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Maximizing the Index of Trees with Given Domination Number

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Abstract. The index of a graph G is the maximum eigenvalue of its adjacency matrix $A(G)$. In this paper we characterize the extremal tree with given domination number that attains the maximum index.

1 Introduction

Suppose that $G = (V, E)$ is a simple graph on n vertices and that $A(G)$ is the adjacency matrix of G. Since $A(G)$ is symmetric, we can write its eigenvalues as $\rho_1(G) \geq \rho_2(G) \geq \cdots \geq \rho_n(G)$, where $\rho_1(G)$ is also the *spectral radius* (or *index*) of G. If G is connected, then, by the Perron–Frobenius Theorem, $\rho_1(G)$ is simple and has a unique positive unit eigenvector $x(G) = \big(x_{v_1}(G), x_{v_2}(G), \dots, x_{v_n}(G)\big)^T$, where $x_{v_i}(G)$ corresponds to the vertex v_i $(i = 1, 2, \dots, n)$. We shall refer to such an eigenvector as the *principal eigenvector* of $A(G)$.

The investigation on the index of graphs is an important topic in the theory of graph spectra. Recently, the problem concerning graphs with maximal index of a given class of graphs has been studied by many authors. J. Guo and S. Tan [2] studied the index of trees. B. Wu, E. Xiao and Y. Hong [5] obtained the index of trees on n vertices with k pendant vertices. J. Guo and J. Shao [3] characterized the index of trees with fixed diameter. G. Xu [6] determined the index of trees with perfect matchings.

A subset S of V is a *dominating set* of G if for each $v \in V\backslash S$, there exists a vertex $u \in S$ such that v is adjacent to u. If $u \in S$ then u is a *dominating vertex*. The *domination number* of G is the minimum cardinality of a dominating set of G. In [1], L. Feng, G. Yu and Q. Li studied the Laplacian spectral radius of trees on n vertices with domination number γ, where $n = k\gamma$, $k \geq 2$ is an integer, and determined the extremal tree that attains the minimal Laplacian spectral radius when $\gamma = 2, 3, 4$.

In this paper we characterize the extremal tree with given domination number that attains the maximum index.

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2 Main Results

Throughout this paper we write $\rho(G)$ for $\rho_1(G)$, and let $N_G(v)$ denote the set of neighbors of vertex v in graph G and $\gamma(G)$ denote the domination number of G.

Lemma 2.1 *[5] Let u and v be two vertices of graph G. Suppose $v_1, v_2, \dots, v_s \in N_G(v) \backslash N_G(u)$. Let $\overline{G}$ be the graph obtained from G by deleting the edges vv_i and adding uv_i $(1 \le i \le s)$. If $x_u(G) \ge x_v(G)$, then $\rho(\overline{G}) > \rho(G)$.*

Lemma 2.2 *Suppose that G_0 is a connected graph with at least two vertices and that $u \in V(G_0)$. Let G and G^* be the graphs shown in Figure 1, where $t \ge 0, s \ge 1$. Then $\rho(G) \le \rho(G^*)$, with equality if and only if $G \cong G^*$.*

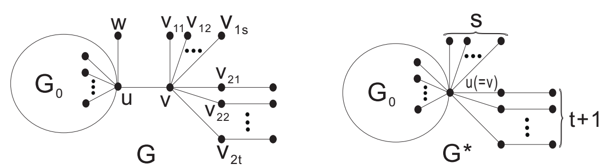

Figure 1: G and G^*.

Proof We can easily observe that

$$G - \sum_{2 \le i \le s} vv_{1i} - \sum_{1 \le i \le t} vv_{2i} + \sum_{2 \le i \le s} uv_{1i} + \sum_{1 \le i \le t} uv_{2i}$$

and

$$G - \sum_{z \in N_{G_0}(u)} uz + \sum_{z \in N_{G_0}(u)} vz$$

are both isomorphic to G^*. Therefore, whether $x_u(G) \ge x_v(G)$ or $x_v(G) \ge x_u(G)$, we always have $\rho(G) < \rho(G^*)$ by Lemma 2.1. ■

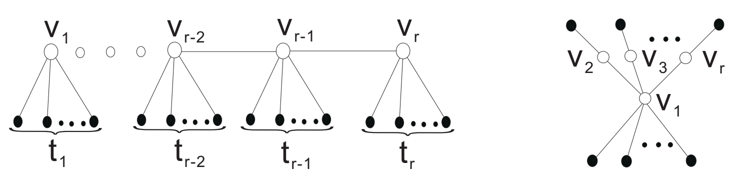

Figure 2: $T^{t_1,t_2,\dots,t_r}_{v_1,v_2,\dots,v_r}$ and T^*.

A *pendant vertex* of a graph is a vertex that only has one neighbor, and a *pendant edge* of a graph is an edge one of whose ends is a pendant vertex. In what follows we let $T^{t_1,t_2,\dots,t_r}_{v_1,v_2,\dots,v_r}$ be one of the trees as in Figure 2, where $\sum_{1 \le i \le r} t_i = n - r$, and let T^* denote the tree obtained from the star $K_{1,r-1}$ by attaching one pendant edge to each pendant vertex of $K_{1,r-1}$ and attaching $n - 2r + 1$ pendant edges to the center vertex v_1 of $K_{1,r-1}$ as in Figure 2. If $\mathscr{T}^r_n$ is the set of trees on n vertices with $\gamma(T) = r$, then it is obvious that $T^* \in \mathscr{T}^r_n$ and $T^{t_1,t_2,\dots,t_r}_{v_1,v_2,\dots,v_r} \subset \mathscr{T}^r_n$.

Lemma 2.3 *Let T^* and $T^{t_1,t_2,\ldots,t_r}_{v_1,v_2,\ldots,v_r}$ be the graphs as above, where $t_i \geq 1$ ($i = 1,2,\ldots,r$). If $T \in T^{t_1,t_2,\ldots,t_r}_{v_1,v_2,\ldots,v_r}$, then $\rho(T) \leq \rho(T^*)$, with equality if and only if $T \cong T^*$.*

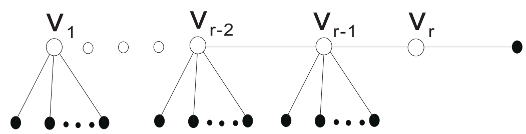

Figure 3: T^1.

Proof In T, by transferring all but one pendant edge from v_r to v_{r-1}, we obtain a graph T^1 as in Figure 3. In T^1, by transferring all pendant paths but one pendant edge from v_{r-1} to v_{r-2}, we obtain a graph T^2. Continuing this process the resulting graph will be isomorphic to T^*. By Lemma 2.2, we know that $\rho(T) < \rho(T^*)$. ∎

Suppose that G is a connected graph and that $e = uv$ is a non-pendant edge of G with $N_G(u) \cap N_G(v) = \varnothing$. Then we call the process of deleting the edge e, identifying u with v and adding a pendant edge to u $(= v)$ the DIA-*transformation* of G for uv, and denote by $\mathrm{DIA}_{uv}(G)$ the resulting graph.

Lemma 2.4 ([4]) *Let G be a connected graph and let $e = uv$ be a non-pendant edge of G with $N_G(u) \cap N_G(v) = \varnothing$. Then $\rho(G) < \rho\big(\mathrm{DIA}_{uv}(G)\big)$.*

Lemma 2.5 *Suppose that $1 \leq r \leq 3$. If $T \in \mathscr{T}_n^r$, then $\rho(T) \leq \rho(T^*)$, with equality if and only if $T \cong T^*$.*

Proof If $T \in \mathscr{T}_n^1$, then $T \cong K_{1,n-1}$. It is easily seen that $\mathscr{T}_n^2$ contains only the following three classes of the trees $\mathfrak{T}_1^2$, $\mathfrak{T}_2^2$ and $\mathfrak{T}_3^2$ as in Figure 4.

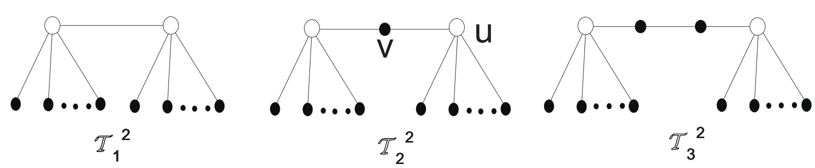

Figure 4: $\mathfrak{T}_i^2$ ($i = 1,2,3$).

It is clear that if $T_{i+1}^2 \in \mathfrak{T}_{i+1}^2$, then there must be some $T_i^2 \in \mathfrak{T}_i^2$ such that $\mathrm{DIA}_{uv}(T_{i+1}^2) \cong T_i^2$ ($i = 1,2$). Thus, we have $\rho(T_{i+1}^2) < \rho(T_i^2)$ by Lemma 2.4 ($i = 1,2$). From Lemma 2.3 we know that if $T \in \mathfrak{T}_1^2$, then $\rho(T) \leq \rho(T^*)$. Therefore, for any $T \in \mathscr{T}_n^2$, $\rho(T) \leq \rho(T^*)$, with equality if and only if $T \cong T^*$.

We can see that $\mathscr{T}_n^3$ contains only the following nine classes of the trees in Figure 5: $\mathfrak{T}_1^3$, $\mathfrak{T}_2^3$, $\mathfrak{T}_3^3$, $\mathfrak{T}_4^3$, $\mathfrak{T}_5^3$, $\mathfrak{T}_6^3$, $\mathfrak{T}_7^3$, $\mathfrak{T}_8^3$, and $\mathfrak{T}_9^3$.

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We can easily see that the following three results are true:

(i) if $T_i^3 \in \mathfrak{T}_i^3$, then there must be some $T_{i-1}^3 \in \mathfrak{T}_{i-1}^3$ such that $\mathrm{DIA}_{u_iv_i}(T_i^3) = T_{i-1}^3$ $(i = 5, 6, 8, 9)$;

(ii) if $T_i^3 \in \mathfrak{T}_i^3$, then there must be some $T_2^3 \in \mathfrak{T}_2^3$ such that $\mathrm{DIA}_{u_iv_i}(T_i^3) = T_2^3$ $(i = 3, 4)$;

(iii) if $T_i^3 \in \mathfrak{T}_i^3$, then there must be some $T_1^3 \in \mathfrak{T}_1^3$ such that $\mathrm{DIA}_{u_iv_i}(T_i^3) = T_1^3$ $(i = 2, 7)$.

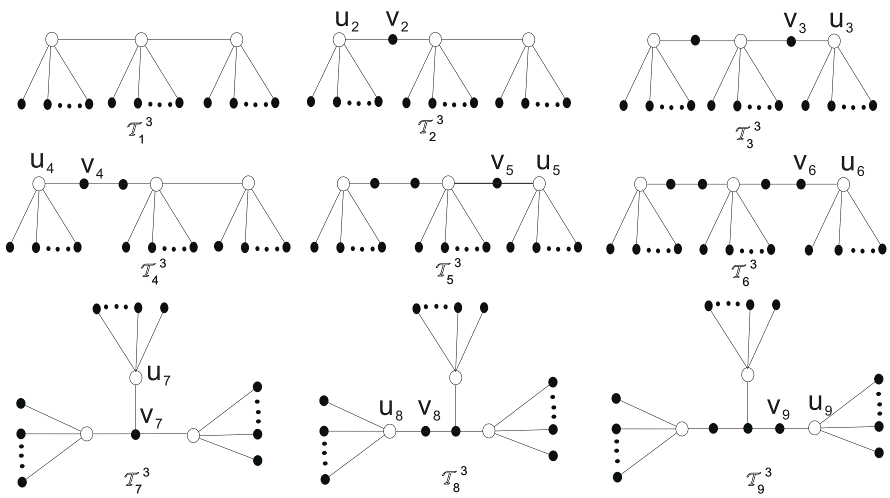

Figure 5: $\mathfrak{T}_i^3$ $(1 \le i \le 9)$.

Thus, by Lemma 2.4, we know that if $T \in \mathfrak{T}_i^3$ $(2 \le i \le 9)$, then there must be some $\widetilde{T} \in \mathfrak{T}_1^3$ such that $\rho(T) \le \rho(\widetilde{T})$. Whereas, for any $T \in \mathfrak{T}_1^3$, $\rho(T) \le \rho(T^*)$ by Lemma 2.3, we have that if $T \in \mathfrak{T}_i^3$ $(1 \le i \le 9)$, then $\rho(T) \le \rho(T^*)$, with equality if and only if $T \cong T^*$. ■

Theorem 2.6 *If $T \in \mathscr{T}_n^r$, then $\rho(T) \le \rho(T^*)$, with equality if and only if $T \cong T^*$.*

Proof If $1 \le r \le 3$, then by Lemma 2.5, the result is true. So assume $r \ge 4$. Now we choose $T_0 \in \mathscr{T}_n^r$ such that $\rho(T_0)$ is as large as possible. Let S be one of the minimum dominating sets of T_0. Then we can require that S contains no pendant vertex, since otherwise we can use its neighbor instead of it. Next we use five facts to characterize S and thus determine that the tree T_0 is isomorphic to T^*.

Fact 1 *If $v \in S$, then v is adjacent to at least one pendant vertex.*

Suppose on the contrary that there is $u \in S$ such that $N_{T_0}(u)$ contains no pendant vertex. Then there must exist a vertex $w \in N_{T_0}(u) \backslash S$ since otherwise $S \backslash u$ is a smaller dominating set of T_0. Let $T' = \mathrm{DIA}_{uw}(T_0)$. Then, by Lemma 2.4, $\rho(T_0) < \rho(T')$. Since S is clearly also a dominating set of T', we have $\gamma(T') \le r$. Suppose that S' is one of the minimum dominating sets of T'. Then $|S'| = \gamma(T')$. If $|S'| = r$, then

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$T' \in \mathscr{T}_n^r$, which contradicts the maximality of $\rho(T_0)$; if $|S'| \le r-2$, then since $S' \cup \{w\}$ is a dominating set of T_0 with $|S' \cup \{w\}| \le r-1$, we obtain a contradiction with the fact that $\gamma(T_0) = r$. Now we suppose that $\gamma(T') = r-1$.

Let T_1 and T_2 be two components of $T_0 \backslash uw$ as in Figure 6.

Figure 6: T_0 and T'.

Since the neighbor of a pendant vertex must be a dominating vertex, we have $w = u \in S'$, and so $|S' \cap V(T_1)| = |S \cap V(T_1)|$. Thus,

$$\left|S' \cap \left(V(T_2)\backslash w\right)\right| = \left|S \cap \left(V(T_2)\backslash w\right)\right| - 1.$$

This shows that $\overline{S} = \left(S \cap V(T_1)\right) \cup \left(S' \cap (V(T_2)\backslash w)\right) \cup \{w\}$ is another minimum dominating set of T_0 containing u, whereas $|N_{T_0}(u)\backslash \overline{S}| < |N_{T_0}(u)\backslash S|$. If $N_{T_0}(u)\backslash \overline{S} = \varnothing$, then $\overline{S}\backslash u$ is also a dominating set of T_0, which contradicts the minimality of S; otherwise, continuing the above process we will obtain a minimum dominating set $\widetilde{S}$ of T_0 containing u such that $N_{T_0}(u)\backslash \widetilde{S} = \varnothing$. This implies that $\widetilde{S}\backslash u$ is also a dominating set of T_0, which contradicts the minimality of S.

Fact 2 *If $v \in S$, then $N_{T_0}(v)$ only consists of pendant and dominating vertices.*

Set $N = \{y \in V(T_0) \mid y \text{ is adjacent to a pendant vertex}\}$. Since the neighbor of a pendant vertex must be a dominating vertex, $N \subseteq S$. By Fact 1, we know that $S \subseteq N$. Thus, we have $N = S$.

Suppose for a contradiction that $u \in N_{T_0}(v)$ is neither a dominating vertex nor a pendant vertex. Let $T' = \mathrm{DIA}_{vu}(T_0)$ and

$$N' = \{z \in V(T') \mid z \text{ is adjacent to a pendant vertex}\}.$$

Then it is clear that $N = N'$. Let S' be one of the minimum dominating sets of T' containing no pendant vertex. Then $N' \subseteq S'$. So far, we obtain that $S \subseteq S'$, that is, $\gamma(T') \ge \gamma(T_0)$. It has been shown in the proof of Fact 1 that $\gamma(T') \le \gamma(T_0)$. Therefore, $\gamma(T') = \gamma(T_0)$, which implies that $T' \in \mathscr{T}_n^r$. By Lemma 2.4, we have $\rho(T_0) < \rho(T')$, which contradicts the maximality of $\rho(T_0)$.

Fact 3 *There is a unique vertex v_1 in S such that $|N_{T_0}(v_1) \cap S| \ge 3$, and any other vertex u in S satisfies $|N_{T_0}(u) \cap S| \le 2$.*

If each vertex $v \in S$ satisfies $|N_{T_0}(v) \cap S| \le 2$, then, by Fact 2, T_0 is isomorphic to some tree in $T_{v_1,v_2,\dots,v_r}^{t_1,t_2,\dots,t_r}$. Whereas by Lemma 2.3, $\rho(T_0) < \rho(T^*)$, this contradicts the maximality of $\rho(T_0)$. Thus, there must be one vertex v_1 such that $|N_{T_0}(v_1) \cap S| \ge 3$.

Suppose on the contrary that there is another vertex $v_2 \in S$ such that $|N_{T_0}(v_2) \cap S| \ge 3$. Let $P_{T_0}(v_1, v_2)$ be the path between v_1 and v_2 in T_0. If $x_{v_1}(T_0) \ge x_{v_2}(T_0)$, then we can choose vertex $u \in \left(N_{T_0}(v_2) \cap S\right) \backslash V\left(P_{T_0}(v_1, v_2)\right)$ and let $\overline{T} = T_0 - v_2u + v_1u$;

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and otherwise we choose vertex $u \in \left(N_{T_0}(v_1) \cap S\right) \backslash V\left(P_{T_0}(v_1, v_2)\right)$ and let $\overline{T} = T_0 - v_1 u + v_2 u$. Obviously, $\overline{T} \in \mathscr{T}_n^r$, and by Lemma 2.1 we have $\rho(T_0) < \rho(\overline{T})$, which contradicts the maximality of $\rho(T_0)$.

Fact 4 *Let v_1 be the vertex in Fact 3. If $v \in S \backslash \{v_1\}$, then $d_{T_0}(v_1, v) = 1$.*

We choose $v_2 \in S$ such that $d_{T_0}(v_1, v_2)$ is maximum. Then $|N_{T_0}(v_2) \cap S| = 1$. Let $P_{T_0}(v_1, v_2) = v_1 w_1 \cdots w_{d-1} v_2$ be the path between v_1 and v_2 in T_0. Then, by Fact 2, we can determine that for each i, $w_i \in S$.

If $d_{T_0}(v_1, v_2) \geq 2$, then in T_0, by transferring all but one pendant edge from v_2 to w_{d-1}, we obtain a graph T_0^1. In T_0^1, by transferring all pendant paths but one pendant edge from w_{d-1} to w_{d-2}, we obtain a graph T_0^2. Continuing this process results in a graph isomorphic to T^*. By Lemma 2.2, we know that $\rho(T_0) < \rho(T^*)$. Note that $T^* \in \mathscr{T}_n^r$. This contradicts the maximality of $\rho(T_0)$.

Fact 5 *Let v_1 be the vertex in Fact 3. If $v \in S \backslash \{v_1\}$, then $N_{T_0}(v)$ contains only one pendant vertex.*

Suppose on the contrary that there is a vertex $v_2 \in S \backslash \{v_1\}$ such that $N_{T_0}(v_2)$ contains at least two pendant vertices. By transferring all but one pendant edge from v_2 to v_1, we get a tree $\overline{\overline{T}}$. Clearly, $\overline{\overline{T}} \in \mathscr{T}_n^r$. By Lemma 2.2, we have $\rho(T_0) < \rho(\overline{\overline{T}})$. This contradicts the maximality of $\rho(T_0)$.

From the above five facts, we can determine that $T_0 \cong T^*$, as acquired. ■

It is well known that $\rho_n(G) = -\rho(G)$ if and only if G is a bipartite graph. Thus, by Theorem 2.6, we obtain the following.

Theorem 2.7 *If $T \in \mathscr{T}_n^r$, then $\rho_n(T) \geq \rho_n(T^*)$, with equality if and only if $T \cong T^*$.*

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