

ON A METHOD OF DESCRIBING FORMAL DEDUCTIONS CONVENIENT FOR THEORETICAL PURPOSES

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Introduction

In my paper [2], I have proposed a method of describing formal deductions which seems to be convenient for practical purposes. In my paper [3], I have employed an index-system to exactly express tree-form configurations of proofs in Gentzen's formalism for sequents. In the present paper, I would like to propose a method of describing formal deductions which seems to be convenient for theoretical purposes. The device employed for this purpose relies mostly on an index-system. Just as in [2] as well as in [3], I propose here also to denote every proposition and every denomination of a variable, or every sequent in Gentzen's formalism, by an index-word.

Although our device, to be illustrated in the present paper, can be applied to a large variety of various formalisms, I will illustrate it in the present paper by taking up two examples. The first example is Gentzen's *LK* (notation in the present paper: *GLK*). It can be applied also to Gentzen's *LJ* without any essential modification. The second example is the lower classical predicate logic *LK* in the form I have dealt with in my former papers. Our device can be naturally extended to other logics such as the intuitionistic logic, Johansson's minimal logic, the positive logic, etc.

Our index-system introduced in the present paper has the strong point for theoretical purposes, that not only the tree-form configuration of each proof is clearly denoted by the index-system but also the inference rules employed for the deduction of steps are expressed exactly and further their reference steps too can be founded out by index-words only. However, this index-system has the weak point that, for practical purposes, proof-note

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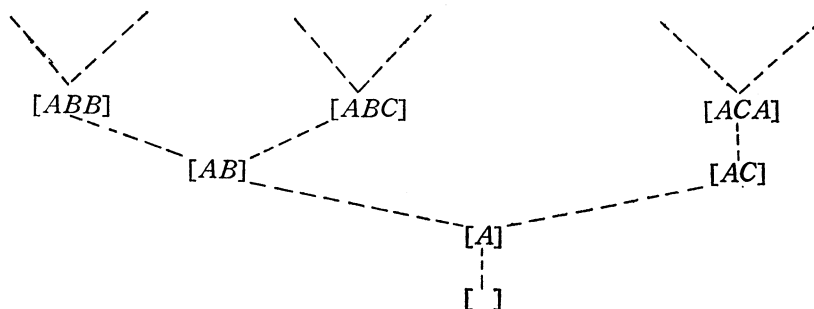
as well as index-words turn out much longer when described by our new method compared with proof-notes or index-words described by the method proposed in [2].

In Section (1), I will illustrate our new index-word system with respect to *GLK* (Gentzen's *LK*), and I will illustrate it with respect to the lower classical predicate logic *LK* in Section (2). Proofs of the logics such as the intuitionistic logic *LJ*, the minimal logics *LM* and *LN*, the positive logics *LP* and *LQ*, and also the primitive logic *LO* in the form I have dealt with in my former papers, can be described along this line without any essential modification. The method of description introduced in Section (2) is referred to by *TD* (theoretical description) and the method of description introduced in my former paper [2] is referred to by *PD* (practical description). When the logic *LK* is described in *TD* or *PD*, it is referred to by *TLK* or *PLK*, respectively. More generally, any logic *LX* can be referred to by *TLX* or *PLX*, when it is described in *TD* or in *PD*, respectively.

In Section (3), I would like to expose the mutual relation between *TLK* and *GLK* which is formulated in Section (1), and also the mutual relation between *TLK* and *PLK*.

(1) Index-system for *GLK* (Gentzen's *LK*)

To denote tree-form configuration of proofs, I have employed in my paper [3] an index-system, in which any index-word is a sequence of letters *A*, *B*, and *C*. Configurations of proofs have been figured out something like



This index-system is enough to show the tree-form configurations of proofs themselves, but it can not show completely, for each step; which inference rule is employed to deduce the step from a step or steps standing above it. In the present section, I will introduce an index-system which

enables us to show, for each step of a proof, from which step or steps and by which inference rule the step is deduced.

For this purpose, I will take up the following list of letters

$F, V, V^*, R, R^*, Q, Q^*, I, I', I^*, C, C', C^*, C^{**}, D', D'', D^*, D^{**}, N, N^*, U, U^*, E, E^*, S, S',$

instead of the list of three letters A, B, C . The letter F does not occur in any index-word except at its tail. It should be occasionally disregarded but occasionally should not be disregarded. When the letter F should be disregarded in an index-word, the index-word is enclosed in “{ }”. Namely, both index-word \underline{s} and $\underline{s}F$ is denoted by $\{s\}$ as well as by $\{sF\}$. Throughout this section, notations of the form \underline{s} (underlined lower case letters) stand for sequences of letters in our list. In any bare index-word having no F explicitly, it is tacitly assumed that no F occurs in the index-word.

For each inference rule, I will give in the following, its notation, its index-word form, and its sequent form. Any sequent indicated by an index-word of the form $\underline{s}F$ should be a fundamental sequent.

Inference rules of GLK:

Notation	Index-word form	Sequent form
(F)	$\underline{s}F$	$\mathfrak{A} \vdash \mathfrak{A}$
(V)	$\frac{\{sV\}}{\underline{s}}$	$\frac{\Gamma \vdash \Delta}{\Gamma, \mathfrak{A} \vdash \Delta}$
(V^*)	$\frac{\{sV^*\}}{\underline{s}}$	$\frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, \mathfrak{A}}$
(R)	$\frac{\{sR\}}{\underline{s}}$	$\frac{\Gamma, \mathfrak{A}, \mathfrak{B}, \Theta \vdash \Delta}{\Gamma, \mathfrak{B}, \mathfrak{A}, \Theta \vdash \Delta}$
(R^*)	$\frac{\{sR^*\}}{\underline{s}}$	$\frac{\Gamma \vdash \Delta, \mathfrak{A}, \mathfrak{B}, \Delta}{\Gamma \vdash \Delta, \mathfrak{B}, \mathfrak{A}, \Delta}$
(Q)	$\frac{\{sQ\}}{\underline{s}}$	$\frac{\Gamma, \mathfrak{A}, \mathfrak{A} \vdash \Delta}{\Gamma \mathfrak{A} \vdash \Delta}$
(Q^*)	$\frac{\{sQ^*\}}{\underline{s}}$	$\frac{\Gamma \vdash \Delta, \mathfrak{A}, \mathfrak{A}}{\Gamma \vdash \Delta, \mathfrak{A}}$
(I', I'')	$\frac{\{sI'\} \{sI''\}}{\underline{s}}$	$\frac{\Gamma \vdash \Delta, \mathfrak{A} \quad \Gamma, \mathfrak{B} \vdash \Delta}{\Gamma, \mathfrak{A} \rightarrow \mathfrak{B} \vdash \Delta}$
(I^*)	$\frac{\{sI^*\}}{\underline{s}}$	$\frac{\Gamma, \mathfrak{A} \vdash \Delta, \mathfrak{B}}{\Gamma \vdash \Delta, \mathfrak{A} \rightarrow \mathfrak{B}}$

(C')	$\frac{\{\underline{s}C'\}}{\underline{s}}$	$\frac{\Gamma, \mathfrak{A} \vdash \Delta}{\Gamma, \mathfrak{A} \wedge \mathfrak{B} \vdash \Delta}$
(C'')	$\frac{\{\underline{s}C''\}}{\underline{s}}$	$\frac{\Gamma, \mathfrak{A} \vdash \Delta}{\Gamma, \mathfrak{B} \wedge \mathfrak{A} \vdash \Delta}$
(C*, C**)	$\frac{\{\underline{s}C^*\} \{\underline{s}C^{**}\}}{\underline{s}}$	$\frac{\Gamma \vdash \Delta, \mathfrak{A} \quad \Gamma \vdash \Delta, \mathfrak{B}}{\Gamma \vdash \Delta, \mathfrak{A} \wedge \mathfrak{B}}$
(D', D'')	$\frac{\{\underline{s}D'\} \{\underline{s}D''\}}{\underline{s}}$	$\frac{\Gamma, \mathfrak{A} \vdash \Delta \quad \Gamma, \mathfrak{B} \vdash \Delta}{\Gamma, \mathfrak{A} \vee \mathfrak{B} \vdash \Delta}$
(D*)	$\frac{\{\underline{s}D^*\}}{\underline{s}}$	$\frac{\Gamma \vdash \Delta, \mathfrak{A}}{\Gamma \vdash \Delta, \mathfrak{A} \vee \mathfrak{B}}$
(D**)	$\frac{\{\underline{s}D^{**}\}}{\underline{s}}$	$\frac{\Gamma \vdash \Delta, \mathfrak{A}}{\Gamma \vdash \Delta, \mathfrak{B} \vee \mathfrak{A}}$
(N)	$\frac{\{\underline{s}N\}}{\underline{s}}$	$\frac{\Gamma \vdash \Delta, \mathfrak{A}}{\Gamma, \sim \mathfrak{A} \vdash \Delta}$
(N*)	$\frac{\{\underline{s}N^*\}}{\underline{s}}$	$\frac{\Gamma, \mathfrak{A} \vdash \Delta}{\Gamma \vdash \Delta, \sim \mathfrak{A}}$
(U)	$\frac{\{\underline{s}U\}}{\underline{s}}$	$\frac{\Gamma, \mathfrak{A}(t) \vdash \Delta}{\Gamma, (x)\mathfrak{A}(x) \vdash \Delta}$
(U*)	$\frac{\{\underline{s}U^*\}}{\underline{s}}$	$\frac{\Gamma \vdash \Delta, \mathfrak{A}(t)}{\Gamma \vdash \Delta, (x)\mathfrak{A}(x)}$
(E)	$\frac{\{\underline{s}E\}}{\underline{s}}$	$\frac{\Gamma, \mathfrak{A}(t) \vdash \Delta}{\Gamma, (\exists x)\mathfrak{A}(x) \vdash \Delta}$
(E*)	$\frac{\{\underline{s}E^*\}}{\underline{s}}$	$\frac{\Gamma \vdash \Delta, \mathfrak{A}(t)}{\Gamma \vdash \Delta, (\exists x)\mathfrak{A}(x)}$
(S', S'')	$\frac{\{\underline{s}S'\} \{\underline{s}S''\}}{\underline{s}}$	$\frac{\Gamma \vdash \Delta, \mathfrak{A} \quad \Gamma, \mathfrak{A} \vdash \Delta}{\Gamma \vdash \Delta}$

Any non-empty finite set \mathfrak{X} of index-words is called a *proof-tree* if and only if it satisfies the following conditions:

(TG 1) For any index-word of the form $\underline{s}F$ in \mathfrak{X} , there is no index-word beginning with \underline{s} other than $\underline{s}F$ in \mathfrak{X} .

(TG 2) For any index-word of the form $\underline{s}X$ in \mathfrak{X} , where X differs from F , the index-word \underline{s} belongs to \mathfrak{X} .

(TG 3) For any index-word \underline{s} in \mathfrak{X} having no F at its tail, there is at least one index-word of the form $\underline{s}X$ in \mathfrak{X} .

(TG 4) For any index-word of the form $\{\underline{s}X\}$ in \mathfrak{X} , there is no index-word beginning with \underline{s} other than \underline{s} and those beginning with $\underline{s}X$ in \mathfrak{X} , where X stands for any one of the letters $V, V^*, R, R^*, Q, Q^*, I^*, C', C'', D^*, D^{**}, N, N^*, U, U^*, E, E^*$.

(TG 5) For any index-word of the form $\{\underline{s}X\}$ in \mathfrak{X} , there is the index-word $\{\underline{s}Y\}$ and no index-word other than \underline{s} and those beginning with $\underline{s}X$ or $\underline{s}Y$ in \mathfrak{X} , where the pair of letters X and Y stands for any one of the non-ordered pairs of letters $\{I', I''\}$, $\{C^*, C^{**}\}$, $\{D', D''\}$, $\{S', S''\}$.

Any function Π which maps a proof-tree into the domain of sequents is called a *proof-note* if and only if Π satisfies the following conditions:

($\Pi G 1$) $\Pi(\underline{s}F)$ is a fundamental sequent.

($\Pi G 2$) $\Pi(\{\underline{s}X\})$ and $\Pi(\underline{s})$ have the forms of the sequents indicated by $\{\underline{s}X\}$ and \underline{s} in the inference rule (X) , respectively, where X stands for any one of the letters $V, V^*, R, R^*, Q, Q^*, I^*, C', C'', D^*, D^{**}, N, N^*, U, U^*, E, E^*$. Furthermore, in the case where X stands for any one of the letters U, U^*, E, E^* , the variable t should never occur in the range of any quantifier of the bound variable x in $\mathfrak{X}(t)$, and in the cases where X stands for U^* or E , the variable t should never occur in $\Pi(\underline{s})$.

($\Pi G 3$) $\Pi(\{\underline{s}X\})$, $\Pi(\{\underline{s}Y\})$, and $\Pi(\underline{s})$ have forms of the sequents indicated by $\{\underline{s}X\}$, $\{\underline{s}Y\}$, and \underline{s} in the inference rule of (X, Y) , respectively, where the pair of X and Y stands for any one of the ordered pairs of letters $\langle I', I'' \rangle$, $\langle C^*, C^{**} \rangle$, $\langle D', D'' \rangle$, $\langle S', S'' \rangle$.

Any proof-note Π is regarded as a proof of $\Pi(\emptyset)$, where \emptyset denotes the null-sequence index-word. Notice that \emptyset belongs to any proof-tree according to the condition (TG 2). It can be seen without difficulty that this system is equivalent to Gentzen's **LK**. Also, it would be easily seen that our device can be extended agreeably to describe Gentzen's **LJ**.

(2) **Index-system for the classical predicate logic LK.**

The logic I am going to formulate here is the classical predicate logic **LK** of my former papers. The logic **LK** has the negation notion, but $\sim\mathfrak{F}$ can be agreeably replaced by $\mathfrak{F} \rightarrow \wedge$ by adopting the proposition constant \wedge . I do not give the inference rule for negation with respect to the logical constant " \sim ", but I will give here the inference rules with respect to the proposition constant \wedge .

The index-system for **LK** is similar to the index-system I have given in the preceding section. Namely, I take up the following list of letters

$F, I', I'', i^*, C', C'', C^*, C^{**}, D, d', d'', D^*, D^{**}, U, U^*, E,$
 $e, E^*, \forall, \wedge, P$

together with an auxiliary symbol " $-$ ". Any index-word contains at most

one auxiliary symbol “-”. Hence, starting from a sequence \underline{s} of letters of the length n , we can make $n+1$ index-words by either inserting “-” just after k letters of the sequence ($k = 0, \dots, n-1$) or not inserting “-”. The index-word obtained by inserting “-” just after k letters of \underline{s} is denoted by \underline{s}^k . In the following, I would like to establish a convention that any symbol of the form \underline{s} denotes an index-word containing no “-”, any symbol of the form \underline{s}^- denotes an index-word having “-” somewhere, and any symbol of the form \underline{s}^m for $m \geq n$ denotes \underline{s} . Hence, symbols of the form \underline{s}^k in general can stand both for \underline{s}^- and for \underline{s} .

The inference rules of **LK** read:

(F) $\langle\langle \underline{s} \rangle \mathcal{A} \rangle$ is deducible from $\langle\langle \underline{s}^j F \rangle \mathcal{A} \rangle$ for any j .

(I', I'') $\langle\langle \underline{s} \rangle \mathcal{A} \rangle$ is deducible from $\langle\langle \underline{s}^j I' \rangle \mathcal{B} \rangle$ and $\langle\langle \underline{s}^k I'' \rangle \mathcal{B} \rightarrow \mathcal{A} \rangle$ for any j and k .

(i*) $\langle\langle \underline{s} \rangle \mathcal{A} \rightarrow \mathcal{B} \rangle$ is deducible from the fact that $\langle\langle \underline{s} i^* \rangle \mathcal{B} \rangle$ is deducible from steps of the form $\langle\langle \underline{s} - i^* g \rangle \mathcal{A} \rangle$.

(C') $\langle\langle \underline{s} \rangle \mathcal{A} \rangle$ is deducible from $\langle\langle \underline{s}^j C' \rangle \mathcal{A} \wedge \mathcal{B} \rangle$ for any j .

(C'') $\langle\langle \underline{s} \rangle \mathcal{A} \rangle$ is deducible from $\langle\langle \underline{s}^j C'' \rangle \mathcal{B} \wedge \mathcal{A} \rangle$ for any j .

(C*, C**) $\langle\langle \underline{s} \rangle \mathcal{A} \wedge \mathcal{B} \rangle$ is deducible from $\langle\langle \underline{s}^j C^* \rangle \mathcal{A} \rangle$ and $\langle\langle \underline{s}^k C^{**} \rangle \mathcal{B} \rangle$ for any j and k .

(D, d', d'') $\langle\langle \underline{s} \rangle \mathcal{A} \rangle$ is deducible from $\langle\langle \underline{s}^j D \rangle \mathcal{B} \vee \mathcal{C} \rangle$ and the facts that $\langle\langle \underline{s} d' \rangle \mathcal{A} \rangle$ is deducible from steps of the form $\langle\langle \underline{s} - d' g \rangle \mathcal{B} \rangle$ and that $\langle\langle \underline{s} d'' \rangle \mathcal{A} \rangle$ is deducible from steps of the form $\langle\langle \underline{s} - d'' h \rangle \mathcal{C} \rangle$.

(D*) $\langle\langle \underline{s} \rangle \mathcal{A} \vee \mathcal{B} \rangle$ is deducible from $\langle\langle \underline{s}^j D^* \rangle \mathcal{A} \rangle$ for any j .

(D**) $\langle\langle \underline{s} \rangle \mathcal{A} \vee \mathcal{B} \rangle$ is deducible from $\langle\langle \underline{s}^j D^{**} \rangle \mathcal{B} \rangle$ for any j .

(U) $\langle\langle \underline{s} \rangle \mathcal{A}(t) \rangle$ is deducible from $\langle\langle \underline{s}^j U \rangle (x)\mathcal{A}(x) \rangle$ for any j .

(U*, \forall) $\langle\langle \underline{s} \rangle (x)\mathcal{A}(x) \rangle$ is deducible from the fact that $\langle\langle \underline{s} U^* \rangle \mathcal{A}(t) \rangle$ is deducible for $\langle\langle \underline{s} \forall \rangle \forall t : \rangle$

(E, e, \forall) $\langle\langle \underline{s} \rangle \mathcal{A} \rangle$ is deducible from $\langle\langle \underline{s}^j E \rangle (\exists x)\mathcal{B}(x) \rangle$ and the fact that $\langle\langle \underline{s} e \rangle \mathcal{A} \rangle$ is deducible from steps of the form $\langle\langle \underline{s} - e g \rangle \mathcal{B}(t) \rangle$ for $\langle\langle \underline{s} \forall \rangle \forall t : \rangle$.

(E*) $\langle\langle \underline{s} \rangle (\exists x)\mathcal{A}(x) \rangle$ is deducible from $\langle\langle \underline{s}^j E^* \rangle \mathcal{A}(t) \rangle$ for any j .

(\wedge) $\langle\langle \underline{s} \rangle \mathcal{A} \rangle$ is deducible from $\langle\langle \underline{s}^j \wedge \rangle \mathcal{A} \wedge \mathcal{B} \rangle$ for any j .

(P) $\langle\langle \underline{s} \rangle \mathcal{A} \rangle$ is deducible from $\langle\langle \underline{s}^j P \rangle (\mathcal{A} \rightarrow \mathcal{B}) \rightarrow \mathcal{A} \rangle$ for any j .

Now, I will define proof-trees in **LK**. Namely, any non-empty finite set \mathfrak{T} of index-words is called a *proof-tree* if and only if it satisfies the following conditions:

(T1) Any index-word in \mathfrak{T} contains at most one “-”, and any “-” in an

index-word stands just before i^, d', d'' , or e . The letter \forall can stand at the end of an index-word, if any. Any index-word ending with \forall can not contain “-”. For any index-word of the form $\underline{s}-\underline{g}$ in \mathfrak{X} , $\underline{s}-\underline{g}$ is the only index-word in \mathfrak{X} which begins with an index-word of the form $(\underline{s}g)^j$.*

(T 2) *For any index-word \underline{s} in \mathfrak{X} , either there is at least one index-word of the form \underline{s}^jX , or \underline{s} ends with \forall .*

(T 3) *For any index-word \underline{s}^jX in \mathfrak{X} , \underline{s} belongs to \mathfrak{X} .*

(T 4) *For any index-word of the form \underline{s}^jX in \mathfrak{X} , any index-word beginning with \underline{s}^k for any k is either \underline{s} or an index-word beginning with $(\underline{s}X)^h$ for some h , where X stands for any one of the letters $F, i^*, C', C'', D^*, D^{**}, U, E^*, \wedge, P$.*

(T 5) *For any index-word of the form \underline{s}^jX in \mathfrak{X} , there is an index-word of the form \underline{s}^kY for some k in \mathfrak{X} , but there is no index-word beginning with an index-word of the form \underline{s}^h in \mathfrak{X} other than \underline{s} and those index-words beginning with index-words of the forms $(\underline{s}X)^p$ or $(\underline{s}Y)^q$ for some p and q , where the pair of X and Y stands either for any one of the non-ordered pairs $\{I', I''\}, \{C^*, C^{**}\}$ or for the ordered pair $\langle U^*, \forall \rangle$.*

(T 6) *For any index-word of the form \underline{s}^jX in \mathfrak{X} , there are index-words of the forms \underline{s}^hY and \underline{s}^kZ for some h and k in \mathfrak{X} , but there can not be any index-word beginning with an index-word of the form \underline{s}^q in \mathfrak{X} other than \underline{s} and those index-words beginning with index-words of the forms $(\underline{s}X)^p, (\underline{s}Y)^q$, or $(\underline{s}Z)^r$ for some p, q, r , where the triple of X, Y , and Z , stands either for the non-ordered triple $\{D, d', d''\}$ or any one of the ordered triples $\langle E, e, \forall \rangle$ and $\langle e, E, \forall \rangle$.*

(T 7) *For any index-word of the form $\underline{s}\forall$ in \mathfrak{X} , there is either the index-word $\underline{s}U^*$ or the index-word of the form $\underline{s}e$ in \mathfrak{X} .*

Any function Π which maps a proof-tree into the domain of propositions and denominations of variables is called a *proof-note* if and only if Π satisfies the following conditions:

(Π 1) $\Pi(\underline{s}\forall)$ is a denomination of the form $\ll \forall t : \gg$. $\Pi(\underline{s}^j)$ is a proposition.

(Π 2) $\Pi(\underline{s}^jX)$ and $\Pi(\underline{s})$ have the forms of the propositions indicated by \underline{s}^jX and \underline{s} in the inference rule (X), respectively, where X stands for any one of the letters $F, C', C'', D^*, D^{**}, U, E^*, \wedge, P$.

(Π 3) $\Pi(\underline{s}i^*)$ and $\Pi(\underline{s})$ have the forms of the propositions indicated by the index-words $\underline{s}i^*$ and \underline{s} in the inference rule (i^*), respectively, and all the values

$\Pi(\underline{s}-i^*g)$ for various g are identical to the same proposition having the form of the proposition indicated by the index-word $\underline{s}-i^*g$ in the same inference rule (i^*).

($\Pi 4$) $\Pi(\underline{s}U^*)$ and $\Pi(\underline{s})$ have the forms of the propositions indicated by the index-words $\underline{s}U^*$ and \underline{s} in the inference rule (U^*, \forall), respectively. The denominated variable t in $\ll \underline{s}\forall \forall t : \gg$ should occur neither in the range of any quantifier of the bound variable x of the proposition $\Pi(\underline{s}U^*)$, i.e. $\mathfrak{A}(t)$, nor in any proposition of the form $\Pi(\underline{s}^-g)$.

($\Pi 5$) $\Pi(\underline{s}^kX)$, $\Pi(\underline{s}^jY)$, and $\Pi(\underline{s})$ have the forms of the propositions indicated by the index-words \underline{s}^kX , \underline{s}^jY , and \underline{s} in the inference rule (X, Y), respectively, where the pair of X and Y stands for either of the ordered pairs $\langle I', I'' \rangle$ or $\langle C^*, C^{**} \rangle$.

($\Pi 6$) $\Pi(\underline{s}^jE)$, $\Pi(\underline{s}e)$, and $\Pi(\underline{s})$ have the forms of propositions indicated by \underline{s}^jE , $\underline{s}e$, and \underline{s} in the inference rule (E, e, \forall), respectively, and all $\Pi(\underline{s}-eg)$ of Π for various g are identical to the same proposition. The denominated variable t of $\ll \underline{s}\forall \forall t : \gg$ should occur neither in the range of any quantifier of the bound variable x of the proposition $\Pi(\underline{s}-eg)$, i.e. $\mathfrak{B}(t)$, nor in the proposition $\Pi(\underline{s}^jE)$, i.e. $(\exists x)\mathfrak{B}(x)$, nor in $\Pi(\underline{s}e)$, i.e. \mathfrak{A} , nor in any proposition of the form $\Pi(\underline{s}^-h)$.

($\Pi 7$) $\Pi(\underline{s}^jD)$, $\Pi(\underline{s}d')$, $\Pi(\underline{s}d'')$, and $\Pi(\underline{s})$ have the forms of the propositions indicated by \underline{s}^jD , $\underline{s}d'$, $\underline{s}d''$, and \underline{s} in the inference rule (D, d', d''), respectively, and all $\Pi(\underline{s}-d'g)$ for various g are identical to the same proposition. Likewise, all $\Pi(\underline{s}-d''h)$ for various h are identical to the same proposition. $\Pi(\underline{s}-d'g)$ and $\Pi(\underline{s}-d''h)$ are propositions of the forms of the propositions indicated by the index-words $\underline{s}-d'g$ and $\underline{s}-d''h$ in the inference rule (D, d', d''), respectively.

Any proposition \mathfrak{B} is called *provable* by a proof-note Π if and only if Π is a correct proof-note and $\Pi(\emptyset)$ is the proposition \mathfrak{B} .

Although I have illustrated our method of description with respect to **LK** only, it can be easily seen that our index-system device can be extended agreeably to descriptions of other logics such as the intuitionistic predicate logic **LJ**, the minimal predicate logics **LM** and **LN** as well as the positive predicate logics **LP** and **LQ**, the primitive logic **LO**, etc.

In the logic **LO**, the inference rules (F), (I', I''), (i^*), (U), and (U^*, \forall) are admitted. In all the other logics **LJ**, **LK**, **LM**, **LN**, **LP**, and **LQ**, the inference rules (C'), (C''), (C^*, C^{**}), (D, d', d''), (D^*), (D^{**}), (E, e, \forall), and (E^*) are further admitted. In the logics **LJ**, **LK**, **LM**, and **LN**, the proposition constant \wedge is assumed. In the logics **LJ** and **LK**, the inference rule (\wedge) is admitted. In the logics **LK**, **LN**, and **LQ**, the inference rule (P) is further admitted.

When any logic LX is described by making use of the index-system illustrated in the present paper, I will refer to it by TLX .

(3) **Mutual relations**

The main purpose of the present paper is to expose the mutual relation between TLK and GLK as has been formulated in Section (1), which I will denote hereafter by $TGLK$. I will also expose the mutual relation between TLK and PLK .

To show the mutual relation between TLK and $TGLK$ very clearly, it seems better to modulate either $TGLK$ or TLK so that it matches better with the other. It was really a splendid idea of Gentzen to deal with any finite number of cases (Δ of any sequent of the form $\Gamma \vdash \Delta$) simultaneously. To match with $TGLK$, we might modulate TLK so that we can deal with any finite number of cases simultaneously. Indeed, this must be an interesting task. In the present paper, however, I will modulate $TGLK$ so as to match with TLK . Namely, by TG^*LK , I will denote the system dealing with only sequents of the form $\Gamma \vdash \mathfrak{A}$ and having the inference rules of (F) , (V) , (R) , (Q) , (I^*) , (C') , (C'') , (C^*, C^{**}) , (D', D'') , (D^*) , (D^{**}) , (U) , (U^*) , (E) , (E^*) with respect to sequents of the form $\Gamma \vdash \mathfrak{A}$, together with the following inference rules:

$$\begin{array}{l} (I', I'') \quad \frac{\{\underline{s}I'\} \{\underline{s}I''\}}{\underline{s}} \quad \frac{\Gamma \vdash \mathfrak{A} \quad \Gamma, \mathfrak{B} \vdash \mathfrak{C}}{\Gamma, \mathfrak{A} \rightarrow \mathfrak{B} \vdash \mathfrak{C}} \\ (S', S'') \quad \frac{\{\underline{s}S'\} \{\underline{s}S''\}}{\underline{s}} \quad \frac{\Gamma \vdash \mathfrak{A} \quad \Gamma, \mathfrak{A} \vdash \mathfrak{B}}{\Gamma \vdash \mathfrak{B}} \end{array}$$

and the two kinds of new fundamental sequents

$$\begin{array}{l} (\wedge) \quad \underline{s}\wedge \quad \wedge \vdash \mathfrak{A}, \\ (P) \quad \underline{s}P \quad (\mathfrak{A} \rightarrow \mathfrak{B}) \rightarrow \mathfrak{A} \vdash \mathfrak{A}. \end{array}$$

The negation “ $\sim\mathfrak{F}$ ” is defined by “ $\mathfrak{F} \rightarrow \wedge$ ”. Accordingly, we need the following list of letters

$$F, V, R, Q, I', I'', I^*, C', C'', C^*, C^{**}, D', D'', D^*, D^{**}, U, U^*, E, E^*, S', S'', \wedge, P.$$

If any index-word \underline{s} is enclosed in “{ }”, the letters F, \wedge , or P in \underline{s} should be disregarded.

Any non-empty finite set of index-words is called a *proof-tree* in TG^*LK , if and only if it satisfies the following conditions:

(TG* 1) For any index-word of the forms, $\underline{s}F$, $\underline{s}\wedge$, or $\underline{s}P$ in \mathfrak{X} , there is no index-word beginning with \underline{s} in \mathfrak{X} other than $\underline{s}F$, $\underline{s}\wedge$, or $\underline{s}P$.

(TG* 2) For any index-word of the form $\underline{s}X$, where X differs from F , \wedge , and P , in \mathfrak{X} , the index-word \underline{s} belongs to \mathfrak{X} .

(TG* 3) For any index-word \underline{s} in \mathfrak{X} having neither F nor \wedge nor P at its tail, there is at least one index-word of the form $\underline{s}X$ in \mathfrak{X} .

(TG* 4) The same as (TG 4), the letters V^* , R^* , Q^* , N , N^* being deleted.

(TG* 5) The same as (TG 5).

Just as in **TGLK**, any function Π which maps a proof-tree into the domain of sequents of the form $\Gamma \vdash \mathfrak{A}$ is called a *proof-note* in **TG*LK** if and only if Π satisfies the following conditions:

(ΠG^* 1) Any sequent of the forms $\Pi(\underline{s}F)$, $\Pi(\underline{s}\wedge)$, and $\Pi(\underline{s}P)$ is a *fundamental sequent*.

(ΠG^* 2) The same as (ΠG 2), the letters V^* , R^* , Q^* , N , N^* being deleted.

(ΠG^* 3) The same as (ΠG 3).

Any sequent of the form $\Gamma \vdash \mathfrak{A}$ is called *provable* in **TG*LK** if and only if there is a proof-note Π of **TG*LK** such that $\Pi(\emptyset)$ is the sequent $\Gamma \vdash \mathfrak{A}$. Now, let $\Gamma \vdash \mathfrak{A}_1, \dots, \mathfrak{A}_s$ be any sequent, and let \mathfrak{A} stand for $\mathfrak{A}_1 \vee \dots \vee \mathfrak{A}_s$. For the case $s=0$, let \mathfrak{A} stand for \wedge . Then, we can see without difficulty that **TGLK** and **TG*LK** are mutually equivalent in the sense that any sequent $\Gamma \vdash \mathfrak{A}_1, \dots, \mathfrak{A}_s$ is provable in **TGLK** if and only if the corresponding sequent $\Gamma \vdash \mathfrak{A}$ is provable in **TG*LK**.

The next step is to show the mutual relation between **TG*LK** and **TLK**. This becomes clear when we associate with every sequent of **TG*LK** not a step of **TLK** but a whole proof-part to the step in **TLK**. To meet this situation, I will at first introduce the notions “*proof-branch*” and “*semi-proof-note*” in **TLK**. Namely, let Π be any proof-note in **TLK**, and \mathfrak{X} be the proof-tree associated with Π . For any index-word \underline{s} in \mathfrak{X} , the set $\mathfrak{X}(\underline{s})$ of index-words beginning with \underline{s} satisfies the conditions (T 1), (T 2), (T 3) (which should be read “For any index-word $\underline{s}r^iX$ in \mathfrak{X} , $\underline{s}r$ belongs to \mathfrak{X} ”), (T 4), (T 5), and (T 6). The set $\mathfrak{X}(\underline{s})$ is called the *proof-branch* of \mathfrak{X} with respect to \underline{s} . The restricted function $\Pi[\text{---}\underline{s}]$ of Π to the domain $\mathfrak{X}(\underline{s})$ is called naturally a *semi-proof-note* of Π . With every semi-proof-note $\Pi[\text{---}\underline{s}]$ of Π , we can associate a sequent $\Sigma(\underline{s})$. If $\Pi(\underline{s})$ is a denomination of the form “ $\forall t:$ ”, we disregard it. If $\Pi(\underline{s})$ is a proposition \mathfrak{A} , and Γ is the sequence of all the propositions of the form $\Pi(\underline{s}^i g)$, we

take the sequent $\Gamma \vdash \mathfrak{A}$ as $\Sigma(\underline{s})$. Any sequent of the form $\Gamma \vdash \mathfrak{A}$ is called *provable in TLK*, if and only if there is such semi-proof-note $\Pi[\text{---}\underline{s}]$ for which $\Sigma(\underline{s})$ is the sequent $\Gamma \vdash \mathfrak{A}$.

To show that *TLK* and *TG*LK* are equivalent, let us assume at first that a sequent $\Gamma \vdash \mathfrak{A}$ is provable in *TLK*. Then, we can associate with each step \underline{s} of the semi-proof-note of the sequent a sequent $\Sigma(\underline{s})$. We can prove without difficulty that the totality of the sequents $\Sigma(\underline{s})$ form a framework of a proof-note of $\Gamma \vdash \mathfrak{A}$ in *TG*LK*, although a pretty long series of verifications are necessary for that. To illustrate this by an example only, let us take up a deduction by the inference rule (E, e, \forall) . In this case, the deduction in *TLK* must have the form

$$\Sigma(\underline{s}): \Gamma \vdash \mathfrak{A}, \quad \Sigma(\underline{s}E): \Gamma \vdash (\exists x)\mathfrak{B}(x), \quad \Sigma(\underline{s}e): \Gamma, \mathfrak{B}(t) \vdash \mathfrak{A},$$

where t occurs neither in Γ , nor in $(\exists x)\mathfrak{B}(x)$, nor in \mathfrak{A} . By making use of the inference rules (E) and (S', S'') of *TG*LK*, we can deduce $\Gamma \vdash \mathfrak{A}$ from $\Gamma \vdash (\exists x)\mathfrak{B}(x)$ and $\Gamma, \mathfrak{B}(t) \vdash \mathfrak{A}$ also in *TG*LK*.

Next, let us assume that we have a proof-note Π of a sequent $\Gamma \vdash \mathfrak{A}$ in *TG*LK*. Then, I can show that there is a semi-proof-note Φ of the sequent $\vdash \Gamma \rightarrow \mathfrak{A}$, where $\Gamma \rightarrow \mathfrak{A}$ stands for the proposition $\mathfrak{C}_1 \rightarrow (\mathfrak{C}_2 \rightarrow (\dots (\mathfrak{C}_n \rightarrow \mathfrak{A}) \dots))$ assuming that Γ is the sequence $\mathfrak{C}_1, \dots, \mathfrak{C}_n$. Accordingly, the sequent $\Gamma \vdash \mathfrak{A}$ corresponds to the index-word \underline{j} of the length n consisting exclusively of i^*s . Any proposition of the form $\Phi(\underline{j}^{k-1}g)$ for $\underline{j}^{k-1}g$ in $\mathfrak{X}(\underline{j})$ is \mathfrak{C}_k ($k \leq n$). Evidently, any fundamental sequent of *TG*LK* can be transformed into a semi-proof-note of this kind by making use of the inference rules (F) , (\wedge) , and (P) . Accordingly, we have only to confirm that, for every deduced sequent (a sequent indicated by an index-word ending with neither F , nor \wedge , nor P), we can produce a semi-proof-note of this kind for the sequent by assuming that there is a semi-proof-note of the same kind for each sequent from which the above sequent is deduced. In reality, we need a long series of easy verifications. It would be enough to illustrate the verification by two examples.

The first example is a deduction by the inference rule (I, I') . Namely, let

$$(\underline{s}) \Gamma, \mathfrak{A} \rightarrow \mathfrak{B} \vdash \mathfrak{C}, \quad (\underline{s}I') \Gamma \vdash \mathfrak{A}, \quad (\underline{s}I'') \Gamma, \mathfrak{B} \vdash \mathfrak{C}$$

be the deduction in question in *TG*LK*, and Φ' and Φ'' be semi-proof-note of the same kind for the sequents $(\underline{s}I')$ and $(\underline{s}I'')$, respectively, and

$\mathfrak{X}'(j)$ and $\mathfrak{X}''(ji^*)$ be the proof-branches associated to the semi-proof-note Φ' and Φ'' , respectively. Here, we assume that Γ is a sequence of n propositions, and j is the sequence of n i^* 's.

Now, we produce a new semi-proof-note Φ of the sequent $\Gamma, \mathfrak{A} \rightarrow \mathfrak{B} \vdash \mathfrak{C}$ and the proof-branch $\mathfrak{X}(ji^*)$ associated with it in **TLK** as follows:

1) All the index-words \underline{s} of $\mathfrak{X}''(ji^*)$ belongs to $\mathfrak{X}(ji^*)$ except such \underline{s} that begins with " $j-i^*$ ". For any index-word \underline{s} of the form $j-i^*g$ in $\mathfrak{X}''(ji^*)$ which surely indicates the proposition \mathfrak{B} , the index-word $\underline{j}i^*g$ belongs to $\mathfrak{X}(ji^*)$. In both cases, $\Phi(\underline{s})$ is the same as $\Phi''(\underline{s})$.

2) All the index-words of the form $\underline{j}^p i^* g I' h$ belong to $\mathfrak{X}(ji^*)$ for any index-word of the form $\underline{j}^p h$ in $\mathfrak{X}'(j)$ and for any index-word of the form $j-i^*g$ in $\mathfrak{X}''(ji^*)$. $\Phi(\underline{j}^p i^* g I' h)$ is the same as $\Phi'(\underline{j}^p h)$.

3) All the index-words of the form $j-i^*g I''$ belong to $\mathfrak{X}(ji^*)$ for any index-word of the form $j-i^*g$ in $\mathfrak{X}''(ji^*)$. $\Phi(j-i^*g I'')$ is the proposition $\mathfrak{A} \rightarrow \mathfrak{B}$.

We can easily confirm that Φ is a semi-proof-note of the requested kind and $\mathfrak{X}(ji^*)$ is a proof-branch associated with it.

Another example is a deduction by the inference rule (U^*). Namely, let

$$(\underline{s}) \Gamma \vdash (x)\mathfrak{A}(x), \quad (\underline{s}U^*) \Gamma \vdash \mathfrak{A}(t)$$

be the deduction in question in **TG*LK**. Then, we can assume that there is a semi-proof-note Φ' of $\Gamma \vdash \mathfrak{A}(t)$ and a proof-branch $\mathfrak{X}'(j)$ associated with it. The variable t does not occur in any $\Phi'(j^-g)$ for any j^-g in $\mathfrak{X}'(j)$. Then, we can produce a semi-proof-note Φ of $\Gamma \vdash (x)\mathfrak{A}(x)$ and a proof-branch $\mathfrak{X}(j)$ associated with it as follows:

1) $\Phi(j)$ is the proposition $(x)\mathfrak{A}(x)$.

2) All the index-words of the form $\underline{j}^p U^* g$ belong to $\mathfrak{X}(j)$ for any index-word $\underline{j}^p g$ in $\mathfrak{X}'(j)$. $\Phi(\underline{j}^p U^* g)$ is the same as $\Phi'(\underline{j}^p g)$.

3) The index-word $j\forall$ belongs to $\mathfrak{X}(j)$. $\Phi(j\forall)$ is naturally the denomination " $\forall t$ ".

We can easily confirm that Φ is a semi-proof-note of the requested kind and $\mathfrak{X}(j)$ is a proof-branch associated with it, respectively.

Lastly, I will show the equivalence of **TLK** and **PLK**.

Namely, let us assume at first that we have a proof-note Π of a proposition \mathfrak{A} in **TLK**, and let \mathfrak{X} be the proof-tree associated with Π . Let us define the rank of any index-word by

$\text{Rank}(\underline{s}) =$ the number of letters i^*, d', d'', e, U^* , and \forall in \underline{s} ,

$\text{Rank}(\underline{s} - \underline{g}) =$ (the number of letters i^*, d', d'', e , and U^* in \underline{s}) + 1.

Let \mathfrak{X}_r be the set of index-words of the rank r in \mathfrak{X} . With any index-word \underline{s}^p in \mathfrak{X}_{r+1} , we can associate an index-word \underline{g} in \mathfrak{X}_r , which is obtained from \underline{s}^p by deleting the letters after “-” (including “-” itself), if any, or by deleting the letters after the last letter i^*, d', d'', e, U^* , or \forall (including the letter itself) in \underline{s}^p . The operation from \underline{s}^p to \underline{g} is denoted by f ($\underline{g} = f(\underline{s}^p)$).

Now, for any index-word of \underline{g} , let $\mathfrak{B}(\underline{g})$ be the set of inverse images of \underline{g} with respect to the function f , *i.e.*, the set of all such \underline{s}^p in \mathfrak{X} that satisfy $\underline{g} = f(\underline{s}^p)$. With any index-word of \mathfrak{X} , I will associate an index-word in **PLK** recursively by the following rules:

1) Arrange all the index-words of \mathfrak{X}_0 in a series, placing longer ones before shorter ones. Let $\underline{s}_1, \dots, \underline{s}_n, \underline{s}$ be the series thus arranged, where \underline{s} is surely the null sequence. Then, we associate the letters b, c, \dots in the lexicographic order of letters up to the n -th letter with $\underline{s}_1, \dots, \underline{s}_n$, respectively, and associate \in to the null sequence \underline{s} . Propositions indicated by the index-words remain unchanged.

2) Assume that we have already associated an index-word of **PLK** with any one of letters belonging to any one of $\mathfrak{X}_0, \dots, \mathfrak{X}_r$. I will give a rule of association of an index-word to any index-word of **PLK** by giving a rule of association of an index-word in **PLK** to every index-word of $\mathfrak{B}(\underline{g})$ for any index-word \underline{g} in \mathfrak{X}_r . Let \underline{s} be the index-word in **PLK** associated with \underline{g} . If $\mathfrak{B}(\underline{g})$ is not empty, only the following four cases are possible:

2.1) $\mathfrak{B}(\underline{g})$ contains $\underline{g}i^*$. In this case, associate $\underline{s}A$ with all the index-words of the form $\underline{g} - i^*v$. This is possible because all the index-words of this form indicate the same proposition by assumption. All the other index-words of $\mathfrak{B}(\underline{g})$ begin with $\underline{g}i^*$. Arrange these index-words in a series placing longer ones before shorter ones. Associate with the series of the index-words of this kind, the series $\underline{s}A, \dots, \underline{s}\in$ of index-words of **PLK** respectively. Propositions indicated by the index-words remain unchanged.

2.2) $\mathfrak{B}(\underline{g})$ contains $\underline{g}d'$ and $\underline{g}d''$. In this case, associate $\underline{s}A'$ with all the index-words of the form $\underline{g} - d'v$ and $\underline{s}A''$ with all the index-words of the form $\underline{g} - d''w$. This is possible because all the index-words of the form $\underline{g} - d'v$ indicate the same proposition, and all the index-words of the

form $\underline{g}-d''w$ indicate the same proposition by assumption. $\mathfrak{B}(\underline{g})$ can be arranged in two series of index-words, the one is a series of index-words beginning with $\underline{g}d'$, and the other beginning with $\underline{g}d''$, in each series placing longer ones before shorter ones. Then, associate the index-words $\underline{s}b', \dots, \underline{s}\in'$ and $\underline{s}b'', \dots, \underline{s}\in''$ of *PLK* with these series of index-words, respectively. Propositions indicated by these index-words remain unchanged.

2.3) $\mathfrak{B}(\underline{g})$ contains $\underline{g}e$ and $\underline{g}v$. In this case, delete $\underline{g}v$, which indicates a denomination of the form “ $\forall t:$ ”, and associate $\underline{s}A$ with all the index-words of the form $\underline{g}-e\underline{v}$. This is possible because all the index-words of this form indicate the same proposition, say $\mathfrak{A}(t)$, by assumption. Arrange the index-words of $\mathfrak{B}(\underline{g})$ beginning with $\underline{g}e$ in a series placing longer ones before shorter ones. Associate with the series of index-words, the series $\underline{s}b, \dots, \underline{s}\in$ of index-words of *PLK*, respectively. The proposition indicated by $\underline{s}A$ becomes the denomination “ $\forall t: \mathfrak{A}(t)$ ” (originally, the proposition $\mathfrak{A}(t)$ of $\underline{g}-e\underline{v}$). Propositions indicated by the other index-words remain unchanged.

2.4) $\mathfrak{B}(\underline{g})$ contains $\underline{g}U^*$. In this case, associate $\underline{s}A$ with $\underline{g}v$, which indicates a denomination of the form “ $\forall t:$ ”. Arrange all the index-words of $\mathfrak{B}(\underline{g})$ beginning with $\underline{g}U^*$ in a series placing longer ones before shorter ones. Then, associate with the series of index-words the series of index-words $\underline{s}b, \dots, \underline{s}\in$ of *PLK*, respectively. Propositions and denominations indicated by the index-words remain unchanged.

It can be confirmed without difficulty that we have a correct proof-note Φ of the proposition \mathfrak{A} in *PLK*.

Now, conversely, let us assume that we have a proof-note Π of a proposition \mathfrak{A} in *PLK*. Then, let us arrange it in the fundamental order of steps. Let the number of steps be n . Starting from the step \in of Π we can make many threads of references. Namely, any sequence of steps $\underline{s}^1, \dots, \underline{s}^m$ is called a *thread* if and only if \underline{s}^1 is \in , \underline{s}^m is an assumption step of the forms $\underline{t}A, \underline{t}A'$, or $\underline{t}A''$, and \underline{s}^{i+1} is referred by \underline{s}^i for any $i = 1, \dots, m-1$. Here we also say that any step of the forms $\underline{t}\in, \underline{t}\in'$, or $\underline{t}\in''$ is referred by \underline{t} . We can see evidently that we can associate a definite index-word of *TLK* with each terms of any thread except for denominations of the form “ $\forall t:$ ”. Especially, the last term \underline{s}^m of any thread must be any one of the forms $\underline{t}A, \underline{t}A'$, or $\underline{t}A''$, and there must be a step \underline{s}^k of the forms $\underline{t}\in, \underline{t}\in'$, or $\underline{t}\in''$ in the thread, respectively. If

the index-word \underline{g} of **TLK** is associated with the step \underline{s}^{k-1} of the thread, then an index-word of the form $\underline{g}X$ must be associated with \underline{s}^k for a letter X standing for any one of the letters i^* , d' , d'' , e , or U^* . If X stands for any one of the letters i^* , d' , or d'' , an index-word of the form $\underline{g}-X\underline{h}$ should be associated with \underline{s}^m . If X stands for the letter e , an index-word of the form $\underline{g}-e\underline{h}$ should be associated with \underline{s}^m and at the same time the index-word $\underline{g}\forall$ should be attached to the thread. If X stands for the letter U^* , we delete \underline{s}^m from the thread after attaching $\underline{g}\forall$ to the thread.

For every index-word \underline{g}^p associated with a step \underline{s} , $\Phi(\underline{g}^p)$ denotes the proposition indicated by the step \underline{s} in the original proof-note Π . If \underline{g}^p is an index-word of the form $\underline{h}-e\underline{k}$, the step \underline{s} indicates a denomination of the form " $\forall t: \mathfrak{A}(t)$ ". In this case, $\Phi(\underline{g}^p)$ denotes the proposition $\mathfrak{A}(t)$. For any index-word \underline{g} , $\Phi(\underline{g}\forall)$ denotes naturally a denomination of the form " $\forall t:$ ", the variable t being the variable of the denomination " $\forall t: \mathfrak{A}(t)$ " of the step in Π with which an index-word of the form $\underline{g}-e\underline{h}$ is associated.

In every thread, steps proceed from the bottom upward in the original proof-note arranged in the fundamental order of steps. Therefore, the length of any thread does not exceed n . Accordingly, the number of all the threads with respect to the proof-note Π must be finite. So, we can thus define a proof-note Φ of the proposition \mathfrak{A} in **TLK**. It would not be necessary to go into detailed examination of the fact that Φ is really a correct proof-note in **TLK**.

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