J. Aust. Math. Soc. **74** (2003), 421–436

ON THE ASYMPTOTIC VALUES OF LENGTH FUNCTIONS IN KRULL AND FINITELY GENERATED COMMUTATIVE MONOIDS

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(Received 4 December 2000; revised 27 March 2002)

Communicated by D. Easdown

Abstract

Let M be a commutative cancellative atomic monoid. We consider the behaviour of the asymptotic length functions $\bar{\ell}(x)$ and $\bar{L}(x)$ on M. If M is finitely generated and reduced, then we present an algorithm for the computation of both $\bar{\ell}(x)$ and $\bar{L}(x)$ where x is a nonidentity element of M. We also explore the values that the functions $\bar{\ell}(x)$ and $\bar{L}(x)$ can attain when M is a Krull monoid with torsion divisor class group, and extend a well-known result of Zaks and Skula by showing how these values can be used to characterize when M is half-factorial.

2000 *Mathematics subject classification*: primary 20M14, 20M25, 13F05, 11Y05.

Introduction

Let $\mathbb{N}$ and $\mathbb{N}^+$ represent the nonnegative integers and positive integers respectively. Call a mapping $\lambda : \mathbb{N}^+ \to \mathbb{N}^+$ *subadditive* if $\lambda(x+y) \le \lambda(x) + \lambda(y)$ for all $x, y \in \mathbb{N}^+$, in which case, by an elementary argument, $\lim_{n\to\infty} \lambda(n)/n$ exists and equals $\inf\{\lambda(n)/n \mid n \in \mathbb{N}^+\}$. Call $\Lambda : \mathbb{N}^+ \to \mathbb{N}^+ \cup \{\infty\}$ *superadditive* if $\Lambda(x+y) \ge \Lambda(x) + \Lambda(y)$ for all $x, y \in \mathbb{N}^+$, in which case $\lim_{n\to\infty} \Lambda(n)/n$ exists and equals $\sup\{\Lambda(n)/n \mid n \in \mathbb{N}^+\}$ (which is possibly ∞). Let M be an atomic monoid, that is,

This work was completed while the first author was on an Academic Leave granted by the Faculty Development Committee at Trinity University. He also wishes to thank the Departmento de Álgebra, Universidad de Granada for their support and hospitality.
The second author was supported by the project DGES PB96-1424.
The authors would like to thank P. A. García-Sánchez, J. I. García-García, W. W. Smith and the referee for their comments and suggestions.

every nonunit can be expressed as a product of irreducible elements, and x a nonunit from M. Define

$$\ell(x) = \inf X \quad \text{and} \quad L(x) = \sup X$$

where $X = \{n \in \mathbb{N}^+ \mid x = x_1 \cdots x_n \text{ with } x_i \in M \text{ irreducible}\}$. Then the mappings $n \mapsto \ell(x^n)$ and $n \mapsto L(x^n)$ are subadditive and superadditive respectively. Thus

$$\lim_{n\to\infty} \frac{\ell(x^n)}{n} = \inf\left\{ \frac{\ell(x^n)}{n} \,\middle|\, n \in \mathbb{N}^+ \right\} \quad \text{and} \quad \lim_{n\to\infty} \frac{L(x^n)}{n} = \sup\left\{ \frac{L(x^n)}{n} \,\middle|\, n \in \mathbb{N}^+ \right\}.$$

Following [2], denote these limits by $\bar{\ell}(x)$ and $\bar{L}(x)$ respectively. Their existence has been observed in [2] for multiplicative monoids of atomic domains and in [11] for a wider class of commutative monoids. Also in [2], the authors conjecture that if R is a Krull or Noetherian domain, then these limits are always positive rational numbers. In [10], this conjecture was proved for Krull domains and several kinds of Noetherian domains, but an example of Noetherian domain R having an irreducible element x with $\bar{\ell}(x) = 0$ was given. In more generality, [10, Theorem 2] actually shows that if H is an atomic monoid and $x \in H$ is a nonunit where the set

$$\{y \in H \mid y \text{ divides } x^n \text{ for some } n \geq 1\}$$

has only a finite number of non-associated irreducible elements, then $\bar{\ell}(x)$ and $\bar{L}(x)$ are positive rational numbers (this result was also independently obtained in [1, Theorem 12]).

Our purpose in writing this paper is twofold. First, in Section 1 we explore the possible values that the functions $\bar{\ell}(x)$ and $\bar{L}(x)$ can attain in Krull monoids with torsion divisor class groups. As a by-product, we obtain (in Theorem 1.4) an extension of a theorem proved independently by Zaks [17, Theorem 3.3] and Skula [15, Theorem 3.1] which characterizes certain Krull monoids which are half-factorial. We also give in Theorem 1.6 and Corollary 1.8 a 'Carlitz type' version of this result for algebraic rings of integers. We close this section by giving bounds for the values of $\bar{\ell}(x)$ and $\bar{L}(x)$ when x is irreducible and show that these bounds are the best possible. In Section 2, we build on the proof of Theorem 2 in [10] and give an algorithmic process which allows us to compute the numbers $\bar{\ell}(x)$ and $\bar{L}(x)$ for a nonidentity x in any finitely generated reduced cancellative monoid. We organize Section 2 into four subsections. After some notation and definitions in Section 2.1, Section 2.2 presents some general properties of the function $\bar{\ell}(x)$. These are then used in Section 2.3 to develop an algorithm for its computation. Section 2.4 is devoted to the development of a similar algorithm for $\bar{L}(x)$.

While the settings in each of the two sections are different, they allow us to emphasize the strong similarities and differences which they present. By [7, Proposition 1], the study of lengths of factorizations in a Krull monoid M can be reduced to that of the same study in an appropriate *block monoid* (see [9] for more information on block monoids). If the divisor class group of M is finite, then the block monoid is finitely generated and our algorithm of Section 2 can be applied. Conversely, in Section 2 we present examples to demonstrate that many of the properties proved for the functions $\bar{\ell}(x)$ and $\bar{L}(x)$ in Section 1 for Krull monoids, fail in the general finitely generated case.

1. $\bar{\ell}(x)$ and $\bar{L}(x)$ in Krull monoids

Unless otherwise noted, we assume that M is a Krull monoid with torsion divisor class group $\mathscr{C}(M)$ and set $\mathscr{A}(M)$ of irreducible elements. Hence, if x is a nonunit of M, then there exist unique prime divisors $p_1, \ldots, p_t$ and natural numbers $n_1, \ldots, n_t$ such that $x = p_1^{n_1} \cdots p_t^{n_t}$. Given a prime divisor p, let $[p]$ represent the divisor class of p in $\mathscr{C}(M)$. For x as above, set

$$k(x) = \sum_{i=1}^{t} \frac{n_i}{|[p_i]|}$$

where $|[p_i]|$ represents the order of the element $[p_i]$ in $\mathscr{C}(M)$. Setting $k(u) = 0$ if u is a unit of M defines a function from M into $\mathbb{Q}_{\geq 0}$ known in the literature as the *Zaks-Skula function* (see [6]). For a nonunit x of M, the value $k(x)$ is also known as the *cross number of* x [12]. If x and y are elements of M, then it is easy to verify that $k(xy) = k(x) + k(y)$. When k is considered as a function, it is merely an example of what is known as a *length function on* M (see [2]). It is well know that M is a half-factorial domain (that is, an atomic domain where the length of factorization of a nonzero nonunit y into irreducibles is constant) if and only if $k(x) = 1$ for every irreducible $x \in M$ (see [16, 17, 15]).

We explore further the functions $\bar{\ell}$ and $\bar{L}$ on Krull monoids with torsion divisor class group, but begin with a few general results.

BASIC LEMMA 1.1. Let M be an atomic commutative monoid, x an irreducible element of M and y a nonunit element of M.

(1) If y can be written as a product of m irreducibles (where $m \in \mathbb{N}$), then $\bar{L}(y) \geq m$ and $\bar{\ell}(y) \leq m$.

(2) $\bar{\ell}(x) \leq 1$ and $\bar{L}(x) \geq 1$. Hence, if $\bar{\ell}(x) = \bar{L}(x)$, then $\bar{\ell}(x) = \bar{L}(x) = 1$.

(3) $\bar{\ell}(y) < 1$ if and only if for some $k \in \mathbb{N}$, y^k can be written as a product of less than k irreducible factors.

(4) $\bar{L}(y) > 1$ if and only if for some $k \in \mathbb{N}$, y^k can be written as a product of more than k irreducible factors.

(5) If $\bar{\ell}(y) = \bar{L}(y)$ then every irreducible factorization of y^n (for any $n \in \mathbb{N}$) has the same length.

PROOF. Parts (1)–(4) are immediate from the definitions and facts noted in the first paragraph. For (5), suppose y^n can be factored as a product of m and t irreducibles where $m < t$. Then

$$\bar{L}(y) = \lim_{s\to\infty} \frac{L((y^n)^s)}{ns} \geq \frac{t}{n} > \frac{m}{n} \geq \lim_{s\to\infty} \frac{\ell((y^n)^s)}{ns} = \bar{\ell}(y). \qquad \square$$

PROPOSITION 1.2. *Let M be an atomic commutative monoid. The following statements are equivalent:*

(1) *M is half-factorial.*
(2) *$\bar{\ell}(x) = \bar{L}(x)$ for every nonunit $x \in M$.*

PROOF. That (1) implies (2) is obvious, and that (2) implies (1) follows from Lemma 1.1 (5). $\square$

Before proceeding, we introduce some notation. If M is a Krull monoid with $\mathscr{C}(M)$ a torsion group and x is a nonunit of M, then write

$$x = p_1 \cdots p_t \tag{1}$$

where the p_i are prime divisors of M. Let $k = \mathrm{lcm}\{|[p_1]|, \dots, |[p_t]|\}$ and for each i set $k = k_i|[p_i]|$. Then

$$x^k = (p_1^{|[p_1]|})^{k_1} \cdots (p_t^{|[p_t]|})^{k_t} \tag{2}$$

and setting $\alpha_i = p_i^{|[p_i]|}$, we have that

$$x^k = \alpha_1^{k_1} \cdots \alpha_t^{k_t} \tag{3}$$

where each $\alpha_i \in \mathscr{A}(M)$ and $k(\alpha_i) = 1$ for each i. Notice that (3) implies that

$$k(x) = \frac{k_1 + \cdots + k_t}{k}. \tag{4}$$

LEMMA 1.3. *Let M be a Krull monoid with $\mathscr{C}(M)$ a torsion group and suppose that x is nonunit element of M. Then $\bar{\ell}(x) = \bar{L}(x) = 1$ if and only if*

(1) *x is irreducible in M, and*
(2) *every irreducible divisor α of the collective powers of x has $k(\alpha) = 1$.*

PROOF. ($\Rightarrow$) That x is irreducible follows from Basic Lemma 1.1 (1). By Basic Lemma 1.1 (5), every irreducible factorization of x^n has the same length. Now, suppose that α is an irreducible divisor of some power of x (say x^t). Then $\alpha^s \mid x^{ts}$ for every $s \in \mathbb{N}$. Since every x^{ts} has unique irreducible factorization length, then so too must each α^s. By writing α^k in the form (3), we have that $k = k_1 + \cdots + k_t$ and $k(\alpha) = (k_1 + \cdots + k_t)/k = 1$.

($\Leftarrow$) We argue that conditions (1) and (2) imply that $\bar{\ell}(x) = \bar{L}(x)$. The result then follows from Basic Lemma 1.1 (2). Suppose that x is irreducible and

$$x^n = \gamma_1 \cdots \gamma_s = \beta_1 \cdots \beta_t$$

with each γ_i and β_j in $\mathscr{A}(M)$. By the properties of the Zaks-Skula function,

$$k(x^n) = nk(x) = k(\gamma_1) + \cdots + k(\gamma_s) = k(\beta_1) + \cdots + k(\beta_t)$$

and condition (2) then implies that $n = s = t$. Thus, for each n, $l(x^n) = L(x^n)$ and hence $\bar{\ell}(x) = \bar{L}(x)$. □

Lemma 1.3 allows us to extend a well known characterization of half-factorial domains (see [16, 17, 15]).

THEOREM 1.4. *Let M be a Krull monoid with $\mathscr{C}(M)$ a torsion group. The following statements are equivalent*:

(1) *M is half-factorial.*
(2) *$\bar{\ell}(x) = \bar{L}(x)$ for every nonunit $x \in M$.*
(3) *$k(x) = 1$ for every $x \in \mathscr{A}(M)$.*
(4) *$\bar{\ell}(x) = \bar{L}(x) = 1$ for every $x \in \mathscr{A}(M)$.*
(5) *$\bar{\ell}(x^t) = \bar{L}(x^t) = t$ for every $t \in \mathbb{N}$ and $x \in \mathscr{A}(M)$.*

PROOF. (1) and (2) are equivalent by Proposition 1.2. The equivalence of (1) and (3) is proved in both [17, Theorem 3.3] and [15, Theorem 3.1]. Lemma 1.3 implies the equivalence of (3) and (4). Clearly (1) implies (5) and (5) implies (4). □

We can also deduce the following from Lemma 1.3.

COROLLARY 1.5. *Let M be a Krull monoid with $\mathscr{C}(M)$ a torsion group.*

(1) *If x is irreducible and primary in M, then $\bar{\ell}(x^t) = \bar{L}(x^t) = t$ for every $t \in \mathbb{N}^+$.*
(2) *If x is primary, then $\bar{\ell}(x)$ and $\bar{L}(x)$ are positive integers.*

PROOF. By [11, Satz 10A ii)], if x is primary in M, then $x = p^r$ where p is a prime divisor of M and $|[p]|$ divides r. Suppose that x is irreducible and primary. Since every irreducible divisor of the powers of x is of the form $\alpha = p^{|[p]|}$, that $\bar{\ell}(x) = \bar{L}(x) = 1$ follows directly from Lemma 1.3, and it follows immediately for each $t \in \mathbb{N}^+$ that $\bar{\ell}(x^t) = \bar{L}(x^t) = t$. (2) now follows directly from (1). □

Theorem 1.4 and Corollary 1.5 are not valid in general (see Examples 2.12 and 2.10). If M is an atomic commutative cancellative monoid, set

$$\bar{\ell}(M) = \{\bar{\ell}(x) \mid x \text{ a nonunit in } M\},$$

and

$$\bar{L}(M) = \{\bar{L}(x) \mid x \text{ a nonunit in } M\}.$$

If M is a Krull monoid, then the results of [10] imply that both $\bar{\ell}(M)$ and $\bar{L}(M)$ are contained in $\mathbb{Q}_{>0}$, a fact that we use below without further comment. If M is half-factorial, then clearly $\bar{\ell}(M) = \bar{L}(M) = \mathbb{N}^+$. While the converse of the last statement is not true in general (see Example 1.7 below), we show that it is true under a certain assumption on a Krull monoid M.

THEOREM 1.6. *Let M be a Krull monoid with torsion divisor class group $\mathscr{C}(M)$ such that every nontrivial divisor class of M contains a prime divisor. Conditions* (1)–(5) *of Theorem* 1.4 *are also equivalent to*:

(6) *$\bar{\ell}(M)$ and $\bar{L}(M)$ are both contained in $\mathbb{N}^+$.*
(7) *$\bar{\ell}(M) = \bar{L}(M) = \mathbb{N}^+$.*

PROOF. Clearly (7) implies (6). Under our hypothesis, M must contain an irreducible primary element, and so (6) implies (7) by Corollary 1.5. We argue that (1) implies (6). If M is half-factorial and $x \in M$ can be written as a product of t irreducibles, then $l(x^n) = L(x^n) = tn$ and $\bar{\ell}(x) = \bar{L}(x) = t$; proving (6). Suppose (7) holds. To see (1), we argue that the divisor class group of M must be trivial or $\mathbb{Z}_2$. From this the result follows very easily. Suppose that $\mathscr{C}(M)$ contains an element g of order greater than 2. Let p_1 and p_2 be prime divisors of M so that $[p_1] = g$ and $[p_2] = -g$. Then $x = p_1p_2$ is irreducible in M and $\ell(x^{n|g|}) = 2n$ so $\bar{\ell}(x) = \lim_{n\to\infty} 2n/(n|g|) = 2/|g| \notin \mathbb{N}^+$. Suppose now that g_1 and g_2 are in $\mathscr{C}(M)$ with $|g_1| = |g_2| = 2$ and $g_1 \neq g_2$. If $g_3 = g_1 + g_2$ and $[p_1] = g_1$, $[p_2] = g_2$ and $[p_3] = g_3$, then $x = p_1p_2p_3$ is an irreducible element, $L(x^{2n}) = 3n$ and so $\bar{L}(x) = 3/2 \notin \mathbb{N}$. Thus $\mathscr{C}(M)$ must be trivial or $\mathbb{Z}_2$. □

EXAMPLE 1.7. We show for a general Krull monoid with torsion class group that (7) does not imply (1). Let $F = \langle p_1, \ldots, p_6 \rangle$ be the free commutative monoid on 6 generators, expressed multiplicatively, and put

$$M = \{p_1^{x_1} \cdots p_6^{x_6} \in F \mid x_1 \equiv \cdots \equiv x_6 \mod 3\}.$$

It is routine to check that M is a Krull monoid with divisor class group isomorphic to $\mathbb{Z}_3^5$ and the irreducibles are $p_1^3, \ldots, p_6^3$ and $p_1 \cdots p_6$. Further, if $\alpha = p_1^{x_1} \cdots p_6^{x_6} \in M$, then one can easily verify that

$$\bar{L}(\alpha) = (x_1 + \cdots + x_6)/3 \quad \text{and} \quad \bar{\ell}(\alpha) = \bar{L}(\alpha) - \min\{x_1, \ldots, x_6\},$$

which are positive integers. Observe also that $k(p_1 \cdots p_6) = 2$ and (as predicted by Theorem 1.4) M is not half-factorial. For example, $p_1^3 \ldots p_6^3 = (p_1 \ldots p_6)^3$ are factorizations of different length.

Theorem 1.6 can be applied to rings of algebraic integers.

COROLLARY 1.8. *Let R be a ring of integers in a finite algebraic extension of the rationals. Denote the multiplicative monoid of R by R^*. Conditions* (1)–(7) *are equivalent to*

(8) $|\mathscr{C}(R)| \leq 2$.

PROOF. Since R^* is a Krull monoid with finite divisor class group with the property that every divisor class of $\mathscr{C}(R^*)$ contains a nonzero prime divisor, we can apply Theorem 1.6. (1) and (8) are equivalent by a well-known theorem of Carlitz [3]. □

Recall that the *elasticity* of M is defined as

$$\rho(M) = \sup\left\{ \frac{n}{m} \,\middle|\, \alpha_1 \cdots \alpha_n = \beta_1 \cdots \beta_m \text{ where each } \alpha_i \text{ and } \beta_j \in \mathscr{A}(M) \right\}.$$

We use the elasticity to provide some bounds for the values $\bar{\ell}(x)$ and $\bar{L}(x)$.

PROPOSITION 1.9. *Let M be a Krull monoid with torsion divisor class group $\mathscr{C}(M)$ and finite elasticity $\rho(M)$. Suppose that x is a nonunit of M. Then,*

(a) $\bar{\ell}(x) \leq k(x) \leq \bar{L}(x)$, *and*

(b) *if x is irreducible, then* $1/\rho(M) \leq \bar{\ell}(x) \leq k(x) \leq \bar{L}(x) \leq \rho(M)$.

PROOF. (a) Let $x = p_1 \cdots p_t$ be as in (1). By (3), $x^k = \alpha_1^{k_1} \cdots \alpha_t^{k_t}$ where the α_i are primary and for every $m \in \mathbb{N}$, $x^{km} = \alpha_1^{k_1 m} \cdots \alpha_t^{k_t m}$. The last equality and Corollary 1.5 imply

$$\frac{l(x^{km})}{km} \leq \sum_{i=1}^{t} \frac{l(\alpha_i^{k_i m})}{km} \leq \sum_{i=1}^{t} \frac{1}{|[p_i]|} \frac{l(\alpha_i^{k_i m})}{k_i m} = \sum_{i=1}^{t} \frac{1}{|[p_i]|} = k(x).$$

Similarly $L(x^{km})/km \geq k(x)$.

(b) Let x be irreducible. Then $l(x^m)$, $L(x^m)$ and m represent factorization lengths of the product x^m. Hence,

$$\frac{1}{\rho(M)} \leq \frac{l(x^m)}{m} \leq \frac{L(x^m)}{m} \leq \rho(M).$$

https://doi.org/10.1017/S1446788700003396 Published online by Cambridge University Press

Applying limits yields $1/\rho(M) \leq \bar{\ell}(x) \leq \bar{L}(x) \leq \rho(M)$ and the result now follows from (a). $\square$

While we have shown previously that condition (7) of Theorem 1.6 does not in general imply condition (1), when $\mathscr{C}(M)$ is finite (7) does imply two interesting properties. Our argument will require the following lemma.

LEMMA 1.10. *Let M be a Krull monoid with torsion divisor class group and suppose that $k(y) \geq 1$ for all atoms (and hence also for all nonunits) $y \in M$. Then $\bar{L}(y) = k(y)$.*

PROOF. Let y be a nonunit of M. If $n \in \mathbb{N}^+$ and $y^n = z_1 \cdots z_t$ is a factorization of maximal length, then $nk(y) = k(y^n) = k(z_1) + \cdots + k(z_t) \geq t = L(y^n)$ so that $L(y^n)/n \leq k(y)$. Thus $\bar{L}(y) \leq k(y)$. By Proposition 1.9 (a) (noting that the elasticity is irrelevant) we get equality. $\square$

PROPOSITION 1.11. *Let M be a Krull monoid where $\mathscr{C}(M)$ is a finite group. Consider the following conditions on M.*

(a) $\bar{\ell}(M) = \bar{L}(M) = \mathbb{N}^+$.
(b) $k(y) \in \mathbb{N}^+$ *for every nonunit* $y \in M$.
(c) $\rho(M) \in \mathbb{N}^+$.

Then (a) $\Rightarrow$ (b) $\Rightarrow$ (c) *and none of these implications are reversible.*

PROOF. Suppose (a) holds. If x is an atom then $k(x) \geq 1$, because otherwise, by Proposition 1.9 (a), $\bar{\ell}(x) \leq k(x) < 1$, which contradicts (a). By Proposition 1.9, (b) holds.

Now suppose (b) holds. The finite divisor class group hypothesis implies that $\{k(x) \mid x \in \mathscr{A}(M)\}$ achieves a maximum $\mu \in \mathbb{N}^+$. If $\alpha_1 \cdots \alpha_m = \beta_1 \cdots \beta_n$, where each $\alpha_i, \beta_j \in \mathscr{A}(M)$, then $m \leq k(\alpha_1) + \cdots + k(\alpha_m) = k(\beta_1) + \cdots + k(\beta_n) \leq n\mu$, so that $m/n \leq \mu$. Hence $\rho(M) \leq \mu$. But taking $x \in \mathscr{A}(M)$ such that $k(x) = \mu$, and using the notation of (1), (3) and (4), we get that $x^k = \alpha_1^{k_1} \cdots \alpha_t^{k_t}$, so that $\rho(M) \geq (k_1 + \cdots + k_t)/k = k(x) = \mu$. Hence $\rho(M) = \mu \in \mathbb{N}^+$, and (c) is proved.

That (c) does not imply (b) follows by considering any algebraic ring of integers whose divisor class group contains an element of even order > 2. That (b) does not imply (a) follows by Example 1.12 below. $\square$

EXAMPLE 1.12. Let M be the set of nonnegative integer solution to the linear Diophantine equation $15x_1 + 10x_2 + 6x_3 + x_4 = 30x_5$. By [5, Theorem 1.3], M is a Krull monoid with $\mathscr{C}(M) = \mathbb{Z}_{30}$ and the prime divisors $p_1, \ldots, p_4$ can be viewed such that $[p_1] = \overline{15}$, $[p_2] = \overline{10}$, $[p_3] = \overline{6}$ and $[p_4] = \overline{1}$ in $\mathscr{C}(M)$. Using elementary number theory (or the algorithm suggested in [17, Chapter 7]) one can verify easily that $v = (1, 2, 4, 1, 2)$ is the only irreducible (corresponding to $p_1p_2^2p_3^4p_4$ and having $k(v) = 2$) with Zaks-Skula value $\neq 1$. Consider $u = (1, 2, 3, 7, 2) \in M$. By considering the 3rd and 5th coordinates of u and v, it is straightforward then to verify that $\ell(u^k) = 2k - \lfloor 3k/4 \rfloor$, so that $\bar{\ell}(u) = 5/4 \notin \mathbb{N}^+$.

For more information on Krull monoids which satisfy condition (b) of Proposition 1.11, the interested reader is referred to [8, Section 4]. While the implications in Proposition 1.11 are not reversible, there is a partial converse involving the first implication.

COROLLARY 1.13. *Let M be a Krull monoid with finite divisor class group. If $k(y) \in \mathbb{N}$ for every nonunit $y \in M$, then $\bar{L}(M) = \mathbb{N}$.*

PROOF. If $k(y) \in \mathbb{N}$ for all nonunits $y \in M$, then $k(y) \geq 1$ for each such y. Now, $\bar{L}(y) = k(y)$ by Lemma 1.10 and the result follows. □

Recall that if G is a finite abelian group, then the *Davenport constant of G* (denoted $D(G)$) is the length of the longest finite sequence of elements of G that sums to 0, which has no nonempty subsum equal to 0.

COROLLARY 1.14. *Let M be a Krull monoid with finite divisor class group $\mathscr{C}(M)$. If x is irreducible in M then,*

$$\frac{2}{D(\mathscr{C}(M))} \leq \bar{\ell}(x) \leq k(x) \leq \bar{L}(x) \leq \frac{D(\mathscr{C}(M))}{2}.$$

PROOF. This follows directly from Proposition 1.9 by using the well-known fact that $\rho(M) \leq D(\mathscr{C}(M))/2$ (see [6, Introduction]). □

EXAMPLE 1.15. In general, the bounds presented in Corollary 1.14 are the best possible. We have already seen in Lemma 1.10 an example where there are irreducibles x with $k(x) = \bar{L}(x)$. It is easy to argue that if x is a primary element in a Krull monoid with finite divisor class group, then $k(x) = \bar{\ell}(x)$. Suppose that M is an algebraic ring of integers. If $\mathscr{C}(M) = \mathbb{Z}_n$ with $n > 2$, then let p_1 and q_1 be prime divisors of M with $[p_1] = \bar{1}$ and $[q_1] = \overline{n-1}$. Setting $x = p_1q_1$, it is easy to see that $l((x^n)^k) = 2k$ for each positive integer k. Thus $\bar{\ell}(x) = \lim_{k\to\infty} l((x^n)^k)/nk = 2/n = 2/D(\mathscr{C}(M))$. Now, suppose that $\mathscr{C}(M)$ is an elementary 2-group of rank $t > 1$. Observe that $D(\mathscr{C}(M)) = t + 1$ [4, Theorem 1.4] and let $p_1, \ldots, p_{t+1}$ be a sequence of prime divisors in M such that $[p_1] + \cdots + [p_{t+1}] = 0$ in $\mathscr{C}(M)$ and no nonempty proper subsum of the $[p_i]$'s is zero. Then $y = p_1 \cdots p_{t+1}$ is irreducible in M and $y^2 = w_1 + \cdots + w_{t+1}$ where the w_i's are irreducibles of the form $w_i = p_i^2$ for $1 \leq i \leq t+1$. An argument similar to that used on x above yields that $\bar{L}(y) = D(\mathscr{C}(M))/2$.

https://doi.org/10.1017/S1446788700003396 Published online by Cambridge University Press

2. The computation of $\bar{\ell}(x)$ and $\bar{L}(x)$ in finitely generated monoids

2.1. Notation and definitions All monoids appearing in this section are commutative and cancellative. By [10] and [13], when considering problems involving lengths of factorizations, we can assume without loss of generality that the monoid we are considering is reduced (it has only one unit, its identity element). Moreover, in [13] an algorithmic method that allows us to compute from the presentation of a finitely generated monoid a presentation of its associated reduced monoid is given. Since in this section we consider quotients of $\mathbb{N}^k$ (for some $k \in \mathbb{N}$), we use additive notation and denote the identity element of a monoid $(S, +)$ by 0. An element s of S is a *unit* if there exists s' such that $s + s' = 0$. Denote by $\mathscr{U}(S)$ the set of units of S.

The basic concepts related with factorizations are translated to additive notation as follows. If $a, b \in S$, then *a divides b*, denoted $a \mid b$, if there exists $s \in S$ such that $a + s = b$. Two elements $a, b \in S$ are *associated*, denoted $a \simeq b$, if there exists $s \in \mathscr{U}(S)$ such that $a + s = b$ (note that $a \simeq b$ if and only if $a \mid b$ and $b \mid a$). An element $s \in S$ is *irreducible* (or an atom) if $a \mid s$ implies that either $a \in \mathscr{U}(S)$ or $a \simeq s$. Denote by $\mathscr{A}(S)$ the set of all the atoms of S. We say that a monoid S is *atomic* if every element which is not a unit can be expressed as a sum of atoms.

It is well known and routine to see that if S is a commutative cancellative reduced monoid with $\{s_1, \dots, s_p\}$ its minimal system of generators, then:

- $a \simeq b$ if and only if $a = b$,
- $\mathscr{A}(S) = \{s_1, \dots, s_p\}$.

As a consequence, we obtain in this setting that S is an atomic monoid.

Consider the monoid epimorphism $\varphi : \mathbb{N}^p \to S$ defined by

$$\varphi(x_1, \dots, x_p) = x_1 s_1 + \cdots + x_p s_p$$

and let σ be the kernel of φ, so S is isomorphic to $\mathbb{N}^p/\sigma$. Note also that if $\varphi(x) = s$, then elements of the set $[x]_\sigma$ correspond to factorizations of s in terms of irreducible elements of S, since $(y_1, \dots, y_p) \in [x]_\sigma$ if and only if $y_1 s_1 + \cdots + y_p s_p = s$.

Given a subgroup H of $\mathbb{Z}^p$, define the congruence $\sim_H$ on $\mathbb{N}^p$ by $a \sim_H b$ if $a - b \in H$. Since $\mathbb{N}^p/\sigma$ is isomorphic to S, the monoid $\mathbb{N}^p/\sigma$ is cancellative and therefore, using [14, Proposition 1.4], we deduce that there exists a subgroup M of $\mathbb{Z}^p$ such that $\sigma = \sim_M$. Under the assumption that S is reduced, $\mathbb{N}^p/\sigma$ is also reduced, so by [14, Propositions 3.6–3.7] we may assume that $M \cap \mathbb{N}^p = \{0\}$. Note also that in this setting $[x]_{\sim_M}$ has finite cardinality (see [14, Lemma 9.1]) and using the results of [14, Chapter 8] we can determine all its elements. Hence, from M we can compute all the factorizations into irreducible elements of any element of S.

Given $a = (a_1, \dots, a_p) \in \mathbb{N}^p$, we denote by $|a| = a_1 + \cdots + a_p$. Using this notation, define $\ell([a]_{\sim_M}) = \min\{|b| \mid b \in [a]_{\sim_M}\}$ and $L([a]_{\sim_M}) = \max\{|b| \mid b \in [a]_{\sim_M}\}$ which are equal to $\ell(\varphi(a))$ and $L(\varphi(a))$, respectively, as defined in the introduction.

2.2. The asymptotic behaviour of l Throughout the remainder of this paper, our standing hypothesis will be that M is a subgroup of $\mathbb{Z}^p$ such that $M \cap \mathbb{N}^p = \{0\}$ (this simply means that the finitely generated cancellative monoid $\mathbb{N}^p/\sigma$ is reduced). The elements of $\mathbb{N}^p/\sim_M$ will be denoted by $[a]$ (this element is equal to the set $\{b \mid a\sim_M b\}$).

Let $x \in \mathbb{N}^p \setminus \{0\}$. Our aim in this section is to describe the behaviour of $\ell([nx])/n$ as n goes to ∞.

Let $\preceq$ represent a graded order on $\mathbb{N}^p$ (a well order compatible with the operation of the monoid such that $|a| < |b|$ implies that $a \preceq b$). One such graded order is given by the lexicographical total degree order on $\mathbb{N}^p$. Let $\mu : \mathbb{N}^p/\sim_M \to \mathbb{N}^p$ be the map defined by $\mu([a]) = \min_{\preceq}([a])$. Note that if $\gamma = \mu([a])$, then $|\gamma| = \ell([a])$.

Let $A = \{\mu([nx]) \mid n \in \mathbb{N}^+\}$. Since A is a subset of $\mathbb{N}^p$, we deduce, by Dickson's Lemma, that this set has only a finite number of minimal elements with respect to the usual order of $\mathbb{N}^p$. Assume that these minimal elements are

$$B = \{\mu([k_1x]), \ldots , \mu([k_rx])\}.$$

LEMMA 2.1. *Let $a, b, c \in \mathbb{N}^p$ and assume that $\mu([a]) = b + c$. Then $b = \mu([b])$.*

PROOF. Observe that $\mu([b]) + c \in [a]$, so $b + c \preceq \mu([b]) + c \preceq b + c$, whence $b = \mu([b])$. □

LEMMA 2.2. *Let $a \in \mathbb{N}^p \setminus \{0\}$ and $k, \bar{k} \in \mathbb{N}$. If $[ka] = [\bar{k}a]$, then $k = \bar{k}$.*

PROOF. Assume that $k \geq \bar{k}$. Then $[(k - \bar{k})a] = [0]$ and therefore $(k - \bar{k})a \in M$. Applying the fact that $M \cap \mathbb{N}^p = \{0\}$, we deduce that $k = \bar{k}$. □

LEMMA 2.3. *Let $n \in \mathbb{N}$. There exist $\lambda_1, \ldots , \lambda_r \in \mathbb{N}$ such that:*

- $\mu([nx]) = \lambda_1\mu([k_1x]) + \cdots + \lambda_r\mu([k_rx])$,
- $n = \lambda_1k_1 + \cdots + \lambda_rk_r$.

PROOF. Since B is the set of minimal elements of A, there exist $i \in \{1, \ldots , r\}$ and $y \in \mathbb{N}^p$ such that $\mu([nx]) = \mu([k_ix]) + y$. Using Lemma 2.1, we deduce that $y = \mu([y]) = \mu([(n - k_i)x])$ (observe that $k_i \leq n$, since $\mathbb{N}^p/\sim_M$ is cancellative and reduced). Performing this process as many times as necessary, we obtain that there exist $\lambda_1, \ldots , \lambda_r \in \mathbb{N}^+$ such that $\mu([nx]) = \lambda_1\mu([k_1x]) + \cdots + \lambda_r\mu([k_rx])$. Finally, $[nx] = [(\lambda_1k_1 + \cdots + \lambda_rk_r)x]$ and applying Lemma 2.2, we have that $n = \lambda_1k_1 + \cdots + \lambda_rk_r$. □

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Set $\gamma_1 = \mu([k_1x]), \ldots, \gamma_r = \mu([k_rx])$ and (by reordering if necessary) assume that $\gamma_1|/k_1 \leq |\gamma_2|/k_2 \leq \cdots \leq |\gamma_r|/k_r$.

LEMMA 2.4. *Under the standing hypothesis we have that* $\ell([n\gamma_1]) = n|\gamma_1|$ *for all* $n \in \mathbb{N}^+$.

PROOF. By Lemma 2.3, there exist $\lambda_1, \ldots, \lambda_r \in \mathbb{N}$ such that

$$\mu([n\gamma_1]) = \mu([nk_1x]) = \lambda_1\gamma_1 + \cdots + \lambda_r\gamma_r$$

with $\lambda_1k_1 + \cdots + \lambda_rk_r = nk_1$. Thus $\ell([n\gamma_1]) = \lambda_1|\gamma_1| + \cdots + \lambda_r|\gamma_r|$. Applying now that $|\gamma_i| \geq k_i|\gamma_1|/k_1$ for all $1 \leq i \leq r$, we get

$$\ell([n\gamma_1]) \geq (\lambda_1k_1 + \cdots + \lambda_rk_r)|\gamma_1|/k_1 = nk_1|\gamma_1|/k_1 = n|\gamma_1|.$$

But $\ell([n\gamma_1]) \leq n|\gamma_1|$, whence $\ell([n\gamma_1]) = n|\gamma_1|$. □

THEOREM 2.5. *Under the standing hypothesis, we have that* $\bar{\ell}([x]) = |\gamma_1|/k_1$.

PROOF. We know that the $\bar{\ell}([x])$ exists and equals $\lim_{n\to\infty} \ell([nk_1x])/nk_1$. Since $[nk_1x] = [n\gamma_1]$, by Lemma 2.4

$$\bar{\ell}([x]) = \lim_{n\to\infty} \frac{\ell([nk_1x])}{nk_1} = \lim_{n\to\infty} \frac{\ell([n\gamma_1])}{nk_1} = \lim_{n\to\infty} \frac{n|\gamma_1|}{nk_1} = \frac{|\gamma_1|}{k_1}. \qquad \square$$

2.3. An algorithm to compute $\bar{\ell}([x])$ Let $x \in \mathbb{N}^p \setminus \{0\}$. In this section, our goal is to give an algorithm to compute $\lim_{n\to\infty}(\ell([nx])/n)$ from x and M.

The algorithm is based in the following two lemmas.

LEMMA 2.6. *Let* $\gamma \in \mathbb{N}^p$ *such that* $\gamma \in [kx]$ *for some* $k \in \mathbb{N}^+$ *and* $\mu([n\gamma]) = n\gamma$ *for all* $n \in \mathbb{N}^+$. *Then* $\bar{\ell}([x]) = |\gamma|/k$.

PROOF. Since $\gamma \in [kx]$, we have that

$$\bar{\ell}([x]) = \lim_{n\to\infty} \frac{\ell([nkx])}{nk} = \lim_{n\to\infty} \frac{\ell([n\gamma])}{nk} = \lim_{n\to\infty} \frac{n|\gamma|}{nk} = \frac{|\gamma|}{k}. \qquad \square$$

The following lemma proves the existence of an element γ with the properties of the previous lemma.

LEMMA 2.7. *There exists* $\gamma \in \mathbb{N}^p$ *such that* $\gamma \in [kx]$ *for some* $k \in \mathbb{N}^+$ *and* $\mu([n\gamma]) = n\gamma$ *for all* $n \in \mathbb{N}^+$.

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PROOF. Let $\gamma_1, \dots, \gamma_r$ be as in Section 2.2 and

$$C = \{(\lambda_1, \dots, \lambda_r) \in \mathbb{N}^r \mid \lambda_1\gamma_1 + \dots + \lambda_r\gamma_r = \mu([nx]) \text{ for some } n \in \mathbb{N}^+\}.$$

By Lemma 2.3, we deduce that the cardinality of C is not finite. Thus, there exists $i \in \{1, \dots, r\}$ such that $\Pi_i(C) = \{\lambda_i \in \mathbb{N} \mid (\lambda_1, \dots, \lambda_r) \in C\}$ is not finite.

Let $n \in \mathbb{N}^+$. We will show now that $\mu([n\gamma_i]) = n\gamma_i$. Since $\Pi_i(C)$ is not finite, there exists $\lambda_i \in \Pi_i(C)$ such that $\lambda_i \geq n$. There exist $\lambda_1, \dots, \lambda_{i-1}, \lambda_{i+1}, \dots, \lambda_r \in \mathbb{N}$ such that $\lambda_1\gamma_1 + \dots + \lambda_r\gamma_r = \mu([mx])$ for some $m \in \mathbb{N}^+$. Applying Lemma 2.1, we deduce that $\lambda_i\gamma_i = \mu([\lambda_i\gamma_i])$. Therefore, $\mu([\lambda_i\gamma_i]) = \lambda_i\gamma_i = (\lambda_i - n)\gamma_i + n\gamma_i$ which, by Lemma 2.1, implies that $n\gamma_i = \mu([n\gamma_i])$. □

In [14, proof of Proposition 8.2], it is proved that $\sim_M$ is a submonoid of $\mathbb{N}^p \times \mathbb{N}^p$ generated by the minimal elements of $\sim_M \setminus \{(0,0)\}$. Denote this set by $\mathscr{A}(\sim_M)$ (in fact this set is the set of atoms of $\sim_M$). Furthermore, [14, Chapter 8] illustrates an algorithm to compute from M the set $\mathscr{A}(\sim_M)$.

Assume that $\mathscr{A}(\sim_M) = \{(\alpha_1, \beta_1), \dots, (\alpha_t, \beta_t)\}$ and let

$$\{\theta_1, \dots, \theta_l\} = \left\{ a \in \mathbb{N}^p \,\middle|\, a = \max_{\preceq}\{\alpha_i, \beta_i\} \text{ for some } i \in \{1, \dots, t\} \text{ with } \alpha_i \neq \beta_i \right\}.$$

Given $a = (a_1, \dots, a_p) \in \mathbb{N}^p$, denote by $\operatorname{Supp}(a)$ the set $\{i \in \{1, \dots, p\} \mid a_i \neq 0\}$.

LEMMA 2.8. *Under the standing hypothesis the following statements are equivalent*:

(1) $\mu([n\gamma]) = n\gamma$ *for all* $n \in \mathbb{N}^+$.
(2) $\operatorname{Supp}(\theta_i) \not\subseteq \operatorname{Supp}(\gamma)$ *for all* $i \in \{1, \dots, l\}$.

PROOF. Assume that $\operatorname{Supp}(\theta_i) \subseteq \operatorname{Supp}(\gamma)$ for some i. There exists $n \in \mathbb{N}^+$ such that $n\gamma - \theta_i \in \mathbb{N}^p$. Without loss of generality, we can assume that $\alpha_j \prec \theta_i = \beta_j = \max_{\preceq}\{\alpha_j, \beta_j\}$. Since $(\alpha_j, \beta_j) \in \sim_M$ and $n\gamma - \beta_j \in \mathbb{N}^p$, we have that $n\gamma - \beta_j + \alpha_j \in [n\gamma]$. Furthermore, $\alpha_j \prec \beta_j$ and thus $n\gamma - \beta_j + \alpha_j \prec n\gamma$, allowing us to assert that $\mu([n\gamma]) \neq n\gamma$.

Suppose now that (2) holds and $a \in [n\gamma]$. Then $(a, n\gamma) \in \sim_M$ and there exist $\lambda_1, \dots, \lambda_t \in \mathbb{N}$ such that $(a, n\gamma) = \lambda_1(\alpha_1, \beta_1) + \dots + \lambda_t(\alpha_t, \beta_t)$. Note that by (2) we can deduce that if $\lambda_i \neq 0$, then $\beta_i \preceq \alpha_i$. Hence

$$a = \lambda_1\alpha_1 + \dots + \lambda_t\alpha_t \succeq \lambda_1\beta_1 + \dots + \lambda_t\beta_t = n\gamma,$$

whence we get $\mu([n\gamma]) = n\gamma$. □

ALGORITHM 2.9. The input is an element $x \in \mathbb{N}^p$ and the output is $\tilde{\ell}([x])$.

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1. $k = 1$.
2. Compute $[kx]$.
3. Check if there exists $\gamma \in [kx]$ such that $\mathrm{Supp}(\theta_i) \not\subseteq \mathrm{Supp}(\gamma)$ for all $i \in \{1, \ldots, l\}$.
4. If there exists such γ, then return $|\gamma|/k$. Else $k = k + 1$ and go to 2.

By Lemma 2.6 and Lemma 2.8, if γ exists, then $\bar{\ell}([x]) = |\gamma|/k$. By Lemma 2.7 the algorithm ends after a finite number of steps.

We illustrate the above algorithm with an example.

EXAMPLE 2.10. Let $S = \mathbb{N} \setminus \{1, 2, 5\}$ be the primitive numerical submonoid of $(\mathbb{N}, +)$ generated by $\{3, 4\}$. Clearly S is a commutative cancellative reduced monoid with minimal system of generators equal to $\{3, 4\}$. Furthermore, S is isomorphic to $\mathbb{N}^2/\sim_M$ with $M = \{(x, y) \in \mathbb{Z}^2 \mid 3x + 4y = 0\}$. Applying the results of [14] we have that $\mathscr{A}(\sim_M) = \{((1,0),(1,0)), ((0,1),(0,1)), ((4,0),(0,3)), ((0,3),(4,0))\}$. Taking $\preceq$ as the lexicographical total degree order on $\mathbb{N}^2$, we get that $l = 1$ and $\{\theta_1\} = \{(4, 0)\}$. We use Algorithm 2.9 to compute $\bar{\ell}(3)$ which is equal to $\lim_{n\to\infty}(\ell([n(1,0)])/n)$. For $k = 1, 2, 3$, we obtain

$$[k(1,0)] = \{(k,0)\} \quad \text{and} \quad \mathrm{Supp}(\theta_1) = \mathrm{Supp}((k,0)).$$

But $[4(1,0)] = \{(4,0),(0,3)\}$ and $\mathrm{Supp}(\theta_1) \not\subseteq \mathrm{Supp}((0,3))$ and therefore we can assert that

$$\bar{\ell}([3]) = \lim_{n\to\infty} \frac{\ell([n(1,0)])}{n} = \frac{|(0,3)|}{4} = 3/4.$$

Notice that 3 is both irreducible and primary in S.

2.4. The asymptotic behaviour of L Let $x \in \mathbb{N}^p \setminus \{0\}$. Our goal in this section is to compute $\bar{L}([x])$. The results and its proofs are analogous to the ones given in the previous sections.

Let $\mathscr{M} : \mathbb{N}^p/\sim_M \to \mathbb{N}^p$ be the map defined by $\mathscr{M}([a]) = \max_{\preceq}([a])$. Note that, as we indicated in Section 2.1, the cardinality of $[a]$ is finite and therefore its maximum exists. Note also that if $\gamma = \mathscr{M}([a])$, then $|\gamma| = L([a])$. We take now

$$A = \{\mathscr{M}([nx]) \mid n \in \mathbb{N}^+\}$$

and let $B = \{\mathscr{M}([k_1x]), \ldots, \mathscr{M}([k_rx])\}$ be its minimal elements. As in Section 2.2 we have that:

- If $\mathscr{M}([a]) = b + c$, then $b = \mathscr{M}([b])$.
- If $n \in \mathbb{N}$, then there exist $\lambda_1, \ldots, \lambda_r \in \mathbb{N}$ such that $\mathscr{M}([nx]) = \lambda_1\mathscr{M}([k_1x]) + \cdots + \lambda_r\mathscr{M}([k_rx])$ and $n = \lambda_1k_1 + \cdots + \lambda_rk_r$.

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• Denote by $\gamma_1 = \mathscr{M}([k_1x]), \ldots, \gamma_r = \mathscr{M}([k_rx])$ and without loss of generality we assume that $|\gamma_1|/k_1 \leq \cdots \leq |\gamma_r|/k_r$.

• $L([n\gamma_r]) = n|\gamma_r|$ for all $n \in \mathbb{N}^+$.

• $\bar{L}([x]) = |\gamma_r|/k_r$.

The results of Section 2.3 can now be restated as follows:

• Let $\gamma \in \mathbb{N}^p$ such that $\gamma \in [kx]$ for some $k \in \mathbb{N}^+$ and $\mathscr{M}([n\gamma]) = n\gamma$ for all $n \in \mathbb{N}^+$. Then $\bar{L}([x]) = |\gamma|/k$.

• There exists $\gamma \in \mathbb{N}^p$ such that $\gamma \in [kx]$ for some $k \in \mathbb{N}^+$ and $\mathscr{M}([n\gamma]) = n\gamma$ for all $n \in \mathbb{N}^+$.

• If $\mathscr{A}(\sim_M) = \{(\alpha_1, \beta_1), \ldots, (\alpha_t, \beta_t)\}$, then we define $\{\theta_1, \ldots, \theta_l\}$ as the set $\{a \in \mathbb{N}^p \mid a = \min_{\preceq}\{\alpha_i, \beta_i\}$ for some $i \in \{1, \ldots, t\}$ with $\alpha_i \neq \beta_i\}$.

• $\mathscr{M}([n\gamma]) = n\gamma$ for all $n \in \mathbb{N}^+$ if and only if $\mathrm{Supp}(\theta_i) \not\subseteq \mathrm{Supp}(\gamma)$ for all $i \in \{1, \ldots, l\}$.

• Finally, with this notation, the algorithm to compute $\lim_{n\to\infty} L([nx])/n$ is identical to the algorithm obtained from Algorithm 2.9 changing l by L.

EXAMPLE 2.11. Let S be as in Example 2.10. We compute now

$$\bar{L}(3) = \lim_{n\to\infty} \frac{L([n(1,0)])}{n}.$$

We have that $\{\theta_1\} = \{(0,3)\}$ and $[(1,0)] = \{(1,0)\}$. Since $\mathrm{Supp}(\theta_1) \not\subseteq \mathrm{Supp}((1,0))$ we can assert that

$$\lim_{n\to\infty} \frac{L([n(1,0)])}{n} = \frac{|(1,0)|}{1} = \frac{1}{1} = 1.$$

We close with an example which relates to behaviour observed in Section 1.

EXAMPLE 2.12. Let $S = \mathbb{N}^5/\sim_M$ where $M = \langle(1,1,1,-1,-1)\rangle$. If e_i represents the ith basis vector of $\mathbb{N}^5$ for $1 \leq i \leq 5$, then we have that $\sim_M$ is generated as a monoid by $\{(e_1, e_1), \ldots, (e_5, e_5), ((1,1,1,0,0), (0,0,0,1,1)), ((0,0,0,1,1), (1,1,1,0,0))\}$. The irreducible elements of S are $\{[e_1], [e_2], \ldots, [e_5]\}$ and an easy application of the formulas in this section shows that $\bar{\ell}([e_i]) = \bar{L}([e_i]) = 1$ when $1 \leq i \leq 5$. Also, $[e_1] + [e_2] + [e_3] = [e_4] + [e_5]$ and S is not half-factorial.

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