CONCERNING BINARY RELATIONS ON CONNECTED ORDERED SPACES

I. S. KRULE

1. Introduction. In a recent paper Mostert and Shields **(4)** showed that if a space homeomorphic to the non-negative real numbers is a certain type of topological semigroup, then the semigroup must be that of the non-negative real numbers with the usual multiplication. Somewhat earlier Faucett **(2)** showed that if a compact connected ordered space is a suitably restricted topological semigroup, then it must be both topologically and algebraically the same as the unit interval of real numbers with its usual multiplication.

In studying certain binary relations on topological spaces there have become known (see, in particular, Wallace **(5)** and the author **(3)**) a number of properties analogous to those possessed by topological semigroups. Because of these analogous properties between relations and semigroups the author was motivated by the general nature of the Faucett and Mostert-Shields results (that is, that the multiplication assumed turned out to be the same as the usual multiplication) to feel that certain relations on a connected ordered space should turn out to be the same as the orders whose order topologies are the topology on the space. (Eilenberg **(1)** showed, among other things, that a connected ordered space consisting of more than one point can be endowed with exactly two orders whose order topologies are the topology on the space, and these orders must be dual to each other.) The main result of this note is a characterization of these orders as reflexive transitive relations satisfying certain topological restrictions. As an immediate consequence of this characterization there is Faucett's result **(2**, Lemma 2**)** that if a compact connected ordered space S is a topological semigroup with zero, if the zero is an endpoint, and if each element of S has a left unit, then the binary relation on S,

$$\{(a, b) \in S \times S \mid a \in S\,b\},$$

is one of the two orders on S whose order topologies are the topology on S.

2. Preliminary definitions and results. *Throughout this paper it is assumed that X is a set consisting of more than a single element.* A set L will be called *a relation on X* provided $L \subset X \times X$ (the dual of L will be denoted by $\sigma\,L$), and (X, L) will be called *an ordered set* provided L is reflexive, transitive, antisymmetric, and satisfies the requirement that if $x, y \in X$ then

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$(x, y) \in L$ or $(y, x) \in L$. If (X, L) is an ordered set, the usual terminology regarding lower bounds, upper bounds, infima, and suprema *with respect to* L of subsets of X will be used. If X is a topological space, then X will be called *an ordered space* provided there is such a relation L on X that (X, L) is an ordered set and the order topology induced by L *is* the topology on X.

If L is any relation on the topological space X, the following terminology and notation (in which, as throughout the paper, * is used to denote topological closure) will be employed:

(1) if $x \in X$ then $L(x) = \{y \in X \mid (y, x) \in L\}$;

(2) if $A \subset X$ then $L(A) = \bigcup\{L(a) \mid a \in A\}$;

(3) L will be called *continuous* (*monotone*) provided $L(A^*) \subset L(A)^*$ for each $A \subset X$ ($L(x)$ is connected for each $x \in X$);

(4) If $k \in X$ then k will be called *L-minimal* provided whenever $x \in X$ and $x \in L(k)$ then $k \in L(x)$; the set of L-minimal elements will be denoted by K_L;

(5) L will be called *closed below* (*closed above*) provided $L(x)$ ($\sigma L(x)$) is closed for each $x \in X$;

(6) if $B \subset X$ then B will be called an *L-ideal* provided $B \neq \phi$ and $L(B) \subset B$.

Lemma. *Let (X, R) be a connected ordered space, and let L be a reflexive monotone continuous relation on X. If $x \in X - K_L$ then either $L(x) \subset R(x)$ or there exists $y \in R(X) - x$ such that $x \in L(y)$.*

Proof. Suppose $y \in R(X) - x$ implies $x \in X - L(y)$. If $y \in R(x) - x$ and if $L(y) \not\subset R(x) - x$, then there exists $z \in L(y)$ such that $x \in R(z)$; thus since L is reflexive and monotone $x \in \sigma R(y) \cap R(z) \subset L(y)$, a contradiction of the supposition. Therefore $y \in R(x) - x$ implies $L(y) \subset R(x) - x$, and it follows that $L(R(x) - x) = R(x) - x$. Hence from the reflexitivity and continuity of L one has

$$L(x) \subset L(R(x)) = L((R(x) - x)^*) \subset L(R(x) - x)^* = (R(x) - x)^* = R(x),$$

which completes the proof.

Corollary. *Let (X, R) be a connected ordered space and let L be a reflexive transitive closed above monotone continuous relation on X. If $x \in X$ and if $\inf \sigma L(x)$ exists but does not belong to K_L, then $L(x) \subset R(x)$.*

Proof. Let $x \in X$ and suppose $L(x) \not\subset R(x)$ although $x_0 = \inf \sigma L(x)$ exists and $x_0 \in X - K_L$. By the lemma $x_0 \in R(x) - x$, and $x_0 \in \sigma L(x)$ since L is closed above. Thus $L(x_0) \not\subset R(x_0)$ so that again using the lemma, there exists $z \in R(x_0) - x_0$ such that $x_0 \in L(z)$. Because L is transitive it follows that $x \in L(z)$, that is, $z \in \sigma L(x)$; therefore $x_0 \neq \inf \sigma L(x)$, a contradiction. Hence it must be true that $L(x) \subset R(x)$.

3. Main result. It is well known that if L is a transitive closed below relation on the T_1 − space X and if A is a compact L-ideal, then $A \cap K_L \neq \phi$. This fact will be used in the proof of the following

THEOREM. *Let (X, R) be a connected ordered space, and let L be a relation on X. If L is reflexive, transitive, closed above and below, monotone, and continuous with $K_L = K_R$ or $K_L = K_{\sigma R}$, then $L = R$ or $L = \sigma R$ (not necessarily respectively). The converse is also true.*

Proof. The truth of the converse is obvious. The proof of the first statement is divided into two cases. It is assumed that $K_L = K_R$, for a completely dual proof holds in the dual case.

Case 1: $K_L \neq \phi$. Then K_R consists of a single endpoint of X, say e. It will be shown in this case that $L = R$, and for this it suffices to show that $L(x) = R(x)$ for each $x \in X$. Let $x \in X$. If $x = e$, then $e = K_R = K_L$ implies $L(e) = e = R(e)$. Suppose $x \neq e$, and let $x_0 = \inf \sigma L(x)$. Then $x_0 \in \sigma L(x)$, so that if $x_0 = e$ then $x \in L(e) = e$, a contradiction. Hence $x_0 \neq e$ and by the corollary $L(x) \subset R(x)$. Therefore since $L(x)$ is closed and $R(x)$ is compact, $L(x)$ is a compact L-ideal and thus $e = L(x) \cap K_L$. From the monotonicity of L it follows that $R(x) \subset L(x)$, and hence $L(x) = R(x)$.

Case 2: $K_R = \phi$.

(i) σL is monotone. To see this let $x \in X$ and suppose that $\sigma L(x)$ is not connected. Then there exists $c \in X - \sigma L(x)$ such that $A = \sigma L(x) \cap R(c)$ and $B = \sigma L(x) \cap \sigma R(c)$ are both non-void. Thus $a = \sup A$ and $b = \inf B$ exist, and it is easily seen that $a, b \in \sigma L(x)$ and

$$U = (\sigma R(a) \cap R(b)) - \{a, b\} \subset X - \sigma L(x).$$

Clearly $L(U)$ is connected so that $a, b \in X - U$ implies $L(U) \subset U$, that is, U is an L-ideal. But because L is continuous, U^* is a compact L-ideal and thus meets K_L, contrary to hypothesis. Consequently, $\sigma L(x)$ is connected and σL is monotone.

(ii) For each $x \in X$, either $L(x) = R(x)$ or $L(x) = \sigma R(x)$. Let $x \in X$. It suffices to show $L(x) \subset R(x)$ or $L(x) \subset \sigma R(x)$, for if $L(x) \subset R(x)$ but $L(x) \neq R(x)$, then $L(x)$ has a lower bound and is thus a compact L-ideal, implying $K_L \neq \phi$; and similarly, if $L(x) \subset \sigma R(x)$ then $L(x) = \sigma R(x)$. Suppose now that $L(x) \not\subset R(x)$. If $\sigma L(x)$ has a lower bound, then $\inf \sigma L(x)$ exists and by the corollary it follows that $L(x) \subset R(x)$, contrary to supposition. Therefore the monotonicity and reflexitivity of σL give $R(x) \subset \sigma L(x)$. If $\sigma L(x)$ also has no upper bound, then $X \subset \sigma L(x)$, implying $x \in K_L = \phi$. Let $x_0 = \sup \sigma L(x)$. (Note that $x_0 \neq \sup X$, for if $x_0 = \sup X$ then $X \subset \sigma L(x)$.) Then $\sigma L(x_0) \subset R(x_0)$ and hence $L(\sigma R(x_0) - x_0) \subset \sigma R(x_0) - x_0$, whence it follows that $L(x) \subset L(x_0) \subset L((\sigma R(x_0) - x_0)^*) \subset L(\sigma R(x_0) - x_0)^* \subset (\sigma R(x_0) - x_0)^* \subset \sigma R(x_0) \subset \sigma R(x)$.

(iii) If $y \in L(x) = R(x)$, then $L(y) = R(y)$. For $y \in L(x)$ implies $L(y) \subset L(x)$, and hence $L(y)$ has no lower bound since $L(y)$ is a closed L-ideal and $K_L = \phi$. Therefore (ii) implies $L(y) = R(y)$.

(iv) If $y \in L(x) = \sigma R(x)$, then $L(y) = \sigma R(y)$. The proof is similar to that of (iii).

(v) $L = R$ or $L = \sigma R$. Let $A = \{a \in X \mid L(a) = R(a)\}$ and let $B = \{b \in X \mid L(b) = \sigma R(b)\}$. Suppose $L \neq \sigma R$. Then from (ii) and (iii) it follows that A is connected and non-void. If A has no upper bound or if it has an upper bound which is also $\sup X$, then $A = X$ and $L = R$. If A has an upper bound which is not $\sup X$, then let $a_0 = \sup A$. It follows from (ii) and (iv) that B is connected and non-void; and using the continuity of L and the supposition $K_L = \phi$, it is not difficult to verify that $a_0 \in A \cap B$. But in order that $A \cap B \neq \phi$, it must be true that X consists of a single point, contrary to hypothesis. Hence $L = R$, and the proof of the theorem is complete.

4. Examples. A reflexive transitive relation satisfying all but one of the hypotheses of the theorem need be neither R nor σR. That this is indeed a fact is proved by the following set of examples in which X is, or is a subset of, the real numbers, and $R = \{(x, y) \in X \times X \mid x \leqq y\}$.

Example 1. Let X be the real numbers, and let $L = \{(x, y) \in X \times X \mid |x| \leqq |y|\}$. Then L is a reflexive transitive monotone continuous relation with closed graph (hence closed above and below), but $K_L = \{0\} \neq K_R = \phi = K_{\sigma R}$ and $R \neq L \neq \sigma R$.

Example 2. Let X and L be as in Example 1, and let $M = \sigma L$. Then M is a reflexive transitive continuous relation with closed graph, and $K_M = \phi = K_R$. However M is not monotone, and $R \neq M \neq \sigma R$.

Example 3. Let X be the real numbers, and let $L = \{(x, y) \in X \times X \mid x \leqq y \leqq 0\} \cup \{(x, y) \in X \times X \mid 0 \leqq y \leqq x\}$. Then L is a reflexive transitive monotone relation with closed graph, and $K_L = \phi = K_R$. However $R \neq L \neq \sigma R$ and L is not continuous.

Example 4. Let X be the set of real numbers x such that $0 < x \leqq 1$, and let

$$L_1 = \{(x, y) \in X \times X \mid x \leqq y < 1\} \cup \{(1, 1)\}.$$

If $L_2 = \sigma L_1$ then both L_1 and L_2 are reflexive transitive monotone continuous relations. Further, L_1 is closed below and L_2 is closed above, but L_1 is not closed above and L_2 is not closed below. Also

$$K_{L_1} = K_{L_2} = \{1\} = K_{\sigma R}$$

although $R \neq L_1 \neq \sigma R$ and $R \neq L_2 \neq \sigma R$.

5. Concerning a possible generalization. Let it be said (see Eilenberg **(1)** that a topological space X *can be ordered* provided there is a relation L on X such that (X, L) is an ordered set and the sets defined as open by the order topology are open in X under its original topology. It would be interesting to know if in the above theorem it is possible to replace "ordered space" by "space which can be ordered" and still have a true statement.

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Louisiana State University