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OSCULATORY PACKINGS BY SPHERES

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If U is an open set in Euclidean N-space E_N which has finite Lebesgue measure $|U|$, then a complete packing of U by open spheres is a collection $C=\{S_n\}$ of pairwise disjoint open spheres contained in U and such that $\sum_{n=1}^{\infty} |S_n|=|U|$. Such packings exist by Vitali's theorem. An osculatory packing is one in which the spheres S_n are chosen recursively so that from a certain point on S_{n+1} is the largest possible sphere contained in $R_n=U\backslash\bigcup_{k=1}^{n} S_k^-$. (Here S^- will denote the closure of a set S). We give here a simple proof of the "well-known" fact that an osculatory packing is a complete packing. Our method of proof shows also that for osculatory packings, the Hausdorff dimension of the residual set $R=U\backslash\bigcup_{n=1}^{\infty} S_n^-$ is dominated by the exponent of convergence of the radii of the S_n.

In case U is a curvilinear triangle bounded by mutually tangent circular arcs in the plane, proofs of our first theorem have appeared in the literature, for example by Kasner and Supnick [5] and by Melzak [9]. These proofs depend rather heavily on geometry and it is not entirely clear that they would generalize to the situation considered here.

In the following we shall denote by C a complete packing of U and by C_0 an osculatory packing of U; r_n is the radius of the sphere S_n in the packing $\{S_n\}$. The exponent of convergence $e(C, U)$ was introduced by Melzak in [9], and is defined by:

$$e(C, U) = \inf\left(a\colon \sum_{n=1}^{\infty} r_n^a < \infty\right). \tag{1}$$

Mergelyan [11] and Wesler [12] have shown that for $U=B_N$, the solid unit N-sphere, and any complete packing C (other than the trivial one $C=\{B_N\}$), one has $\sum_{n=1}^{\infty} r_n^{N-1}=\infty$.

Let T_2 denote a curvilinear triangle in E_2, bounded by mutually tangent circular arcs. Melzak in [9] showed that there are packings C for which $e(C, T_2)=2$, and hence packings of B_2 for which $e(C, B_2)=2$. However, for simple osculatory packings (as defined in our Definition 1), one has

$$1.035 < e(C_0, T_2) < 1.999971. \tag{2}$$

Wilker [13] showed that $e(C_0, T_2)$ is a constant independent of the radii of the circular arcs in T_2. Numerical evidence in [10] suggests that $e(C_0, T_2) \approx 1.306951$. Z. A. Melzak, in a private communication, informed me that D. M. E. Foster had

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improved the lower bound in (2) to 1.218. The writer, in [1], improved the lower bound and upper bound in (2) so that now

$$1.28467 < e(C_0, T_2) < 1.93113. \tag{2'}$$

The reader of [1], [9], [10] and [13] will note that the definitions of osculatory packing used there are somewhat less general than the definition we are proposing here. They deal with osculatory packings of disks which can be reduced to a finite number of simple osculatory packings of curvilinear triangles. This is the reason we have used $e(C_0, T_2)$ in (2) and (2′) above rather than $e(C_0, B_2)$. Using our definition of osculatory packing it is an open question as to whether $e(C_0, B_2)$ depends on the particular packing C_0.

A related constant is the Hausdorff dimension $d(C, U)$ of the residual set R. Let I_N denote the unit cube in N-space. Hirst [4] show that

$$1.001 < d(C_0, I_2) < 1.43113. \tag{3}$$

In [7], Larman proved $d(C, I_2) > 1.03$, and in [8] showed that $d(C, I_N) > (N-1) + 0.03$.

Some connection between the constants $e(C, U)$ and $d(C, U)$ was given by Larman in [6] where he showed that if

$$d^*(I_N) = \inf_C d(C, I_N), \tag{4}$$

then for any complete packing, one has

$$d^*(I_N) \le e(C, I_N). \tag{5}$$

Here we show that one has $d(C_0, U) \le e(C_0, U)$. This would follow from (5) in case $U = I_N$, if we could prove that $d^*(I_N) = d(C_0, I_N)$ for every osculatory packing of I_N.

Complete packings have been used by Davis [2] in obtaining quadrature formulae of a special type and we refer the reader to that paper for references to other applications of packings.

Main results. Our main goal is Lemma 2 from which Theorems 1 and 2 follow directly. We begin with some definitions and the preliminary Lemma 1. We denote by x, y points in E_N. For a set $S \subset E_N$, S^- is the closure of S and S^c the complement of S. The notations dist (x, y), dist (x, S) refer to the distance between the points x and y, and the distance between the point x and the set S. We write $S(a, r)$, $S^-(a, r)$ for the open and closed spheres with center a and radius r; V_N is the volume of the unit N-sphere.

LEMMA 1. *Let U be a non-empty open set of finite measure $|U|$, and let*

$$r(U) = \sup_{y \in U} \operatorname{dist}(y, U^c). \tag{6}$$

Then $r(U) < \infty$, and there are points $x \in U$ such that dist $(x, U^c) = r(U)$.

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Proof. For $y \in U^-$ let $f(y) = \text{dist}(y, U^c)$. Then f is uniformly continuous and vanishes on $U^- \backslash U$. For $y \in U$, $S(y, f(y)) \subset U$ and thus $V_N f(y)^N \leq |U|$ so $r(U) \leq (|U| V_N^{-1})^{1/N}$.

To show that the supremum in (6) is a maximum we need only show that, outside a compact set, $f(y) \leq \frac{1}{2} r(U)$. To see this, let m be such that $|S(0, m) \cap U| > (1-\alpha)|U|$ where $\alpha < 1$ is to be chosen. Then for $y \in U \backslash S(0, m)$, the volume of $S(y, f(y)) \cap S(0, m)^c$ exceeds $\frac{1}{2} V_N f(y)^N$. But this set is contained in $U \cap S(0, m)^c$ hence has volume less than $\alpha |U|$. If α is chosen appropriately,

$$f(y)^N \leq 2\alpha\, |U|\, V_N^{-1} < (\tfrac{1}{2} r(U))^N.$$

Thus f attains its maximum on the compact set $U^- \cap S^-(0, m)$, so say $f(x) = r(U)$, for $x \in U^- \cap S^-(0, m)$. But x must be in U since $f(x) = 0$ for $x \in U^- \backslash U$.

We shall continue to use the notation $r(U)$ for the quantity defined in (6) and call it the inradius of U.

Definition 1. *Let U be a non-empty open set in E_N of finite measure. A simple osculatory packing of U is a sequence (possibly finite) of N-spheres $\{S_n\}$ with radii $\{r_n\}$ such that*

(i) *$S_1 \subset U$ and $r_1 = r(U)$*
(ii) *for $n \geq 1$, $S_{n+1} \subset R_n = U \backslash \bigcup_{k=1}^n S_k^-$, and $r_{n+1} = r(R_n)$.*

Note that the existence of S_{n+1} at each step follows from Lemma 1 unless R_n is empty in which case the sequence $\{S_n\}$ is finite.

Definition 2. *An osculatory packing of U is a sequence of pairwise disjoint N-spheres $\{S_n\}$ contained in U such that for some $m \geq 1$, $\{S_n\}_{n \geq m}$ is a simple osculatory packing of R_m.*

Lemma 2. *Let U be a non-empty open set of finite measure and let $\{S_n\}$ be an osculatory packing of U, with $S_n = S(a_n, r_n)$, and m as in Definition 2. Then for any $t = m, m+1, \ldots,$*

$$U \subset \bigcup_{n=1}^{t-1} S^-(a_n, r_n) \cup \bigcup_{n=t}^{\infty} S^-(a_n, 2r_n). \tag{7}$$

Proof. It suffices to show that if $\{S_n\}$ is a simple osculatory packing of U, then

$$U \subset \bigcup_{n=1}^{\infty} S^-(a_n, 2r_n). \tag{8}$$

To obtain (7), apply (8) to $R_{t-1} = U \backslash \bigcup_{n=1}^{t-1} S^-(a_n, r_n)$ for which $\{S_n : n \geq t\}$ is a simple osculatory packing.

With this assumption, let $x \in U$ and let $b = \text{dist}(x, U^c) > 0$. Since the S_n are disjoint and contained in U, we have $\sum_{n=1}^{\infty} |S_n| \leq |U|$ so that $r_n \to 0$. Note also that the r_n form a decreasing sequence. Choose n so that $r_n < b$. If $x \in \bigcup_{k=1}^n S_k^-$ we are through, so assume this is not the case, i.e., that $x \in R_n$. Let $s = \text{dist}(x, R_n^c)$, so $s \le r(R_n) = r_{n+1}$. Then

$$\begin{aligned} s &= \text{dist}\,(x, (U \setminus \bigcup_{k=1}^{n} S_k^-)^c) \\ &= \min\{\text{dist}\,(x, U^c), \text{dist}\,(x, S_1^-), \ldots, \text{dist}\,(x, S_n^-)\} \qquad (9) \\ &= \min\{\text{dist}\,(x, S_k^-) : 1 \le k \le n\}. \end{aligned}$$

We can drop $\text{dist}\,(x, U^c)$ in the last step, since $s \le r_{n+1} \le r_n$, and we chose n so $r_n \le \text{dist}\,(x, U^c)$.

From (9) we see that there is a k with $1 \le k \le n$ such that $\text{dist}\,(x, a_k) = s + r_k$ and thus

$$\text{dist}\,(x, a_k) = s + r_k \le r_{n+1} + r_k \le 2r_k,$$

proving that $x \in S^-(a_k, 2r_k)$ and finally proving (8).

THEOREM 1. *Let $U \subset E_N$ be an open set of finite measure. Then an osculatory packing of U is a complete packing.*

Proof. Let $\{S_n\}$ be the osculatory packing. Then $\bigcup_{n=1}^{\infty} S_n \subset U$ implies $\sum_{n=1}^{\infty} |S_n| \le |U|$ so that $\sum_{n=t}^{\infty} |S_n| \to 0$ as $t \to \infty$. But Lemma 2 shows that for $t \ge m$,

$$\begin{aligned} |U| &\le \sum_{n=1}^{t-1} |S_n| + \sum_{n=t}^{\infty} 2^N |S_n| \\ &= \sum_{n=1}^{\infty} |S_n| + (2^N - 1) \sum_{n=t}^{\infty} |S_n| \qquad (10) \\ &\to \sum_{n=1}^{\infty} |S_n| \text{ as } t \to \infty, \end{aligned}$$

completing the proof.

For Theorem 2, we require the concept of Hausdorff dimension. Given a set $S \subset E_N$, and $\alpha > 0$, $\delta > 0$, let

$$H_\delta^\alpha(S) = \inf\left\{\sum_{k=1}^{\infty} (\text{diam}\,(F_k))^\alpha : \text{diam}\,(F_k) \le \delta, \bigcup_{k=1}^{\infty} F_k \supset S\right\}$$

where the F_k are closed sets. Let $H^\alpha(S) = \sup_{\delta > 0} H_\delta^\alpha(S)$, and then the Hausdorff dimension of S is the supremum of those α for which $H^\alpha(S) = \infty$.

Given a set U with a complete packing $C = \{S_n\}$ we shall denote by $d(C, U)$ the Hausdorff dimension of the residual set $R = U \setminus \bigcup_{n=1}^{\infty} S_n^-$. The exponent of convergence $e(C, U)$ is defined by (1).

THEOREM 2. *Let $C_0 = \{S_n\}$ be an osculatory packing of U. Then*

$$d(C_0, U) \le e(C_0, U).$$

Proof. By Lemma 2,

$$
\begin{aligned}
R &= \bigcap_{t=m}^{\infty} \Big(U \setminus \bigcup_{n=1}^{t-1} S_n^-\Big) \\
&\subset \bigcap_{t=m}^{\infty} \Big(\bigcup_{n=t}^{\infty} S^-(a_n, 2r_n)\Big) \qquad (11) \\
&\subset \bigcup_{n=t}^{\infty} S^-(a_n, 2r_n) \quad \text{for any } t \geq m.
\end{aligned}
$$

Given $\delta > 0$, there is a t such that $2r_n < \delta$ for $n \geq t$. Let a with $e(C, U) \leq a \leq N$ be chosen so that $\sum_{n=1}^{\infty} r_n^a < \infty$. Then

$$
\sum_{n=t}^{\infty} \{\text{diam } S^-(a_n, 2r_n)\}^a \leq 4^a \sum_{n=1}^{\infty} r_n^a = A < \infty. \tag{12}
$$

Hence (11) implies that $H_\delta^a(R) \leq A$, and since this holds for all $\delta > 0$, we have $H^a(R) \leq A$. Hence $d(C_0, U) \leq a$. Since $a \geq e(C_0, U)$ is arbitrary, we have $d(C_0, U) \leq e(C_0, U)$.

Remarks. The reader will no doubt notice the similarity between the proof of Lemma 2 and the usual proofs of Vitali's covering theorem. The proof is still valid if instead of spheres we use homothetic images of other convex bodies for our packing, with appropriate modifications in Definition 1.

If U is an arbitrary open subset of E_N with finite measure we cannot expect results such as the Mergelyan-Wesler result or Melzak's results (2), since U could be a countable disjoint union of spheres with radii r_n such that $\sum_{n=1}^{\infty} r_n^N < \infty$, but otherwise arbitrary. Even excluding this trivial case one cannot generally expect upper bounds on $e(C_0, U)$ other than $e(C_0, U) \leq N$. For an example in E_2, let $\{s_n\}$ be a decreasing sequence of numbers such that $\sum s_n^2 < \infty$ but $\sum s_n^a = \infty$ for $a < 2$. Construct a function $f(x)$ for $0 \leq x < \infty$ by setting $f(0) = f(s_1) = f(2s_1) = s_1$, $f(2s_1 + s_2) = f(2s_1 + 2s_2) = s_2, \ldots, f(2s_1 + \cdots + 2s_{n-1} + s_n) = f(2s_1 + \cdots + 2s_n) = s_n, \ldots$ and f linear between these points. Let U be the following set $U = \{(x, y) : 0 < x < \infty, |y| < f(x)\}$. Then U has finite measure, but an osculatory packing of U is easily seen to contain disks of radii $s_1, s_2, \ldots$ and hence $e(C_0, U) = 2$.

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