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FREE ACTION OF FINITE GROUPS ON SPACES OF COHOMOLOGY TYPE (0, b)

K. SOMORJIT SINGH, HEMANT KUMAR SINGH
and TEJ BAHADUR SINGH

Department of Mathematics,
University of Delhi
Delhi 110007, India
e-mail: ksomorjitmaths@gmail.com, hemantksingh@maths.du.ac.in, tej_b_singh@yahoo.co.in

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Abstract. Let G be a finite group acting freely on a finitistic space X having cohomology type $(0, b)$ (for example, $\mathbb{S}^n \times \mathbb{S}^{2n}$ is a space of type $(0, 1)$ and the one-point union $\mathbb{S}^n \vee \mathbb{S}^{2n} \vee \mathbb{S}^{3n}$ is a space of type $(0, 0)$). It is known that a finite group G that contains $\mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \mathbb{Z}_p$, p a prime, cannot act freely on $\mathbb{S}^n \times \mathbb{S}^{2n}$. In this paper, we show that if a finite group G acts freely on a space of type $(0, 1)$, where n is odd, then G cannot contain $\mathbb{Z}_p \oplus \mathbb{Z}_p$, p an odd prime. For spaces of cohomology type $(0, 0)$, we show that every p-subgroup of G is either cyclic or a generalized quaternion group. Moreover, for n even, it is shown that $\mathbb{Z}_2$ is the only group that can act freely on X.

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1. Introduction. It has been an interesting problem in the theory of transformation groups to find finite groups that can occur as the fundamental groups of the spaces that have nice universal covering spaces, such as the n-sphere $\mathbb{S}^n$, $\mathbb{S}^n \times \mathbb{S}^m$, the complex projective space $\mathbb{C}P^n$, etc. This is equivalent to determining the finite groups that can act freely on these spaces and to determine the orbit spaces in those cases. The first result in this direction is due to Smith [**14**]. It has been proved that every abelian subgroup of a finite group G that acts freely on a sphere is cyclic. Further, it was shown by Milnor [**12**] that any element of order 2 in a finite group G acting freely on a mod 2 homology n-sphere lies in $Z(G)$, the centre of the group. It follows that every subgroup of order p^2 or $2p$, p a prime, of a finite group acting freely on $\mathbb{S}^n$ is cyclic. In Madsen et al. [**9**], using surgery on manifolds, it is shown that these conditions are also sufficient for the existence of a free action of G on $\mathbb{S}^n$. Thus, we have a complete solution of the problem that is known in the case of $\mathbb{S}^n$. Conner [**15**] has shown that a group containing $\mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \mathbb{Z}_p$, p a prime, cannot act freely on $\mathbb{S}^n \times \mathbb{S}^n$. A generalization of this result for free actions of finite group on the product $\mathbb{S}^n \times \mathbb{S}^m$ was obtained by Heller [**3**]. It has been shown that a finite group G that contains $\mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \mathbb{Z}_p$, p a prime, cannot act freely on $\mathbb{S}^n \times \mathbb{S}^m$.

In this paper, we study for free actions of finite group G on a space of cohomology type $\mathbb{S}^n \times \mathbb{S}^{2n}$ and $\mathbb{S}^n \vee \mathbb{S}^{2n} \vee \mathbb{S}^{3n}$. The cohomology structure of the fixed point set of a periodic map of odd order on the spaces of latter type was first studied by Dotzel and Singh [**18**]. It has been shown that $\mathbb{Z}_p$ can act freely on such spaces. Further investigations of $\mathbb{Z}_p$, p a prime, action on these spaces were done by Dotzel and Singh

[19] and Pergher et al. [16]. Recently, using the results of Pergher et al. [16], Mattos et al. [5] proved Borsuk–Ulam type theorems and their parametrized versions for $\mathbb{Z}_2$-action. For free actions of finite groups on a space of cohomology type $\mathbb{S}^n \vee \mathbb{S}^{2n} \vee \mathbb{S}^{3n}$, we show here that every p-subgroup of G is either cyclic or a generalized quaternion group. Moreover, for such spaces, it is proved that $\mathbb{Z}_2$ is the only group that can act freely on X when n is even. Moreover, if a finite group G acts freely on $\mathbb{S}^n \times \mathbb{S}^{2n}$, then G cannot contain $\mathbb{Z}_p \oplus \mathbb{Z}_p$, for all odd prime p. This improves the result of Heller [3]. All spaces considered here are assumed to be finitistic: A paracompact Hausdorff space is finitistic if every open covering has a finite-dimensional refinement. Moreover, we will use Čech cohomology throughout the paper.

2. Preliminaries. Suppose that a compact Lie group G acts on a space X. If $G \hookrightarrow E_G \to B_G$ is the universal principal G-bundle, then the *Borel construction* on X is defined as the orbit space $X_G = (X \times E_G)/G$, where G acts diagonally on the product $X \times E_G$. The projection $X \times E_G \to E_G$ gives a fibration (called the *Borel fibration*)

$$X_G \longrightarrow B_G$$

with fibre X. We will exploit the Leray–Serre spectral sequence associated with the Borel fibration $X \hookrightarrow X_G \longrightarrow B_G$. The E_2-term of this spectral sequence is given by

$$E_2^{k,l} = H^k(B_G; \mathcal{H}^l(X; \Lambda)),$$

where $\mathcal{H}^l(X; \Lambda)$ is a locally constant sheaf with stalk $H^l(X; \Lambda)$, Λ a field, and it converges to $H(X_G; \Lambda)$ as an algebra. If $\pi_1(B_G)$ acts trivially on $H^*(X; \Lambda)$, then the coefficient sheaf $\mathcal{H}(X; \Lambda)$ is constant so that

$$E_2^{k,l} = H^k(B_G; \Lambda) \otimes H^l(X; \Lambda).$$

For further details about the Leray–Serre spectral sequence, refer to Davis and Kirk [10] and McCleary [11]. For $G = \mathbb{Z}_p$, p a prime, we take $\Lambda = \mathbb{Z}_p$ and write $H^*(X)$ to mean $H^*(X; \mathbb{Z}_p)$. We recall that

$$H^*(B_G) = \begin{cases} \mathbb{Z}_p[t] & \deg t = 1 \text{ for } p = 2 \text{ and} \\ \mathbb{Z}_p[s, t] & \deg s = 1,\ \deg t = 2 \text{ for } p > 2 \text{ and } \beta_p(s) = t, \end{cases}$$

where β_p is the mod-p Bockstein homomorphism associated with the coefficient sequence $0 \to \mathbb{Z}_p \to \mathbb{Z}_{p^2} \to \mathbb{Z}_p \to 0$. We also recall that if X is a paracompact Hausdorff free G-space, G a compact Lie group, then $X/G \simeq X_G$. Note that if X is connected G-space, then $E_2^{*,0} = H^*(B_G)$. Volovikov [4] introduced the following notion of numerical index of a G-space.

Definition 2.1 ([4]). The index $i(X)$ is the smallest r such that for some k, the differential $d_r : E_r^{k-r,r-1} \longrightarrow E_r^{k,0}$ in the cohomology Leray–Serre spectral sequence of the fibration $X \overset{i}{\hookrightarrow} X_G \overset{\pi}{\longrightarrow} B_G$ is nontrivial.

Clearly, $i(X) = r$ if $E_2^{k,0} = E_3^{k,0} = \ldots = E_r^{k,0}$ for all k, and $E_r^{k,0} \neq E_{r+1}^{k,0}$ for some k. If $E_2^{*,0} = E_\infty^{*,0}$, then $i(X) = \infty$.

Proposition 2.2 ([4, Proposition 2.1]). *If there exists an equivariant map between G-spaces X and Y, then $i(X) \leq i(Y)$.*

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Given two integers a and b, a space X is said to have cohomology type (a, b) if $H^i(X, \mathbb{Z}) \cong \mathbb{Z}$ for $i = 0, 2n$, and $3n$ only. Also, the generators $x \in H^n(X; \mathbb{Z})$, $y \in H^{2n}(X; \mathbb{Z})$ and $z \in H^{3n}(X; \mathbb{Z})$ satisfy $x^2 = ay$ and $xy = bz$. For example, $\mathbb{S}^n \times \mathbb{S}^{2n}$ has type $(0, 1)$, $\mathbb{C}P^3$ and $\mathbb{Q}P^3$ have type $(1, 1)$, $\mathbb{C}P^2 \vee \mathbb{S}^6$ has type $(1, 0)$ and $\mathbb{S}^n \vee \mathbb{S}^{2n} \vee \mathbb{S}^{3n}$ has type $(0, 0)$. Such spaces were first investigated by James [**8**] and Toda [**7**].

Proposition 2.3 ([**16**, Theorem 4.1]). *If $G = \mathbb{Z}_2$ acts freely on a space X of cohomology type (a, b), where a and b are even, characterized by an integer $n > 1$, then*

$$H^*(X/G) = \mathbb{Z}_2[u, w]/\langle u^{3n+1}, w^2 + \alpha u^n w + \beta u^{2n}, u^{n+1} w\rangle,$$

where deg $u = 1$, deg $w = n$ and $\alpha, \beta \in \mathbb{Z}_2$.

Proposition 2.4 ([**19**, Theorem 2]). *Suppose that $G = \mathbb{Z}_p$, $p > 2$ a prime, act freely on a space X of cohomology type (a, b), where $a = 0$ (mod p) $= b$. Then,*

$$H^*(X/G) = \mathbb{Z}_p[u, v, w]/\langle u^2, w^2, v^{\frac{n+1}{2}} w, v^{\frac{3n+1}{2}}\rangle,$$

where deg $u = 1$, deg $w = n$, $v = \beta_p(u)$ (β_p being the mod-p Bockstein) and n is odd.

3. Main results. Let X be a space of cohomology type (a, b), characterized by an integer $n > 1$, where $a = 0$ (mod p) and $b = 0$ (mod p) or $b \neq 0$ (mod p), p a prime. We show that the group $G = \mathbb{Z}_p \oplus \mathbb{Z}_p$ cannot act freely on a space X and, for even n and $a = 0$ (mod p) $= b$, we shall show that the only finite group that acts freely on X is $\mathbb{Z}_2$. We also construct a space X of cohomology type (a, b), where $a = 0$ (mod p) $= b$, $n > 1$ and an example of free involution on X. Recall that $G = \mathbb{Z}_p$, p an odd prime, can act freely on a space X of cohomology type $(0, 0)$ [**18**].

Theorem 3.1. *Let X be a space of cohomology type (a, b), characterized by an integer $n > 1$. Then, the group $G = \mathbb{Z}_p \oplus \mathbb{Z}_p$, $p > 2$ a prime, cannot act freely on X if $a = 0$ (mod p).*

Theorem 3.2. *Let X be a space of cohomology type (a, b), characterized by an integer $n > 1$. If a and b are even integers, then the group $G = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ cannot act freely on X.*

We first prove the following propositions.

Proposition 3.3. *Let X be a space of cohomology type (a, b), characterized by an integer $n > 1$. If $G = \mathbb{Z}_p$, where $p > 2$ a prime, acts freely on X and $a = 0$ (mod p) $= b$, then n is odd and $i(X) = 3n + 1$.*

Proof. To prove this proposition, we recapitulate the proof of Theorem 2 [**19**]. Suppose G acts freely on X. Then, n must be odd, otherwise $\chi(X^G) = \chi(X) \neq 0$ mod p (by Floyd's formula). Moreover, $H^*(X_G) = 0$ in higher degree, by [**6**, Theorem 1.5, p. 374]. Clearly, the induced action of G on $H^*(X)$ is trivial, so we have $E_2^{k,l} = H^k(B_G) \otimes H^l(X)$. Let $x \in H^n(X)$, $y \in H^{2n}(X)$ and $z \in H^{3n}(X)$ be the generators. Then, $x^2 = 0$ and $xy = 0$. If $d_{n+1}(1 \otimes x) = t^{\frac{n+1}{2}} \otimes 1$, then $d_{n+1}(1 \otimes y) = 0$, and we have $0 = d_{n+1}((1 \otimes x)(1 \otimes y)) = t^{\frac{n+1}{2}} \otimes y$, a contradiction. Therefore, $d_{n+1}(1 \otimes x) = 0$. Assume now that $d_{n+1}(1 \otimes y) = 0$. And, if $d_{n+1}(1 \otimes z) = 0$, it implies that $E_{2n+1}^{*,*} = E_2^{*,*}$. Further, if $d_{2n+1}(1 \otimes y) = st^n \otimes 1$, then $0 = d_{n+1}((1 \otimes x)(1 \otimes y)) = st^n \otimes x$, a contradiction. Therefore, $d_{2n+1}(1 \otimes y) = 0$, then $E_{3n+1}^{*,*} = E_2^{*,*}$. In this case, at least nth and $2n$th lines

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of the spectral sequence survive to infinity, contradicting our hypothesis. On the other hand, if $d_{n+1}(1 \otimes z) = t^{\frac{n+1}{2}} \otimes y$, then $E_{n+2}^{k,l} = \mathbb{Z}_p$ for $k \geq 0$ and $l = 0, n$; $E_{n+2}^{k,l} = \mathbb{Z}_p$ for $0 \leq k \leq n$ and $l = 2n$ and zero otherwise. Clearly, $E_\infty^{*,*} = E_{n+2}^{*,*}$, and thus the nth and the bottom lines of the spectral sequence survive to infinity, a contradiction. Therefore, we must have $d_{n+1}(1 \otimes y) = t^{\frac{n+1}{2}} \otimes x$. We have, $E_{n+2}^{k,l} = \mathbb{Z}_p$ for $k \geq 0$ if $l = 0, 3n$, $E_{n+2}^{k,l} = \mathbb{Z}_p$ for $0 \leq k \leq n$ if $l = n$ and zero otherwise. So, we have, $E_{3n+1}^{*,*} = E_{n+2}^{*,*}$. Obviously, the differential $d_{3n+1} : E_{3n+1}^{0,3n} \to E_{3n+1}^{3n+1,0}$ must be nontrivial, otherwise the top and bottom lines of the spectral sequence survive to infinity. Hence, $i(X) = 3n + 1$. □

The proof of the following proposition is similar to the proof of previous proposition.

Proposition 3.4. *Let X be a space of cohomology type (a, b), characterized by an integer $n > 1$. If $G = \mathbb{Z}_2$ acts freely on X and a and b are even integers, then $i(X) = 3n + 1$.*

Proposition 3.5. *Suppose that $G = \mathbb{Z}_p$, $p > 2$ a prime, act freely on a space X of cohomology type (a, b), where $a = 0 (mod\, p)$ and $b \neq 0$ $(mod\, p)$. Then,*

$$H^*(X/G) = \mathbb{Z}_p[u, v, w]/\langle u^2, w^2, v^{\frac{n+1}{2}}\rangle,$$

where $\deg\ u = 1$, $\deg\ w = 2n$, $v = \beta_p(u)$ and n is odd (Thus, $X/G \sim_p L_p^n \times \mathbb{S}^{2n}$). Moreover, $i(X) = n + 1$.

Proof. As in Proposition 3.3, we see that n is odd and $E_2^{k,l} = H^k(B_G) \otimes H^l(X)$. Let $x \in H^n(X)$, $y \in H^{2n}(X)$ and $z \in H^{3n}(X)$ be the generators. Then, $x^2 = 0$ and $xy = bz$, where $0 \neq b \in \mathbb{Z}_p$. Assume that $d_{n+1}(1 \otimes x) = 0$. If $d_{n+1}(1 \otimes y) = t^{\frac{n+1}{2}} \otimes x$, then $0 = d_{n+1}((1 \otimes y)(1 \otimes y)) = 2(t^{\frac{n+1}{2}} \otimes xy)$, a contradiction. Therefore, $d_{n+1}(1 \otimes y) = 0$ and so $d_{n+1}(1 \otimes z) = 0$. Therefore, $E_{2n+1}^{*,*} = E_2^{*,*}$. Now, if $d_{2n+1}(1 \otimes y) = st^n \otimes 1$, then $0 = d_{2n+1}((1 \otimes y)(1 \otimes y)) = 2(st^n \otimes y)$, a contradiction. On the other hand, if $d_{2n+1}(1 \otimes y) = 0$, then $d_{2n+1}(1 \otimes z) = 0$. It is also obvious that $d_{3n+1}(1 \otimes z) = 0$. Thus, in this case, spectral sequence collapses and hence $H^*(X_G) \neq 0$ in higher degree, a contradiction. Therefore, $d_{n+1}(1 \otimes x) = t^{\frac{n+1}{2}} \otimes 1$. Then, $d_{n+1}(1 \otimes y) = 0$ and $d_{n+1}(1 \otimes z) = t^{\frac{n+1}{2}} \otimes \frac{1}{b}y$. We have

$$E_\infty^{k,l} = \begin{cases} \mathbb{Z}_p & 0 \leq k \leq n \text{ and } l = 0, 2n. \\ 0 & \text{otherwise.} \end{cases}$$

Consequently,

$$H^j(X_G) = \begin{cases} \mathbb{Z}_p & 0 \leq j \leq n \text{ and } 2n \leq j \leq 3n. \\ 0 & \text{otherwise.} \end{cases}$$

Let $u = \pi^*(s)$ and $v = \pi^*(t)$ be determined by $s \otimes 1$ and $t \otimes 1$, respectively. Clearly, $u^2 = v^{\frac{n+1}{2}} = 0$. Since $1 \otimes y$ is a permanent cocycle, so it determines element $w \in H^{2n}(X_G)$ such that $i^*(w) = y$. Therefore, the cohomology ring of X_G is given by

$$\mathbb{Z}_p[u, v, w]/\langle u^2, v^{\frac{n+1}{2}}, w^2\rangle,$$

where $\deg u = 1$, $\beta_p(u) = v$, $\deg w = 2n$ and n is odd. This completes the proof. □

Proof of Theorem 3.1. Suppose that $G = H \oplus K$, where $H = K = \mathbb{Z}_p$, acts freely on the space X. Then, there is a free action of K on the orbit space $Y = X/H$ via the canonical isomorphism $K \approx G/H$; in fact, for an element $Hx = [x]$ in Y, one defines $k[x] = [kx]$ for all $k \in K$. Obviously, the restriction of the action of G on X to K is free. With these actions of K on X and Y, the orbit map $\pi_H : X \to Y$ is an equivariant. So, by Proposition 2.2, $i(X) \leq i(Y)$ and by Propositions 3.3 and 3.5, we have $i(X) = n + 1$ or $3n + 1$. However, we show that $i(Y) = 2$, which contradicts the above inequality and hence the theorem. The proof of this fact is divided in two parts depending upon whether $b = 0 \pmod p$ or $b \neq 0 \pmod p$.

First, consider the case $b \neq 0 \pmod p$. By Proposition 3.5, we have $H^*(Y) = \mathbb{Z}_p[u, v, w]/\langle u^2, w^2, v^{\frac{n+1}{2}}\rangle$, where $\deg u = 1$, $\deg w = 2n$ and $v = \beta_p(u)$. Thus,

$$H^j(Y) = \begin{cases} \mathbb{Z}_p & 0 \leq j \leq n \text{ and } 2n \leq j \leq 3n \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, induced action of K on $H^*(Y)$ is trivial. Therefore, the E_2-term of the Leray–Serre spectral sequence of the fibration $Y \hookrightarrow Y_K \to B_K$ can be written as $E_2^{*,*} = H^*(B_K) \otimes H^*(Y)$. Since the action of K on Y is free, $d_r \neq 0$ for some $r \geq 2$. If $d_2(1 \otimes u) = 0$, then $1 \otimes u$ is a permanent cocycle. Hence, there exists a nonzero element $u' \in H^1(Y_G)$ such that $i^*(u') = u$. If $d_2(1 \otimes v) \neq 0$, then $E_\infty^{0,2} = E_3^{0,2} = 0$, and we see that the homomorphism $i^* : H^2(Y_G) \to H^2(Y)$ is trivial. Now, by the naturality of p-Bockstien homomorphism, we have $v = \beta_p(i^*(u')) = i^*(\beta_p(u')) = 0$, a contradiction. Therefore, $d_2(1 \otimes v) = 0$. For the same reason, we obtained $d_3(1 \otimes v) = 0$. Also, it is obvious that $d_r(1 \otimes w) = 0$ for $r \leq n$. Moreover, since $w^2 = 0$, and $\deg w$ is even, it is easily seen that $d_r(1 \otimes w) = 0$ for all $r \geq n + 1$. Thus, in this case, the spectral sequence collapses to E_2-term, contrary to fact that the action of K on Y is free. Hence, we find that $d_2(1 \otimes u) \neq 0$ and we have $i(Y) = 2$.

Next, consider the case $b = 0 \pmod p$. By Proposition 2.4, we have

$$H^*(Y) = \mathbb{Z}_p[u, v, w]/\langle u^2, w^2, v^{\frac{n+1}{2}}w, v^{\frac{3n+1}{2}}\rangle,$$

where $\deg u = 1$, $\deg w = n$, and $v = \beta_p(u)$ (β_p being the mod-p Bockstein). Thus,

$$H^j(Y) = \begin{cases} \mathbb{Z}_p & 0 \leq j \leq n-1 \text{ and } 2n+1 \leq j \leq 3n \\ \mathbb{Z}_p \oplus \mathbb{Z}_p & n \leq j \leq 2n \\ 0 & \text{otherwise.} \end{cases}$$

We observe that the action of K induced on $H^*(Y)$ is trivial. Let g be a generator of $K = \mathbb{Z}_p$. By naturality of cup product, we get $g^*(uv^j w) = g^*(u)g^*(v^j)g^*(w)$ and $g^*(v^j) = (g^*(v))^j$. Clearly, $g^*(u) = u$ and $g^*(v) = v$. If the induced action of K is nontrivial, we get $g^*(w) = uv^{\frac{n-1}{2}}$ or $uv^{\frac{n-1}{2}} + w$. If $g^*(w) = uv^{\frac{n-1}{2}}$, then $w = g^{*p}(w) = g^{*p-1}(uv^{\frac{n-1}{2}}) = uv^{\frac{n-1}{2}}$, a contradiction. If $g^*(w) = uv^{\frac{n-1}{2}} + w$, then $0 = g^*(v^{\frac{n+1}{2}}w) = v^{\frac{n+1}{2}}(uv^{\frac{n-1}{2}} + w) = uv^n$, which is again a contradiction. Therefore, it follows that the induced action of K on $H^*(Y)$ is trivial. Thus, the fibration $Y \hookrightarrow Y_K \to B_K$ has a simple local coefficient. Thus, $E_2^{k,l} = H^k(B_K) \otimes H^l(Y)$. As above, we see that if $d_2(1 \otimes u) = 0$, then the spectral sequence collapses to E_2-term, contrary to fact that the action of K on Y is free. Thus, we have $d_2(1 \otimes u) \neq 0$ and $i(Y) = 2$. □

Proof of Theorem 3.2. Suppose that $G = H \oplus K$, where $H = K = \mathbb{Z}_2$, acts freely on the space X. As in case of Theorem 3.1, there is a free action of K on $Y = X/H$, such that the map $\pi_H : X \to Y$, $x \mapsto Hx$, is K-equivariant map. So, $i(X) \leq i(Y)$. By Proposition 2.3, we have

$$H^*(Y) = \mathbb{Z}_2[u, w]/\langle u^{3n+1}, w^2 + \alpha u^n w + \beta u^{2n}, u^{n+1}w\rangle,$$

where deg $u = 1$, deg $w = n$, and $\alpha, \beta \in \mathbb{Z}_2$. Thus,

$$H^j(Y) = \begin{cases} \mathbb{Z}_2 & 0 \leq j \leq n-1 \text{ and } 2n+1 \leq j \leq 3n \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 & n \leq j \leq 2n \\ 0 & \text{otherwise.} \end{cases}$$

As in Theorem 3.1, the induced action of K on $H^*(Y)$ is trivial. So, $E_2^{k,l} = H^k(B_K) \otimes H^l(Y)$. Since K acts freely on Y, some differential $d_r : E_r^{k,l} \longrightarrow E_r^{k+r,l-r+1}$ must be nontrivial. Clearly, either $d_2(1 \otimes u) \neq 0$ or $d_2(1 \otimes u) = 0$ and $d_r(1 \otimes w) \neq 0$, for some $2 \leq r \leq n+1$. In the latter case, suppose that $d_r(1 \otimes w) = t^r \otimes u^{n+1-r}$, for some $2 \leq r \leq n+1$. Then, we have $0 = d_r((1 \otimes u^{n+1})(1 \otimes w)) = t^r \otimes u^{2n+2-r}$, a contradiction. Therefore, we must have $d_2(1 \otimes u) \neq 0$ and $i(Y) = 2$, so that $i(X) \leq 2$. This contradicts Proposition 3.4. □

Now, we prove the following corollary.

COROLLARY 3.6. *Let G be a finite group that acts freely on a space X of cohomology type (a, b), characterized by an integer $n > 1$. If $p > 2$ a prime and $a = 0$ (mod p), then every p-subgroup of G is cyclic.*

Proof. Let p be an odd prime and H a p-subgroup of a group G. Then, centre of H, $Z(H) \neq \{1\}$. Let $K \subset Z(H)$ such that $|K| = p$. If $K' \subset H$ is another subgroup such that $|K'| = p$, then $K \cap K' = \{1\}$ so that $K \oplus K' \subset H$. By Theorem 3.1, this is not possible, and hence there is only one subgroup of order p in H. From ([**13**], Theorem 5.46, p. 121), it follows that every p-subgroup of G is cyclic. □

Again, by Theorem 3.2, we have the following.

COROLLARY 3.7. *Let G be a group that acts freely on a space X of cohomology type (a, b), characterized by an integer $n > 1$, where a and b are even.*

(I) If G is finite, then every 2-subgroup of G is either cyclic or a generalized quaternion group.
(II) If G is infinite, then G cannot contain the rotation group $SO(3)$ as a subgroup.

Proof of Corollary 3.7 is similar to the proof of Corollary 3.6.

THEOREM 3.8. *Let X be a space of cohomology type (a, b), characterized by an integer $n > 1$. If n is even and $a = 0$ (mod p) $= b$, p a prime, then the only finite group that acts freely on X is $\mathbb{Z}_2$.*

Proof. Suppose that G is finite group acting freely on X. If p is an odd prime and $p \mid |G|$, then $\mathbb{Z}_p$ can be regarded as a subgroup of G, and by Flyod's formula, we have $\chi(X) = \chi(X^{\mathbb{Z}_p})$ (mod p). Since n is even, we have $\chi(X) = 4$ and therefore $X^{\mathbb{Z}_p} \neq \phi$, a contradiction. Therefore, G contains no element of odd prime order. Hence, $|G| = 2^k$, for some integer $k \geq 1$. If $k > 1$, then either G has cyclic subgroup of order 4 or G has exponent 2. In either case, there is a free action of $\mathbb{Z}_2$ on $X/\mathbb{Z}_2 = Y$. With the notations as in Theorem 3.2, we must have $d_2(1 \otimes u) = t^2 \otimes 1$. Since n is even, $0 = d_2((1 \otimes u)(1 \otimes u^{3n})) = t^2 \otimes u^{3n}$, a contradiction. Hence, G must be $\mathbb{Z}_2$. □

Now, we construct example of spaces of cohomology type (0, 0) and show that $\mathbb{Z}_2$ acts freely on these spaces.

Example. Consider the antipodal actions of $\mathbb{Z}_2$ on $\mathbb{S}^{2n}$ and $\mathbb{S}^{3n}$, where $n > 1$. Then, $\mathbb{S}^{n-1} \subset \mathbb{S}^{2n} \cap \mathbb{S}^{3n}$ is invariant under this action. So, we have a free $\mathbb{Z}_2$-action on $X = \mathbb{S}^{2n} \cup_{\mathbb{S}^{n-1}} \mathbb{S}^{3n}$, obtained by attaching the sphere $\mathbb{S}^{2n}$ and $\mathbb{S}^{3n}$ along $\mathbb{S}^{n-1}$. Let $A = X - \{p\}$ and $B = X - \{q\}$, where $p \in \mathbb{S}^{2n} - \mathbb{S}^{n-1}$ and $q \in \mathbb{S}^{3n} - \mathbb{S}^{n-1}$. Then, $A \simeq \mathbb{S}^{3n}$, $B \simeq \mathbb{S}^{2n}$ and $A \cap B \simeq \mathbb{S}^{n-1}$. By Mayer–Vietoris cohomology exact sequence, we have $H^i(X; \mathbb{Z}_p) = \mathbb{Z}_p$ for $i = 0, n, 2n, 3n$ and trivial group otherwise. Let $x \in H^n(X; \mathbb{Z}_p)$, $y \in H^{2n}(X; \mathbb{Z}_p)$ and $z \in H^{3n}(X; \mathbb{Z}_p)$ be generators. Obviously, $j^*(x) = 0$ so that $j^*(x^2) = 0$ and $j^*(xy) = 0$, where $j^* : H^k(X; \mathbb{Z}_p) \to H^k(A; \mathbb{Z}_p) \oplus H^k(B; \mathbb{Z}_p)$. Since j^* is an isomorphism for $k = 2n, 3n$, we have $x^2 = xy = 0$. Hence, X is a space of type (a, b), where $a = 0 \pmod p = b$ and $n > 1$.

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