Matrix Differentiation of S-Functions

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1. It has been shown (1; 2, 136) that if S_r, a_r, h_r denote respectively the symmetric functions $\Sigma\lambda_i^r$, $\Sigma\lambda_1\lambda_2\ldots\lambda_r$, and the homogeneous product sum of degree r of the latent roots $\lambda_1, \lambda_2, \ldots, \lambda_n$ of the matrix $X = [x_{ij}]$, then

$$\Omega S_r = rX^{r-1}, \tag{1}$$

$$\Omega a_r = (-1)^{r-1}\{X^{r-1} - a_1 X^{r-2} + a_2 X^{r-3} - \ldots + (-1)^{r-1} a_{r-1}\}, \tag{2}$$

$$\Omega h_r = X^{r-1} + h_1 X^{r-2} + h_2 X^{r-3} + \ldots + h_{r-1}, \tag{3}$$

where $\Omega = \left[\dfrac{\partial}{\partial x_{ji}}\right]$.

The symmetric functions known as Schur-functions, or S-functions, are defined (3, 82) in terms of group characters and are each associated with a partition of a number. In particular the S-functions denoted by $\{1^r\}$ and $\{r\}$ are respectively a_r and h_r. Hence (2) and (3) are particular cases of a more general expression for $\Omega\{\lambda\}$ where $\{\lambda\}$ is the S-function corresponding to any partition (λ). In this note an expression for $\Omega\{\lambda\}$ is obtained in the form of a polynomial in X with coefficients which are linear functions of S-functions, and which can be found without recourse to tables of group characters.

In addition, generalisations of Turnbull's (4) theorems

$$\Omega^2 h_{r+1} = (n+r)\,\Omega h_r,$$

and

$$\Omega^2 a_{r+1} = -(n-r)\,\Omega a_r,$$

are also obtained. It is found that in the general theorems h_{r+1} and a_{r+1} have to be replaced by certain linear functions of S-functions, and the iterative property begins with a higher power of Ω.

2. Since $$\frac{\partial S_r^a}{\partial x_{ji}} = \frac{\partial S_r^a}{\partial S_r}\frac{\partial S_r}{\partial x_{ji}}$$

then $$\Omega S_r^a = \frac{\partial S_r^a}{\partial S_r}\,\Omega S_r.$$

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Similarly $\quad \Omega\,(S_r^a\, S_s^b\, S_t^c \ldots)$

$$= \frac{\partial\,(S_r^a S_s^b S_t^c \ldots)}{\partial S_r}\,\Omega S_r + \frac{\partial\,(S_r^a S_s^b S_t^c \ldots)}{\partial S_s}\,\Omega S_s + \frac{\partial\,(S_r^a S_s^b S_t^c \ldots)}{\partial S_t}\,\Omega S_t + \ldots.$$

By definition,

$$\{\lambda\} = \frac{1}{m\,!} \sum_\rho h_\rho\, \chi_\rho^{(\lambda)} S_1^a\, S_2^b \ldots$$

where h_ρ is the order of the class ρ determined by the partition $1^a\, 2^b \ldots$ of m, and $\chi_\rho^{(\lambda)}$ is the characteristic of the class ρ in the irreducible representation of the symmetric group associated with the partition (λ) of m. Hence

$$\Omega\{\lambda\} = \frac{\partial\{\lambda\}}{\partial S_1}\,\Omega S_1 + \frac{\partial\{\lambda\}}{\partial S_2}\,\Omega S_2 + \ldots + \frac{\partial\{\lambda\}}{\partial S_n}\,\Omega S_n$$

$$= \frac{\partial\{\lambda\}}{\partial S_1} + 2\,\frac{\partial\{\lambda\}}{\partial S_2}\,X + 3\,\frac{\partial\{\lambda\}}{\partial S_3}\,X^2 + \ldots + n\,\frac{\partial\{\lambda\}}{\partial S_n}\,X^{n-1}.$$

The coefficients of the various powers of X could be found by differentiation and expressed in terms of S-functions if the appropriate tables of group characters are available. They can, however, be found more easily without the use of the tables. Any operator $\partial/\partial S_r$ can be written as

$$\frac{1}{j}\left[\chi_r^{(\lambda_1)}\, D_{\lambda_1} + \chi_r^{(\lambda_2)}\, D_{\lambda_2} + \ldots\, \chi_r^{(\lambda_j)} D_{\lambda_j}\right]$$

where $\chi_r^{(\lambda_i)}$ is the characteristic of the class (r) of the symmetric group of order $r\,!$ corresponding to the partition (λ_i), and D_{λ_i} is an operator defined as (5)

$$\sum_\rho \chi_\rho^{(\lambda_i)}\, \frac{1}{a\,!\; b\,! \ldots}\; \frac{\partial^{a+b+\ldots}}{\partial S_1^a\, \partial S_2^b \ldots}\,.$$

It so happens that the characteristics of the class (r) are $(-1)^{k-1}$ for every partition $(r-k+1,\, 1^{k-1})$, $\;k = 1, 2, \ldots, r$, and are zero for all other partitions (6, 134). Hence

$$r\,\frac{\partial}{\partial S_r} = D_r - D_{r-1,1} + D_{r-2,1^2} - D_{r-3,1^3} + \ldots + (-1)^{r-1} D_{1^r}\,.$$

The effect of D_μ on $\{\lambda\}$ is (5)

$$D_\mu\{\lambda\} = \Sigma\, g_{\mu\nu\lambda}\{\nu\}$$

where $g_{\mu\nu\lambda}$ is the coefficient of $\{\lambda\}$ in $\{\mu\}\{\nu\}$. Hence the coefficients in

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$\Omega\{\lambda\}$ can be found from a knowledge of products of S-functions, without using the tables of group characters. For example,

$$\Omega\{2^21^2\} = D_1\{2^21^2\} + (D_2 - D_{1^2})\{2^21^2\}X + (D_3 - D_{21} + D_{1^3})\{2^21^2\}X^2$$
$$+ (D_4 - D_{31} + D_{21^2} - D_{1^4})\{2^21^2\}X^3 + (D_5 - D_{41} + D_{31^2} - D_{21^3} + D_{1^5})\{2^21^2\}X^4$$
$$= (\{21^3\} + \{2^21\}) - (\{1^4\} + \{2^2\})X + \{2\}X^3 - \{1\}X^4.$$

When the number of parts in a partition exceeds n, the corresponding S-function vanishes identically (3, 91), and the expression for $\Omega\{\lambda\}$ then reduces to a polynomial identity in X, which must of course contain the characteristic function of X as a factor. Thus in the above example, if $n = 3$, then

$$\{1\}X^4 - \{2\}X^3 + \{2^2\}X - \{2^21\} = 0,$$

which is equivalent to

$$(X^3 - \{1\}X^2 + \{1^2\}X - \{1^3\})(\{1\}X + \{1^2\}) = 0.$$

3. To extend Turnbull's results

$$\Omega^2\{p\} = (n + p - 1)\Omega\{p - 1\}, \tag{4}$$

$$\Omega^2\{1^q\} = -(n - q + 1)\Omega\{1^{q-1}\} \tag{5}$$

a stage further, operate twice with Ω on the relation

$$\{p1^q\} + \{p + 1, 1^{q-1}\} = \{p\}\{1^q\},$$

obtaining

$$\Omega\{p1^q\} + \Omega\{p + 1, 1^{q-1}\} = \{p\}\Omega\{1^q\} + \{1^q\}\Omega\{p\}$$

and

$$\Omega^2\{p1^q\} + \Omega^2\{p + 1, 1^{q-1}\} = 2\Omega\{p\}\Omega\{1^q\} + \{p\}\Omega^2\{1^q\} + \{1^q\}\Omega^2\{p\}$$
$$= 2\Omega\{p\}\Omega\{1^q\} - (n - q + 1)\{p\}\Omega\{1^{q-1}\} + (n+p-1)\{1^q\}\Omega\{p-1\}. \tag{6}$$

When $q = 1$, this becomes

$$\Omega^2\{p1\} = 2\Omega\{p\} + (n + p - 1)\{1\}\Omega\{p - 1\} - (n + p)\Omega\{p\}$$
$$= (n + p - 1)\{1\}\Omega\{p - 1\} - (n + p - 2)\Omega\{p\}.$$

Again applying Ω, we have

$$\Omega^3\{p1\} = (n + p - 1)[\Omega\{p - 1\} + (n + p - 2)\{1\}\Omega\{p - 2\}]$$
$$- (n + p - 2)(n + p - 1)\Omega\{p - 1\}$$
$$= (n + p - 1)[(n + p - 2)\{1\}\Omega\{p - 2\} - (n + p - 3)\Omega\{p-1\}]$$
$$= (n + p - 1)\Omega^2\{p - 1, 1\}.$$

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Hence (4), which may be written as

$$\Omega^r\{p\} = (n+p-1)\,\Omega^{r-1}\{p-1\}, \qquad p > 0,\ r \geqq 2,$$

has as an extension the theorem that

$$\Omega^r\{p1\} = (n+p-1)\,\Omega^{r-1}\{p-1,\ 1\}, \qquad p > 1,\ r \geqq 3. \tag{7}$$

Similarly, by taking $p = 1$ in (6), we have

$$\Omega^2\{2,\ 1^{q-1}\} = (n-q+2)\,\Omega\{1^q\} - (n-q+1)\{1\}\,\Omega\{1^{q-1}\}$$

and

$$\begin{aligned}\Omega^3\{2,\ 1^{q-1}\} &= -(n-q+2)(n-q+1)\,\Omega\{1^{q-1}\} \\ &\qquad - (n-q+1)\,[\Omega\{1^{q-1}\} - (n-q+2)\{1\}\,\Omega\{1^{q-2}\}] \\ &= -(n-q+1)\,[(n-q+3)\,\Omega\{1^{q-1}\} - (n-q+2)\{1\}\,\Omega\{1^{q-2}\}] \\ &= -(n-q+1)\,\Omega^2\{2,\ 1^{q-2}\}.\end{aligned}$$

If $\{2,\ 1^{q-1}\}$ is written as $\{1^q\,1\}$ in order to bring the form of the result into correspondence with (7), then (5), written as

$$\Omega^r\{1^q\} = -(n-q+1)\,\Omega^{r-1}\{1^{q-1}\}, \qquad q > 0,\ r \geqq 2,$$

has as an extension the theorem that

$$\Omega^r\{1^q\,1\} = -(n-q+1)\,\Omega^{r-1}\{1^{q-1}1\}, \qquad q > 1,\ r \geqq 3. \tag{8}$$

Taking $q = 2$ and $p = 2$ in (6) does not lead to corresponding expressions for $\Omega^r\{p1^2\}$ and $\Omega^r\{1^q1^2\}$, but the simple recursive property shown in (4), (5), (7), (8) appears again in the results of the next section, in which a new type of operand is considered.

4. Consider the expressions defined as

$$\begin{aligned}
&(p,\ 0) = \{p\}, && p > 0,\\
&(p,\ 1) = \{p\ 1\}, && p > 1,\\
&(p,\ 2) = \{p\ 2\} + \{p\ 1^2\}, && p > 2,\\
&(p,\ 3) = \{p\ 3\} + 2\{p\ 21\} + \{p1^3\}, && p > 3,\\
&(p,\ 4) = \{p\ 4\} + 3\{p\ 31\} + 2\{p\ 2^2\} + 3\{p\ 21^2\} + \{p\ 1^4\}, && p > 4,
\end{aligned}$$

. . . .

in which the coefficient of $\{p\,\lambda_1\,\lambda_2\,\lambda_3\ldots\}$ in $(p,\ m)$ is the characteristic of the class (1^m) in the irreducible representation of the symmetric group of order $m!$ corresponding to the partition $(\lambda_1\,\lambda_2\,\lambda_3\ldots)$ of m. The set of coefficients can be written down at once from the character table, as they constitute the first column of the table as usually presented (3).

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These expressions are such that

$$\{1\}(p, m) = (p, m+1) + (p+1, m).$$

Hence

$$\Omega(p, m+1) = (p, m) + \{1\}\,\Omega(p, m) - \Omega(p+1, m),$$

$$\Omega^2(p, m+1) = 2\Omega(p, m) + \{1\}\,\Omega^2(p, m) - \Omega^2(p+1, m),$$

$$\Omega^3(p, m+1) = 3\,\Omega^2(p, m) + \{1\}\,\Omega^3(p, m) - \Omega^3(p+1, m),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\Omega^r(p, m+1) = r\,\Omega^{r-1}(p, m) + \{1\}\,\Omega^r(p, m) - \Omega^r(p+1, m).$$

From (4), $\Omega^2(p, 0) = (n+p-1)\,\Omega(p-1, 0)$, and so for $m = 0$

$$\Omega^2(p, 1) = (n+p-1)\{1\}\,\Omega(p-1, 0) - (n+p-2)\,\Omega(p, 0),$$

$$\begin{aligned}\Omega^3(p, 1) &= (n+p-1)\,[(n+p-2)\{1\}\,\Omega(p-2, 0) - (n+p-3)\,\Omega(p-1, 0)]\\ &= (n+p-1)\,\Omega^2(p-1, 1),\end{aligned}$$

which is (7). For $m = 1$,

$$\begin{aligned}\Omega^4(p, 2) &= 4\Omega^3(p, 1) + \{1\}\,\Omega^4(p, 1) - \Omega^4(p+1, 1)\\ &= 3\Omega^3(p, 1) + (n+p-1)\{1\}\,\Omega^3(p-1, 1) - (n+p)\,\Omega^3(p, 1) + \Omega^3(p, 1)\\ &= (n+p-1)\,[3\Omega^2(p-1, 1) + \{1\}\,\Omega^3(p-1, 1) - \Omega^3(p, 1)]\\ &= (n+p-1)\,\Omega^3(p-1, 2).\end{aligned}$$

The generalisation is now evident and the proof follows by induction. If $\Omega^r(p, m) = (n+p-1)\,\Omega^{r-1}(p-1, m)$ for $r \geqq m+2$, then

$$\begin{aligned}\Omega^r(p, m+1) &= (r-1)\,\Omega^{r-1}(p, m) + \{1\}\,\Omega^r(p, m) - \Omega^r(p+1, m)\\ &\qquad + \Omega^{r-1}(p, m)\\ &= (n+p-1)\,[(r-1)\,\Omega^{r-2}(p-1, m) + \{1\}\,\Omega^{r-1}(p-1, m) - \Omega^{r-1}(p, m)]\\ &\qquad \text{for } r-1 \geqq m+2,\\ &= (n+p-1)\,\Omega^{r-1}(p-1, m+1), \qquad \text{for } r \geqq m+3.\end{aligned}$$

5. A corresponding result can be obtained for the conjugate partitions. For convenience in notation write the conjugate of $\{q\,\lambda_1\,\lambda_2\ldots\}$ as $\{1^q\mu_1\,\mu_2\ldots\}$, where $\{\mu_1\,\mu_2\ldots\}$ and $\{\lambda_1\,\lambda_2\ldots\}$ are conjugates. Then the expressions

$$(1^q, 0) = \{1^q\}, \qquad q > 0,$$

$$(1^q, 1) = \{1^q\,1\}, \qquad q > 1,$$

$$(1^q, 2) = \{1^q\,2\} + \{1^q\,1^2\}, \qquad q > 2,$$

$$(1^q, 3) = \{1^q\,3\} + 2\{1^q\,21\}, + \{1^q\,1^3\}, \qquad q > 3,$$

$$(1^q, 4) = \{1^q\,4\} + 3\{1^q\,31\} + 2\{1^q\,2^2\} + 3\{1^q\,21^2\} + \{1^q\,1^4\}, \qquad q > 4,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

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are such that

$$\{1\}(1^q, m) = (1^q, m+1) + (1^{q+1}, m).$$

Hence

$$\Omega^r(1^q, m+1) = r\,\Omega^{r-1}(1^q, m) + \{1\}\,\Omega^r(1^q, m) - \Omega^r(1^{q+1}, m).$$

Assume

$$\Omega^r(1^q, m) = -(n-q+1)\,\Omega^{r-1}(1^{q-1}, m) \qquad \text{for } r \geqq m+2.$$

Then

$$\begin{aligned}\Omega^r(1^q, m+1) &= (r-1)\,\Omega^{r-1}(1^q, m) + \{1\}\,\Omega^r(1^q, m) \\ &\qquad - \Omega^r(1^{q+1}, m) + \Omega^{r-1}(1^q, m) \\ &= -(n-q+1)\,[(r-1)\,\Omega^{r-2}(1^{q-1}, m) + \{1\}\,\Omega^{r-1}(1^{q-1}, m) \\ &\qquad - \Omega^{r-1}(1^q, m)] \qquad \text{for } r-1 \geqq m+2 \\ &= -(n-q+1)\,\Omega^{r-1}(1^{q-1}, m+1), \qquad \text{for } r \geqq m+3.\end{aligned}$$

Since $m = 0$ gives

$$\Omega^r(1^q, 0) = -(n-q+1)\,\Omega^{r-1}(1^{q-1}, 0) \qquad \text{for } r \geqq 2,$$

which is known to be true by (5), then the result is proved by induction.

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