Preface

This book discusses an approach to the study of global properties of solutions to the equations of general relativity, the Einstein field equations, in which the notion of conformal transformation plays a central role. The use of conformal transformations in differential geometry dates back, at least, to the work of *Hermann Weyl* in the 1920s.[1] Their application to global questions in general relativity, as presented in this book, stems from the seminal work of *Roger Penrose* in the 1960s in which the close connection between the global causal structure of the solutions to the equations of general relativity and conformal geometry was established.[2] Penrose's key insights are that the close relation between the propagation of the gravitational field and the structure of light cones which holds locally in a spacetime is also preserved in the case of large scales and that the asymptotic behaviour of the gravitational field can be conveniently analysed in terms of conformal extensions of the spacetime. In the following decade Penrose's ideas were polished, extended and absorbed into the mainstream research of general relativity by a considerable number of researchers[3] – finally leading to the influential notion of *asymptotic simplicity*. The subject reached its maturity when this *formal* theory was combined with the methods of the theory of partial differential equations (PDEs). This breakthrough is mainly due to the work of *Helmut Friedrich* in the early 1980s, who – through the *conformal Einstein field equations*[4] – showed that ideas of conformal geometry can be used to establish the existence of large classes of solutions to the Einstein field equations satisfying Penrose's notion of asymptotic simplicity. As a result of this work it is now clear that Penrose's original insights hold for large classes of spacetimes and not only for special explicitly known solutions.

This book develops the theory of the conformal Einstein field equations from the ground up and discusses their applications to the study of asymptotically simple spacetimes. Special attention is paid to results concerning the existence and stability of *de Sitter-like spacetimes*, the semiglobal existence and stability of *Minkowski-like spacetimes* using hyperboloidal Cauchy problems and the

[1] See Weyl (1968).
[2] See Penrose (1963, 1964).
[3] See e.g. Hawking and Ellis (1973); Geroch (1976).
[4] See Friedrich (1981a,b, 1983).

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construction of *anti-de Sitter-like spacetimes* from initial boundary value problems. These results belong to the canon of modern mathematical relativity. In addition to their mathematical interest, they are of great physical relevance as they express, among other things, the internal consistency of general relativity and provide an approach for the global evaluation of spacetimes by means of numerical methods.

Why a book on the subject? The applications of conformal methods in general relativity constitute a mature subject with a number of *core results* which will withstand the pass of time. Still, it provides a number of challenging open questions whose resolution will strengthen its connections with other research strands in general relativity. This book aims at making the subject accessible to physicists and mathematicians alike who want to make use of conformal methods to analyse the global structure and properties of spacetimes. Hopefully, this book will provide an alternative to the use of original references while learning the subject or doing research.

Anyone who wants to engage with the subject of this book faces a number of challenges. To begin with, one has a vast literature spreading over more than 50 years. As it is to be expected from a living subject, the perspectives change through time, the importance of certain problems rise and wane and it is sometimes hard to differentiate the fundamental from the subsidiary. The combination of results from various references is often hindered by changing notation and conflicting conventions. Moreover, to appreciate and understand the results of the theory one requires a considerable amount of background material: conformal geometry, spinors, PDE theory, causal theory, etc. These methods are an essential part of the toolkit of a modern mathematical relativist. This book endeavours to bring together in a single volume all the required background material in a concise and coherent manner.

As a cautionary note, it should be mentioned what this book is not intended to be. This book is not an introductory book to general relativity. A certain familiarity with the subject is assumed from the outset – ideally at the level of Part I of R. Wald's book *General Relativity*.[5] This is also not a book on the applications of the theory of PDEs in general relativity. For this, there are other books available.[6] Also, although the Cauchy problem in general relativity is a leading theme, this book should not be viewed as a monograph on the topic – for this, I refer the interested reader to H. Ringström's monograph.[7]

I have endeavoured to write a book which not only serves as an *introduction to the subject* but also is a *tool for research*. With this idea in mind, I have striven to provide as much detail as possible of the arguments and calculations. However, at some stages supplying further details is neither possible nor desirable. Indeed, quoting the preface of J. L. Synge's classical book on general relativity: "There

[5] See Wald (1984).
[6] See e.g. Choquet-Bruhat (2008); Rendall (2008).
[7] See Ringström (2009).

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are heavy calculations in the book, but there are places where the reader will find me sitting on the fence, whistling, instead of rushing into the fray"; see Synge (1960).[8] In an attempt to keep the readability and the length of the text under control, I have not endeavoured to provide completely general or optimal theorems – the attentive reader will realise this and is referred to the literature for further details, if required. As a picture is better than a thousand words, I have complemented the text with a considerable number of figures and diagrams which, I hope, will help to explain the content of the main text and provide useful visual models.

In writing this book, I have assumed the reader to have a certain mathematical maturity. Some basic knowledge of topology is needed – Appendix A in Wald's book contains the required background – as well as familiarity with basic tensorial calculus. I have, however, not assumed any prior knowledge of 2-spinors. The necessary toolkit is developed in the course of two chapters. Readers looking for a supplementary source on the topic are referred to J. Stewart's book.[9] The applications of conformal methods discussed in this book require certain knowledge of the theory of PDEs. I provide all the required material in a chapter of its own – nevertheless, some previous exposure to the basic ideas of the theory of PDEs is an advantage. Some arguments in the book make use of very concrete results of analysis. In these cases, I have included the necessary ideas in appendices to the various chapters.

[8] I am thankful to R. Beig for bringing my attention to this quote.
[9] See Stewart (1991).