have dimension >1, so the first step in our strategy is not yet complete. Lemma 28 remedies this.

LEMMA 28. *There exists a set Y_N containing X_N of Proposition 27 such that* $\dim H^1_{\tilde{\mathcal{L}}^\perp}(G_{Y_N}, Ad\bar{\rho}^*) = 0$ *and* $\dim H^1_{\tilde{\mathcal{L}}}(G_{Y_N}, Ad\bar{\rho}) = 1$, *so* $R^{ord, Y_N\text{-}new} \simeq W(\mathbb{F}_{p^f})[[U]]$. *Furthermore,*

$$\begin{aligned} W(\mathbb{F}_{p^f})[[U]] \simeq R^{ord,Y_N\text{-}new} &\twoheadrightarrow R_2^{ord,Y_N\text{-}new} \\ &= W(\mathbb{F}_{p^f})[[U]]/(w_{2,N}(U)) \twoheadrightarrow W(\mathbb{F}_{p^f})[[U]]/(p^N, U^{Ne}) \\ &\twoheadrightarrow \mathcal{O}/(\pi^n). \end{aligned} \tag{5.2}$$

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Proof. The proof is similar to that of Lemma 24. Let $\rho_N : G_{X_N} \to GL_2\left(W(\mathbb{F}_{p^f})[[U]]/(p^N, U^{Ne})\right)$ be the deformation of Proposition 27. We will need that the image of ρ_N is full.

The ring $W(\mathbb{F}_{p^f})[[U]]/(p^N, U^{Ne})$ has the quotient $W(\mathbb{F}_{p^f})/(p^2)$, and there is the extension

$$1 \to Ad\bar{\rho} \to GL_2\left(W(\mathbb{F}_{p^f})/(p^2)\right) \to GL_2(\mathbb{F}_{p^f}) \to 1, \tag{5.3}$$

which is well known to be nonsplit for $p \geqslant 5$. The cohomology class f also gives rise to the split extension

$$1 \to Ad^0\bar{\rho} \to GL_2\left(\mathbb{F}_{p^f}[\epsilon]/(\epsilon^2)\right) \to GL_2(\mathbb{F}_{p^f}) \to 1. \tag{5.4}$$

Thus there are distinct extensions $K_1/\mathbb{Q}(\bar{\rho})$ and $K_2/\mathbb{Q}(\bar{\rho})$ with $Gal(K_i/\mathbb{Q}(\bar{\rho})) \simeq Ad^0\bar{\rho}$ as $\mathbb{F}_{p^f}[Gal(\mathbb{Q}(\bar{\rho})/\mathbb{Q}]$-modules for $i = 1, 2$. This gives the fullness of image of $\rho_{R_2^{ord,Y_N\text{-}new}} \mod (p, U)^2$, the deformation to $GL_2(W(\mathbb{F}_{p^f})[[U]]/(p, U)^2)$. Lemma 18 implies that $\rho_{R_2^{ord,Y_N\text{-}new}} \mod (p^N, U^{Ne})$ has full image.

Now take $\{\phi_1, \ldots, \phi_s\}$ and $\{h_1, \ldots, h_s, f\}$ as bases for $H^1_{\tilde{\mathcal{L}}^\perp}(G_{X_N}, Ad\bar{\rho}^*)$ and $H^1_{\tilde{\mathcal{L}}}(G_{X_N}, Ad\bar{\rho})$. As before, $f \in H^1_{\mathcal{L}}(G_{X_N}, Ad^0\bar{\rho})$ is the cohomology class arising from $\rho_n \mod (\pi^2)$. Let Q_i be the Chebotarev set of primes q_i satisfying

- $\phi_i|_{G_{q_i}} \neq 0$ (equivalently $\phi_{i,Ad^0\bar{\rho}^*}|_{G_{q_i}} \neq 0$) ,
- $h_i|_{G_{q_i}} \notin \tilde{\mathcal{L}}_{q_i}$ (equivalently $h_{i,Ad^0\bar{\rho}}|_{G_{q_i}} \neq 0$) and $h_{i,sc}|_{G_{q_i}} = 0$,
- for $j \neq i$, $\phi_j, h_j|_{G_{q_i}} = 0$, and
- q_i is ρ_N-nice; that is, $\rho_N(Fr_{q_i}) = \left(\begin{smallmatrix} q_i & 0 \\ 0 & 1 \end{smallmatrix}\right)$, where this element has order prime to p.

Setting

$$\Gamma = \{A \in \mathrm{Image}(\rho_N) \mid A \equiv I \mod (p, U)\}$$

and using the fullness of ρ_N established above, one can easily adapt the proof of Lemma 20 to show that $\dim H^1(\mathrm{Image}(\rho_N), Ad^0\bar{\rho}) = 1$. One then modifies Lemma 24 to show the above bullet points are independent Chebotarev conditions.

Let $q_i \in Q_i$ and set $Y_N = X_N \cup \{q_1, \ldots q_s\}$; then, by part (2) of Proposition 23, $\dim H^1_{\tilde{\mathcal{L}}^\perp}(G_{Y_N}, Ad\bar{\rho}^*) = 0$ and $\dim H^1_{\tilde{\mathcal{L}}}(G_{Y_N}, Ad\bar{\rho}) = 1$, and this last group has basis $\{f\}$. The ordinary ring $R^{ord,Y_N\text{-}new} \simeq W(\mathbb{F}_{p^f})[[U]]$. The fourth condition on the Q_i implies that the deformation ρ_N arises from the weight-2 quotient of $R^{ord,Y_N\text{-}new}$, so $R_2^{ord,Y_N\text{-}new} \twoheadrightarrow W(\mathbb{F}_{p^f})[[U]]/(p^N, U^{Ne})$. $\square$

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The second surjection of (5.2) implies that $w_{2,N}(U) \in (p^N, U^{Ne})$, and Lemma 26 implies that $w_{2,N}(U)$ is a distinguished polynomial. It thus has degree at least Ne.

5.4. Cutting down the size of weight-2 deformation rings via local obstructions. Our next step is to add more nice primes of ramification so that the new weight-2 ordinary ring is a quotient of $W(\mathbb{F}_{p^f})[[U]]$ by a polynomial $v_{2,N}(U)$ of degree *exactly* e. Furthermore, $v_{2,N}(U)$ will have a root $y_{N,1}$ such that $U \mapsto y_{N,1}$ gives rise to ρ_n.

Let C be a positive number smaller than both $|\pi^n|$ and the minimum of half the distances between any pairs of roots of g_π, its Krasner bound.

Recall that $U \mapsto \pi$ in

$$G_{Y_N} \xrightarrow{\rho_{R^{ord,Y_N\text{-}new}}} GL_2(W(\mathbb{F}_{p^f})[[U]]) \to GL_2\left(W(\mathbb{F}_{p^f})[[U]]/(p^N, U^{Ne})\right) \\ \to GL_2\left(\mathcal{O}/(\pi^n)\right).$$

Denote by $\rho_{g_\pi,N}$ the deformation

$$G_{Y_N} \to GL_2\left(R^{ord,Y_N\text{-}new} = W(\mathbb{F}_{p^f})[[U]]\right) \to GL_2\left(R_2^{ord,Y_N\text{-}new}\right) \\ \to GL_2\left(W(\mathbb{F}_{p^f})[[U]]/(p^N, U^{Ne})\right) \\ \to GL_2\left(W(\mathbb{F}_{p^f})[[U]]/(p^N, g_\pi(U), U^{Ne})\right).$$

Let $\rho_{p,k}$ be the reduction of the deformation $G_{Y_N} \xrightarrow{\rho_{R^{ord,Y_N\text{-}new}}} GL_2(W(\mathbb{F}_{p^f})[[U]]/(p^N, U^{Ne}))$ mod (p, U^k).

Note that

- $\dim H^1_{\tilde{\mathcal{L}}}(G_{Y_N}, Ad\bar{\rho}) = 1$, and this space has basis $\{f\}$,
- $\dim H^1_{\mathcal{L}}(G_{Y_N}, Ad^0\bar{\rho}) = 1$, and this space has basis $\{f\}$,
- $\dim H^1_{\tilde{\mathcal{L}}^\perp}(G_{Y_N}, Ad\bar{\rho}^*) = 0$, and
- $\dim H^1_{\mathcal{L}^\perp}(G_{Y_N}, Ad^0\bar{\rho}^*) = 1$, and this space has some basis, say $\{\phi\}$.

Let Q be the set of primes q satisfying

- $H^1(G_{Y_N}, Ad^0\bar{\rho})|_{G_q} = 0$,
- q is $\rho_{g_\pi,N}$-nice,
- $\phi|_{G_q} \neq 0$, and

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- $\rho_{p,e+1}(Fr_q) = \begin{pmatrix} q(1+U^e) & 0 \\ 0 & 1-U^e \end{pmatrix}$.

PROPOSITION 29. *The Chebotarev conditions defining Q above are independent for $p \geqslant 5$.*

Proof. The first two conditions are complete splitting conditions in fields above $\mathbb{Q}(\bar{\rho})$ and can therefore be satisfied simultaneously. They are both $Ad\,\bar{\rho}$ conditions and thus are independent of the third condition, an $Ad\,\bar{\rho}^*$ condition. It remains to show the independence of the fourth condition from the previous three. Since it is a succession of $Ad\,\bar{\rho}$ conditions, we only have to check independence with the first two conditions and, since $e > 1$, independence with the first condition follows from Lemma 20.

Finally we show the independence of the fourth and second conditions. Let $K_{g_{\pi,N}}$, K_e and K_{e+1} be the fields fixed by the kernels of $\rho_{g_{\pi,N}}$, $\rho_{p,e}$ and $\rho_{p,e+1}$, respectively.

$$\begin{tikzcd}
K_{g_{\pi,N}} & & K_{e+1} \\
& K_e \arrow[ul, dash] \arrow[ur, dash] & \\
& \mathbb{Q}(\bar{\rho}) \arrow[u, dash] & \\
& \mathbb{Q} \arrow[u, dash] &
\end{tikzcd}$$

As $g_\pi(U)$ is distinguished of degree e, $K_{g_{\pi,N}} \supset K_e$. We show that $K_{g_{\pi,N}} \not\supseteq K_{e+1}$. The same argument given in the proof of Lemma 28 implies that $\rho_{R_2^{ord,Y_N}\text{-new}}$ mod $(p, U)^2$ has full image. Lemma 18 then implies that the image of $\rho_{R_2^{ord,Y_N}\text{-new}}$ mod (p^N, U^{Ne}) contains $\begin{pmatrix} 1+g_\pi(U) & 0 \\ 0 & (1+g_\pi(U))^{-1} \end{pmatrix}$. When we reduce mod $(p^N, g_\pi(U), U^{Ne})$ to $\rho_{g_{\pi,N}}$, this element becomes trivial. But when we reduce mod (p, U^{e+1}) to $\rho_{p,e+1}$, bearing in mind that $g_\pi(U) \equiv U^e \bmod p$, the image is $\begin{pmatrix} 1+U^e & 0 \\ 0 & 1-U^e \end{pmatrix}$. So $K_{g_{\pi,N}} \not\supseteq K_{e+1}$. Thus $K_{g_{\pi,N}}$ and K_{e+1} are linearly disjoint over K_e. The second condition is a complete splitting condition in $K_{g_{\pi,N}}$, while the fourth is a complete splitting condition in K_e, but *not* in K_{e+1}. $\square$

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5.5. Proof of Theorem 2. Choose $q_1 \in Q$. Part (4) of Proposition 22, using the first and third bullet points on q_1, implies that

$$\dim H^1_{\mathcal{L}}(G_{Y_N \cup \{q_1\}}, Ad^0 \bar{\rho}) = 1 = \dim H^1_{\mathcal{L}^\perp}(G_{Y_N \cup \{q_1\}}, Ad^0 \bar{\rho}^*),$$

and $H^1_{\mathcal{L}}(G_{Y_N \cup \{q_1\}}, Ad^0 \bar{\rho})$ is spanned by $\{f\}$ and $H^1_{\mathcal{L}^\perp}(G_{Y \cup \{q_1\}}, Ad^0 \bar{\rho}^*)$ is spanned by some $\{\tilde{\phi}\}$ ramified at q_1. Thus $R_{2^*}^{ord, Y_N \cup \{q_1\}\text{-}new}$ is a quotient of $W(\mathbb{F}_{p^f})[[U]]$ with one-dimensional tangent space.

By part (1) of Proposition 23 there are two possibilities:

(1) $\dim H^1_{\tilde{\mathcal{L}}}(G_{Y_N \cup \{q_1\}}, Ad \bar{\rho}) = 1$, or

(2) $\dim H^1_{\tilde{\mathcal{L}}}(G_{Y_N \cup \{q_1\}}, Ad \bar{\rho}) = 2$.

In case (1), $R^{ord, Y_N \cup \{q_1\}\text{-}new} \simeq W(\mathbb{F}_{p^f})[[U]]$, and its weight-$2^*$ quotient is formed by quotienting by the one determinant relation $v_{2^*,N}(U)$, which we can assume is a distinguished polynomial by Lemma 26. In case (2), it is possible that $R_{2^*}^{ord, Y_N \cup \{q_1\}\text{-}new}$ is a quotient of $W(\mathbb{F}_{p^f})[[U]]$ by either multiple relations or that it might not be finite and flat over $W(\mathbb{F}_{p^f})$. We will deal with this case by adding another prime of ramification.

While each $q_1 \in Q$ puts us in one of the two cases above, it is an open (and difficult!) question to determine if both cases do occur. The length of the argument below is due to this phenomenon.

5.5.1. Case (1). In the first case, we have deformations associated to the ring homomorphisms

$$\begin{aligned} W(\mathbb{F}_{p^f})[[U]] &\simeq R^{ord, Y_N \cup \{q_1\}\text{-}new} \twoheadrightarrow R_2^{ord, Y_N \cup \{q_1\}\text{-}new} \\ &\simeq W(\mathbb{F}_{p^f})[[U]]/\left(v_{2,N}(U)\right) \twoheadrightarrow W(\mathbb{F}_{p^f})[[U]]/\left(p^N, g_\pi(U), U^{Ne}\right). \end{aligned}$$

The last surjection above implies that $v_{2,N}(U) \in (p^N, g_\pi(U), U^{Ne})$, so its degree is at least e. We claim that it is exactly e.

If the degree is greater than e, then $R_2^{ord, Y_N \cup \{q_1\}\text{-}new} \twoheadrightarrow \mathbb{F}_{p^f}[[U]]/(U^{e+1})$. Call the corresponding deformation α, and let β be the deformation induced by the composite

$$R_2^{ord, Y_N\text{-}new} \twoheadrightarrow W(\mathbb{F}_{p^f})[[U]]/(p^N, U^{Ne}) \twoheadrightarrow \mathbb{F}_{p^f}[[U]]/(U^{Ne}) \twoheadrightarrow \mathbb{F}_{p^f}[[U]]/(U^{e+1}).$$

Note that $\alpha|_{G_{q_1}} \in \mathcal{C}_{q_1}$ and

$$\beta(Fr_{q_1}) = \begin{pmatrix} q_1(1+U^e) & 0 \\ 0 & 1-U^e \end{pmatrix} \implies \beta|_{G_{q_1}} \notin \mathcal{C}_{q_1};$$

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that is, β is not Steinberg at q_1. As both α and β are deformations of $\rho_{g_{\pi,N}}$ mod p to $GL_2\left(\mathbb{F}_{p^f}[[U]]/(U^{e+1})\right)$, they differ by a 1-cohomology class $k \in H^1(G_{Y\cup\{q_1\}}, Ad^0\bar{\rho})$; that is,

$$\alpha = (I + U^e k)\beta.$$

If k is unramified at q_1, then k inflates from $H^1(G_{Y_N}, Ad^0\bar{\rho})$. But q_1 was chosen so that $H^1(G_{Y_N}, Ad^0\bar{\rho})|_{G_{q_1}} = 0$. Thus k cannot change the local at q_1 deformation where $\beta(Fr_{q_1}) = \left(\begin{smallmatrix} q_1(1+U^e) & 0 \\ 0 & 1-U^e \end{smallmatrix}\right)$ to one in $\mathcal{C}_{q_1}$. So k is ramified at q_1. But we chose q_1 such that $\phi|_{G_{q_1}} \neq 0$, where ϕ spanned $H^1_{\mathcal{L}^\perp}(G_{Y_N}, Ad^0\bar{\rho}^*)$. Parts (1) and (2) of Proposition 22 then imply that the map

$$H^1(G_{Y_N\cup\{q_1\}}, Ad^0\bar{\rho}) \to \bigoplus_{v\in Y_N} \frac{H^1(G_v, Ad^0\bar{\rho})}{\mathcal{L}_v}$$

has image one dimension larger than the map

$$H^1(G_{Y_N}, Ad^0\bar{\rho}) \to \bigoplus_{v\in Y_N} \frac{H^1(G_v, Ad^0\bar{\rho})}{\mathcal{L}_v}.$$

For all $v \in Y_N$ we have that $\alpha|_{G_v}$ belongs to our deformable class $\mathcal{C}_v$, as does $\beta|_{G_v}$. But for at least one $v \in Y_N$ we have $k_v|_{G_v} \notin \mathcal{L}_v$, so $\alpha|_{G_v} = (I + U^e k)\beta|_{G_v} \notin \mathcal{C}_v$, a contradiction. Thus k can be neither ramified nor unramified at q_1. This contradiction implies that $v_{2,N}(U)$ has degree e.

Recall that $v_{2,N}(U) \in (p^N, g_\pi(U), U^{Ne})$, so

$$v_{2,N}(U) = a(U)p^N + b(U)g_\pi(U) + c(U)U^{Ne}. \tag{5.5}$$

As $g_\pi(U)$ is the minimal polynomial over $W(\mathbb{F}_{p^f})$ of π, its roots are distinct. Since both $g_\pi(U)$ and $v_{2,N}(U)$ are degree e, $b(U)$ is a unit. Let $\{y_{N,1}, y_{N,2}, \ldots, y_{N,e}\}$ be the roots of $v_{2,N}(U)$. As $v_{2,N}(U)$ is distinguished of degree e, $v_p(y_{N,i}) \geqslant 1/e$. Observe that

$$0 = v_{2,N}(y_{N,i}) = p^N a(y_{N,i}) + b(y_{N,i})g_\pi(y_{N,i}) + c(y_{N,i})y_{N,i}^{Ne}.$$

The outside terms on the right have valuation at least N and $b(y_{N,i})$ is a unit, so $v_p(g_\pi(y_{N,i})) \geqslant N$. Thus $y_{N,i}$ is very close to a root of $g_\pi(U)$. For N large enough, this closeness is closer than the common Krasner bound C on the roots of $g_\pi(U)$. We claim that each $y_{N,i}$ is close to a different root of $g_\pi(U)$. If this were false, then a root of g_π would be missed; that is, there would be a root x_0 of $g_\pi(U)$ with $|x_0 - y_{N,i}| > C$ for all i. As $v_{2,N}(U) = \prod(U - y_{N,i})$, we would have $|v_{2,N}(x_0)| > C^e$. Evaluating (5.5) at x_0 gives $|v_{2,N}(x_0)| < p^{-N}$, a contradiction for large N, so the claim is true. After relabeling, we may assume that $y_{N,1}$ is close to π.

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The composite deformations corresponding to

$$W(\mathbb{F}_{p^f})[[U]] \simeq R^{ord,Y_N\cup\{q_1\}\text{-}new} \twoheadrightarrow R_2^{ord,Y_N\cup\{q_1\}\text{-}new} \\ \twoheadrightarrow W(\mathbb{F}_{p^f})[[U]]/(p^N, g_\pi(U), U^{Ne})$$

and

$$W(\mathbb{F}_{p^f})[[U]] \simeq R^{ord,Y_N\text{-}new} \twoheadrightarrow R_2^{ord,Y_N\text{-}new} \twoheadrightarrow W(\mathbb{F}_{p^f})[[U]]/(p^N, g_\pi(U), U^{Ne})$$

are the same, as the latter is nice at q_1. We know $U \mapsto \pi$ in the latter to give ρ_n, so sending U to π in the former gives ρ_n as well. As $y_{N,1}$ is close enough to π, Krasner's lemma implies that $W(\mathbb{F}_{p^f})[\frac{1}{p}](\pi) \subset W(\mathbb{F}_{p^f})[\frac{1}{p}](y_{N,1})$. As $[W(\mathbb{F}_{p^f})[\frac{1}{p}](y_{N,1}) : W(\mathbb{F}_{p^f})[\frac{1}{p}]] \leqslant \deg(v_{2,N}(U)) = e$, the fields $W(\mathbb{F}_{p^f})[\frac{1}{p}](y_{N,1})$ and $W(\mathbb{F}_{p^f})[\frac{1}{p}](\pi)$ are equal. Recall that C is smaller than both $|\pi^n|$ and the Krasner bound on the roots of $g_\pi(U)$. We chose N large enough so that $|y_{N,1} - \pi| < C < |\pi^n|$, so sending U to $y_{N,1}$ in the former sequence gives ρ_n as well. As $R_2^{ord,Y_N\cup\{q_1\}\text{-}new} \simeq W(\mathbb{F}_{p^f})[[U]]/(v_{2,N}(U))$, we see that ρ_n lifts to an $\mathcal{O}$-valued weight-2 Galois representation. This proves Theorem 2 in the case where we assumed that $\dim H^1_{\mathcal{L}}(G_{Y_N\cup\{q_1\}}, Ad\bar{\rho}) = 1$ which implied that $R^{ord,Y_N\cup\{q_1\}\text{-}new} \simeq W(\mathbb{F}_{p^f})[[U]]$. In this case we set $T_2 = T \cup \{q_1\}$.

5.5.2. Case (2). This case can be dealt with by a more intricate purely Galois cohomological argument that takes two pages and allows ramification at yet another prime q_2 to reduce the question to the point where we can cite case (1). Alternatively, if we allow ourselves standard '$R = \mathbb{T}$' theorems (the only place we do so in this paper), we know that $R_2^{Y_N\cup\{q_1\}}$ is a finite flat complete intersection and thus isomorphic to $W(\mathbb{F}_{p^f})[[U]]/(v_2(U))$ for a distinguished polynomial $v_2(U) \in (p^N, g_\pi(U), U^{Ne})$. Now we just note that the argument in case (1) used only the mod-p reductions of the weight-2 rings $R_2^{ord,Y_N\cup\{q_1\}\text{-}new}$ and $R_2^{ord,Y_N\text{-}new}$, so we can proceed as we did there.

5.6. Proof of Theorem 3.

COROLLARY 30. *Let $\bar{\rho} : G_{\mathbb{Q}} \to GL_2(\mathbb{F}_{p^f})$ be odd, full, ordinary, weight 2, and have determinant ϵ. Let $\mathcal{O}$ be any totally ramified extension of $W(\mathbb{F}_{p^f})$. There exists a set of primes $Y \supset S_0$ such that*

$$W(\mathbb{F}_{p^f})[[U]] \simeq R^{ord,Y} \twoheadrightarrow R_2^{ord,Y\text{-}new} \simeq W(\mathbb{F}_{p^f})[[U]]/(h(U)) \simeq \mathcal{O}.$$

The degree of the map to weight space along the Hida family $R^{ord,Y\text{-}new}$ is $[\mathcal{O} : W(\mathbb{F}_{p^f})]$ when $\bar{\rho}$ is as in cases (1), (2), and (4) of Proposition 11. In the other

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cases, the degree is strictly greater than $[\mathcal{O} : W(\mathbb{F}_{p^f})]$. *There exists a weight-2 form associated to* $\bar{\rho}$ *whose completed field of Fourier coefficients has ring of integers* $\mathcal{O}$.

Proof. Use [14] to get a nontrivial weight-2 deformation of $\bar{\rho}$ to

$$W(\mathbb{F}_{p^f})[[U]]/(p, U^2) \simeq \mathcal{O}/(\pi^2);$$

that is, the corresponding cohomology class in this deformation to the dual numbers is nonzero. Now apply Theorem 2. □

THEOREM 31. *Let* $\bar{\rho} : G_{\mathbb{Q}} \to GL_2(\mathbb{F}_{p^f})$ *be odd, full, ordinary, weight* 2, *and have determinant* ϵ. *Let* $g(U) \in W(\mathbb{F}_{p^f})[U]$ *be a distinguished polynomial of degree* e *with distinct roots, and let* $\varepsilon > 0$ *be given. Then there exists a set* $Y \supset S_0$ *such that*

$$W(\mathbb{F}_{p^f})[[U]] \simeq R^{ord,Y\text{-}new} \twoheadrightarrow R_2^{ord,Y\text{-}new} \simeq W(\mathbb{F}_{p^f})[[U]]/(w_2(U)),$$

where $w_2(U)$ *has degree* e *and each root of* $w_2(U)$ *is within* ε *of a root of* $g(U)$. *Furthermore, if* $g(U) = \prod g_i(U)$ *where* $g_i(U)$ *is irreducible over* $W(\mathbb{F}_{p^f})$ *of degree* e_i, *then* $w_2(U) = \prod w_{2,i}(U)$, *where* $w_{2,i}(U)$ *is irreducible over* $W(\mathbb{F}_{p^f})$ *of degree* e_i, *and its roots are within* ε *of the roots of* $g_i(U)$.

Proof. First choose ε to be less than half the distance between any pair of roots of $g(U)$. Use [14] to get a nontrivial weight-2 deformation of $\bar{\rho}$ to $W(\mathbb{F}_{p^f})[[U]]/(p, U^2)$. Now proceed as in Proposition 27 to get a weight-2 deformation ring surjecting onto $W(\mathbb{F}_{p^f})[[U]]/(p^N, U^{Ne})$ with one-dimensional tangent space. Then add more primes so that the ordinary ring is $W(\mathbb{F}_{p^f})[[U]]$ and its weight-2 quotient is $W(\mathbb{F}_{p^f})[[U]]/(w_2(U))$, where $w_2(U) \in (p^N, g(U), U^{Ne})$ is degree e. The argument in case (1) of Theorem 2 implies that, for N large enough, each root of $w_2(U)$ is within ε of a distinct root of $g(U)$. By the choice of ε, the roots of $w_2(U)$ are distinct. As $g_i(U)$ is a degree-e_i irreducible factor of $g(U)$, let $\{r_{i1}, \ldots, r_{ie_i}\}$ be its roots, and we know that $|r_{ij} - s_{ij}| < \epsilon$, where s_{ij} is a root of $w_2(U)$. We write $r_{ij} = s_{ij} + x_{ij}$. Let σ be an automorphism taking r_{ij} to r_{ik}. Then

$$r_{ik} = \sigma(r_{ij}) = \sigma(s_{ij} + x_{ij}) = \sigma(s_{ij}) + \sigma(x_{ij}).$$

As σ preserves sizes,

$$|r_{ik} - \sigma(s_{ij})| = |\sigma(x_{ij})| = |x_{ij}| < \varepsilon,$$

so $\sigma(s_{ij})$ is the root of $w_2(U) \in W(\mathbb{F}_{p^f})[U]$ close to r_{ik}. Thus $\sigma(s_{ij}) = s_{ik}$, and the roots of $w_2(U)$ break up into Galois orbits corresponding to the Galois

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orbits of the roots of $g(U)$ that are close to them. This proves the factorization statement. $\square$

This theorem has the following corollary (cf. Theorem 3 of the introduction), on using the Lemma 33 below.

COROLLARY 32. *Let $\bar{\rho} : G_{\mathbb{Q}} \to GL_2(\mathbb{F}_{p^f})$ be odd, full, ordinary, weight 2, and have determinant ϵ. Let $g(U) \in W(\mathbb{F}_{p^f})[U]$ be a distinguished polynomial of degree e. Then there exists a set $Y \supset S_0$ such that*

$$W(\mathbb{F}_{p^f})[[U]] \simeq R^{ord,Y\text{-}new} \twoheadrightarrow R_2^{ord,Y\text{-}new} \simeq W(\mathbb{F}_{p^f})[[U]]/g(U)).$$

We learned the proof of the lemma below from M. Nori and N. Fakhruddin.

LEMMA 33. *Let $h(X)$ be a distinguished polynomials in $W(\mathbb{F}_{p^f})[[X]]$ of degree n with distinct roots. Then, for all distinguished polynomials $g(X) \in W(\mathbb{F}_{p^f})[[X]]$ of degree n which are close enough to h, we have an isomorphism of $W(\mathbb{F}_{p^f})$-algebras*

$$W(\mathbb{F}_{p^f})[[X]]/(h(X)) \simeq W(\mathbb{F}_{p^f})[[X]]/(g(X)).$$

Proof. We use the identifications $W(\mathbb{F}_{p^f})[[X]]/(h(X)) \simeq W(\mathbb{F}_{p^f})[X]/(h(X)) \simeq W(\mathbb{F}_{p^f})^n$ (which allows us to work in the polynomial ring rather than power series ring) and consider the map $\alpha : W(\mathbb{F}_{p^f})^n \to W(\mathbb{F}_{p^f})^n$ defined as follows. Given $\gamma \in W(\mathbb{F}_{p^f})^n$, we regard it as an element of $W(\mathbb{F}_{p^f})[X]/(h)$, and send it to the tuple $(a_1, \ldots, a_n)$ in $W(\mathbb{F}_{p^f})^n$, where the characteristic polynomial of the endomorphism of $W(\mathbb{F}_{p^f})[X]/(h(X))$ given by multiplication by γ is $\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_n$. The image of X under α is given by the coefficients of h. Using that h has distinct roots, we see that α is an open mapping in a neighborhood of X, and deduce that all elements in a sufficiently small neighborhood V of $\alpha(X)$, which will correspond to distinguished polynomials g of degree n with distinct roots, are in $\alpha(U)$, where U is the open neighborhood of $X \in W(\mathbb{F}_{p^f})^n$ consisting of elements that are congruent to X mod p.

Choose a $\gamma \in U \subset W(\mathbb{F}_{p^f})[X]/(h(X))$ such that $\alpha(\gamma)$ is given by the coefficients of a g, as in the previous paragraph. Consider the map $W(\mathbb{F}_{p^f})[T] \to W(\mathbb{F}_{p^f})[X]/(h(X))$ given by $T \mapsto \gamma$. We deduce that this map has kernel $g(T)$, so we get a monomorphism $W(\mathbb{F}_{p^f})[T]/(g(T)) \to W(\mathbb{F}_{p^f})[X]/(h(X))$ of $W(\mathbb{F}_{p^f})$-algebras with finite cokernel, as g and h are both monic of degree n. We further deduce this is an isomorphism by reducing mod p, which induces an isomorphism $\mathbb{F}_{p^f}^n = \mathbb{F}_{p^f}[T]/(\bar{g}(T)) \simeq \mathbb{F}_{p^f}[X]/(\bar{h}(X))$. $\square$

One could ask, like in Krasner's lemma, for quantitative refinements of this lemma.

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We end with the following remark. Let $a \in W(\mathbb{F}_{p^f})$ be a nonsquare in $W(\mathbb{F}_{p^f})$. Then if one chooses $g(U) = U^2 - ap^2$ in Theorem 3, the deformation ring will be an *order* in the unramified degree-2 extension of $W(\mathbb{F}_{p^f})$. Thus one can also obtain nontrivial unramified extensions as completed fields of Fourier coefficients of the modular form corresponding to our Galois representation.

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Appendix. Modularity of geometric lifts ρ via p-adic approximation

We apply Theorem 1 to proving modularity of certain representations $\rho : G_{\mathbb{Q}} \to GL_2(\mathcal{O})$, where $\mathcal{O}$ is as in the main text, which reduce to modular $\bar{\rho}$, as in the main text.

THEOREM 34. *Let $\rho : G_{\mathbb{Q}} \to GL_2(\mathcal{O})$ be odd, ordinary, full, balanced, ramified at only finitely many primes, weight* 2*, with determinant ϵ, and have modular reduction. Further, assume that, when $\bar{\rho}$ is split at p, ρ is flat at p. Then ρ is modular.*

The method of proof extends the method of [9] which dealt with the case when K is unramified over $\mathbb{Q}_p$. Of course these results are contained in those of the various '$R = T$' theorems pioneered by Wiles and by Taylor and Wiles. Our point here is to provide a different argument using p-adic approximation. In this appendix, the proofs are merely sketched: we are rederiving known results using Theorem 1 and the strategy of [9].

Using Theorem 1, we first prove that, for each n, $\rho_n = \rho \bmod (\pi^n)$ is modular of a level which depends on ρ_n. We then lower the level of ρ_n, which paves the way to proving modularity of $\rho : G_{\mathbb{Q}} \to GL_2(\mathcal{O})$ by successive approximation.

We have the following corollary of Theorem 1. We keep the notation of the previous sections: for instance, S_0 is the set of primes at which $\bar{\rho}$ is ramified, and S is the set of primes at which ρ is ramified.

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COROLLARY 35. (i) *We consider a set T of nice primes as in Theorem 1 such that $R^{ord,T\text{-}new} \simeq W(\mathbb{F}_{p^f})[[U]]$. Let $\mathbb{T}^{ord,T\text{-}new}$ be the corresponding T-new Hida Hecke algebra. Then the natural surjection $R^{ord,T\text{-}new} \to \mathbb{T}^{ord,T\text{-}new}$ is an isomorphism.*

(ii) *The representation ρ_n is isomorphic as $\mathcal{O}[G_{\mathbb{Q}}]$-representation to a submodule of the ordinary part of the p-divisible group associated to J_T tensored over $\mathbb{Z}_p$ with $\mathcal{O}$. Here, J_T is the Jacobian of the projective modular curve $\Gamma_1(Np^r) \cap \Gamma_0(Q_n)$, N is some fixed integer independent of n, r is an integer which a priori depends on n, and Q_n is the product of primes in the finite set $T \backslash S$ which depends on n.*

We say that ρ_n as in the corollary arises from J_T. We also say that ρ_n arises from the p-divisible group associated to the ordinary factor J_T^{ord} of J_T.

Proof. This follows from Theorem 1, level-raising results of [5], and Hida's theory. These results yield that $R^{ord,T\text{-}new}$ surjects onto the T-new ordinary Hecke algebra $\mathbb{T}^{ord,T\text{-}new}$, which is finite and torsion free over $\Lambda = \mathbb{Z}_p[[T]]$. But using that $R^{ord,T\text{-}new} \simeq W(\mathbb{F}_{p^f})[[U]]$, we deduce that we have an isomorphism of $R^{ord,T\text{-}new}$ with $\mathbb{T}^{ord,T\text{-}new}$. Part (ii) of the corollary follows by standard arguments. □

Proposition 36 follows from arguments in [9], with the twist that, as we allow primes $q \in T \backslash S_0$ that are -1 mod p, we have to keep track of the Atkin–Lehner operators W_q for $q \in Q_n = T \backslash S_0$. We have the relation $W_q^2 = \langle q \rangle$.

PROPOSITION 36. *The representation ρ_n arises from the Q_n-old subvariety of J_T^{ord}, and furthermore all the Hecke operators T_r, for r a prime not in T, act on ρ_n by Trace$(\rho_n(\mathrm{Frob_r}))$.*

Proof. This is an application of Mazur's principle (see §8 of [16]) and uses that $q \not\equiv 1$ mod p. The principle relies on the fact that the Frob_q-action on unramified finite $G_{\mathbb{Q}_q}$-submodules of the torsion points of J_T whose reduction mod q is in the 'toric part' of the reduction mod q of J_T is constrained. Namely, on the 'toric part' the Frobenius Frob_q acts by $-qW_q$, where W_q is the Atkin–Lehner involution. We flesh this out this below.

Consider a prime $q \in Q_n$, where $Q_n = T \backslash S$. Then decompose $\rho_n|_{D_q}$ (which is unramified by hypothesis) into $\mathcal{O}/(\pi^n) \oplus \mathcal{O}/(\pi^n)$, with basis $\{e_n, f_n\}$ with $\mathrm{Frob}_q(e_n) = -W_q.e_n$ and $\mathrm{Frob}_q(f_n) = -qW_q.f_n$ for some character ϵ as above. The action of W_q is by a scalar α_q, and so we have $\mathrm{Frob}_q(e_n) = -\alpha_q e_n$ and $\mathrm{Frob}_q(f_n) = -q\alpha_q f_n$.

Using irreducibility of ρ, Burnside's lemma gives that $\rho(\mathbb{F}_{p^f}[G_{\mathbb{Q}}]) = M_2(\mathbb{F}_{p^f})$, and hence by Nakayama's lemma $\rho_n(\mathcal{O}[G_{\mathbb{Q}}]) = M_2(\mathcal{O}/(\pi^n))$. Thus, using the surjection from the Hecke algebra acting on J_T^{ord} to $\mathcal{O}/(\pi^n)$, we deduce that ρ_n arises from an eponymous submodule of $J := J_T^{ord}$ (see also Theorem 3.1 of [18]).

The fact that the q-old subvariety is stable under the Galois and Hecke action will allow us to deduce that ρ_n arises from the q-old subvariety of J if we can show that e_n is contained in the q-old subvariety of J.

Let $\mathcal{J}$ be the Néron model at q of J. Note that as ρ_n is unramified at q it maps injectively to $\mathcal{J}_{/\mathbb{F}_p}(\bar{\mathbb{F}}_p)$ under the reduction map. Now, if the claim were false, as the group of connected components of $\mathcal{J}$ is Eisenstein, we would deduce that the reduction of e_n in $\mathcal{J}^0(\bar{\mathbb{F}}_p)$ maps nontrivially (and hence its image has order divisible by p) to the $\bar{\mathbb{F}}_p$-points of the torus which is the quotient of $\mathcal{J}^0$ by the image of the q-old subvariety (in characteristic q). But, as we recalled above, it is well known (see §8 of [16]) that Frob_q acts on the $\bar{\mathbb{F}}_p$-valued points of this toric quotient (isogenous to the torus T of $\mathcal{J}^0$, the latter being a semiabelian variety that is an extension of an abelian variety by T) by $-qW_q$, which gives a contradiction. Now taking another prime $q' \in Q_n$ and working within the q-old subvariety of J, by the same argument we see that ρ_n occurs in the $\{q, q'\}$-old subvariety of J, and eventually that ρ_n occurs in the Q_n-old subvariety of J. The last part of the proposition is then clear. □

We finish the proof of the main theorem of the appendix, Theorem 34.

Proof. For an integer N prime to p, denote by $J_1(Np^\infty)^{ord}$ the direct limit of the ordinary parts of $J_1(Np^r)$ as r varies. From Proposition 36 it is easy to deduce that ρ_n, the mod (π^n) reduction of ρ, arises from $J_1(Np^\infty)^{ord}$ for some fixed integer N that is independent of n. Let $\mathbb{T}$ be the Hida Hecke algebra acting on $J_1(Np^\infty)^{ord}$, generated by the Hecke operators T_r with r prime and prime to Np. We claim that the ρ_n give compatible morphisms from $\mathbb{T}$ to the $\mathcal{O}/(\pi^n)$. To get these morphisms, let V_n denote a realization of the representation ρ_n in $J_1(Np^\infty)^{ord}$ which exists by Proposition 36. Then V_n is $G_{\mathbb{Q}}$-stable, and hence $\mathbb{T}$-stable (because of the Eichler–Shimura congruence relation mod r, which gives an equality of correspondences $T_r = \mathrm{Frob}_r + r\langle r\rangle \mathrm{Frob}_r{}^{-1}$, where Frob_r is the Frobenius morphism at r). So V_n is a $\mathbb{T}$-module, and because of the absolute irreducibility (only the scalars commute with the $G_{\mathbb{Q}}$-action) $\mathbb{T}$ acts via a morphism $\alpha_n : \mathbb{T} \to \mathcal{O}/(\pi^n)$ as desired, and the α_n are compatible again because of the congruence relation. This gives a morphism $\alpha : \mathbb{T} \to \mathcal{O}$ such that the representation associated to α is isomorphic to ρ, which finishes the proof of the theorem. Then using that the determinant of ρ is ϵ and Hida's control theorem, we deduce that ρ arises from a weight-2 newform. □

Improvements to the method. The weight-2 assumption on $\bar{\rho}$ and the lifts we consider (and the assumption on the determinant) is for convenience, and our methods apply to ρ of weight $k \geqslant 2$ (the Hodge–Tate weights are $(k-1, 0)$), provided that $\bar{\rho}$ is distinguished at p.

The *fullness* assumption on ρ and ρ_n used in the proof of Theorem 1 arises from the fact that in its absence Lemma 20 is not true. On the other hand one can prove a more qualitative but less restrictive version of this lemma.

LEMMA 37. *Let $p \geqslant 3$. Recall that $\mathcal{O}$ is the ring of integers of a finite extension of $\mathbb{Q}_p$, with uniformizer π and residue field $\mathbb{F}_{p^f}$. Let $G \subset GL_2(\mathcal{O})$ be a closed subgroup. Assume that the image G_1 of $G \to GL_2(\mathbb{F}_{p^f})$ contains $SL_2(\mathbb{F}_{p^f})$ and satisfies hypotheses of Lemma 19. Then* $\dim H^1(G, Ad^0\bar{\rho})$ *is a finite abelian group.*

The proof uses the following: (i) $H^1(G_1, Ad^0\bar{\rho}) = 0$ (cf. Lemma 19), and (ii) the kernel of the homomorphism $G \to G_1$ is a finitely generated pro-p group.

G. Böckle has observed that, using such a lemma, one can remove the assumption on fullness of image of ρ, made in the arguments in the appendix, by using base change to totally real solvable extensions $F/\mathbb{Q}$ and considering nearly ordinary deformations of $\bar{\rho}|_{G_F}$ and Hida's nearly ordinary Hecke algebras. Choose a totally real solvable extension $F/\mathbb{Q}$ disjoint from the field cut out by ρ, and whose degree $d = [F : \mathbb{Q}]$ is $> \dim H^1(G, Ad^0\bar{\rho})$. Then, by choice of F and the technique of killing dual Selmer groups, and obtaining smooth quotients of deformation rings of the expected dimension of this paper, one obtains nearly ordinary deformation rings that are power series rings in $d + \delta + 1$ variables, where d is the degree of F over $\mathbb{Q}$ and δ the Leopoldt defect for F and p, and such that ρ mod π^n arises from the corresponding universal deformation. One then would exploit the fact that Hida's nearly ordinary Hecke algebra is finite flat over $\mathbb{Z}_p[[X_0, \ldots, X_{d+\delta}]]$. By more elaborate level-lowering methods (as in [18]) one would then by a similar strategy as above prove that $\rho|_{G_F}$ is automorphic, which suffices, as $F/\mathbb{Q}$ is solvable.

To make the present method of modularity lifting more robust, one would ultimately hope to also remove the conditions of *ordinarity* and being *balanced* on geometric ρ, and show, assuming that $\bar{\rho}$ is modular and irreducible, that ρ mod (π^n) is modular of some level for each n, and hence by level-lowering techniques deduce that ρ itself arises for a new form. For this again base change to solvable totally real extension of $\mathbb{Q}$ and the *trivial primes* of [8] might be useful to remove the *balanced condition*, and to remove ordinarity one would have to use Coleman families and work on the eigenvariety.

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