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ON THE DERIVATIVES OF THE Γ-FUNCTION

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1. The coefficients of the two series

$$\Gamma(1+z) = \sum_{n=0}^{\infty} c_n z^n, \qquad |z| < 1,$$
(1)
$$1/\Gamma(1+z) = \sum_{n=0}^{\infty} g_n z^n, \qquad |z| < \infty,$$

are recursively given by Nielsen [1]: $c_0 = g_0 = 1$ and

$$(n+1)c_{n+1} = \sum_{j=0}^{n} (-1)^{j+1} s_{j+1} c_{n-j},$$

$$(n+1)g_{n+1} = \sum_{j=0}^{n} (-1)^{j} s_{j+1} g_{n-j},$$

where s_1 is the Euler constant γ and for $n>1$ $s_n = \zeta(n)$. We have then from (1): $c_n = (n!)^{-1}\Gamma^{(n)}(1)$. Nielsen (loc. cit., p. 40) remarks that no simple direct representation of the coefficients c_n is known. Using the Faa di Bruno formula for the n-th derivative of a compound function, he shows, following Schlömilch, that

$$c_n = (-1)^n \sum_{k=1}^{n} \frac{1}{k!} \sum \prod_{i=1}^{k} s_{r_i}/r_i$$

where the second summation is over all positive solutions of $r_1 + \cdots + r_k = n$ and the s_1 are as above. In this note we use a completely self-contained elementary method to calculate the coefficients c_n or, what amounts to the same thing, the derivatives of the Γ-function. We show that

$$\Gamma^{(k)}(1) = 2 \lim_{n\to\infty} \left[(-1)^n \frac{n^{n+2}}{(n+1)!} \sum_{j=0}^{k/2} (2j)!\, \zeta(2j)(1-2^{1-2j}) \binom{k}{2j} \cdot \frac{d^n(\log^{k-2j} a/a)}{da^n} \bigg|_{n=a} \right]. \tag{2}$$

2. Starting with the Euler integral

$$\Gamma(1+x) = \int_0^{\infty} e^{-t} t^x \, dt$$

we have by differentiating k times and putting $x=0$

$$\Gamma^{(k)}(1) = \int_0^{\infty} e^{-t} \log^k t \, dt.$$

To evaluate this we begin by setting

$$I_k(a) = \int_0^\infty \frac{\log^k x}{(a+x)^2}\,dx, \qquad k = 0, 1, \ldots . \tag{3}$$

The reason for this indirection is following. Once $I_k(a)$ is known we have, differentiating n times with respect to a and setting $a=n$,

$$\int_0^\infty \frac{n^{n+2}}{(n+x)^{n+2}} \log^k x\,dx = (-1)^n \frac{n^{n+2}}{(n+1)!} \frac{d^n I_k(a)}{da^n}\bigg|_{a=n}.$$

Since

$$\lim_{n\to\infty} \frac{n^{n+2}}{(n+x)^{n+2}} = e^{-x},$$

justifying the passage to the limit on n under the integral above we find

$$\int_0^\infty e^{-x} \log^k x\,dx = \lim_{n\to\infty}\left[(-1)^n \frac{n^{n+2}}{(n+1)!} \frac{d^n I_k(a)}{da^n}\bigg|_{a=n}\right].$$

3. To evaluate $I_k(a)$ we put $x=ay$ in (3) getting

$$I_k(a) = a^{-1}\sum_{j=0}^{k}\binom{k}{j} b_j \log^{k-j} a \tag{4}$$

where

$$b_j = \int_0^\infty \frac{\log^j y}{(1+y)^2}\,dy. \tag{5}$$

The integral (5) is evaluated by being broken up into two parts corresponding to the integrals $0-1$ and $1-\infty$; letting $y=1/u$ in the first one we get

$$b_j = [1+(-1)^j]\int_1^\infty \frac{\log^j y}{(1+y)^2}\,dy. \tag{6}$$

This is evaluated by the substitution $x=\exp(y)$ and observing that

$$(1+e^x)^{-2}e^x = (1+e^{-x})^{-2}e^{-x};$$

expanding in series and integrating term by term we have

$$b_j = 0 \quad \text{for } j \text{ odd}, \qquad b_j = 2(j!)(1-2^{1-j})\zeta(j) \quad \text{for } j \text{ even}. \tag{7}$$

This holds for all j including $j=0$ as can be verified, either from (6) directly or recalling that $\zeta(0)=-\frac{1}{2}$. Now, putting together (3)–(7) we get (2).

4. It may be verified that for the first few values of k (2) gives us the correct c_k's in (1). For instance, with $k=1$

$$\Gamma'(1) = \lim_{n\to\infty}\left[(-1)^n \frac{n^{n+2}}{(n+1)!} \frac{d^n(\log a/a)}{d^n a}\bigg|_{a=n}\right].$$

The n-th derivative of $a^{-1}\log a$ is computed by the Leibniz rule and we get

$$\Gamma'(1) = -\lim \frac{n}{(n+1)}\left(\log n - \sum_{j=1}^{n} 1/j\right) = \gamma.$$

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Similarly, we verify that

$$\Gamma''(1) = \gamma^2+\pi^2/6, \qquad \Gamma'''(1) = -\gamma^3-2\zeta(3)+\pi^2\gamma/6, \text{ etc.}$$

Here it is to be observed that the quantity $\zeta(3)$ does not arise from the b_j of (7) but from the limit of the n-th derivative of $a^{-1}\log^3 a$.

5. The foregoing allows us to evaluate some improper integrals. For instance integrating (3) with respect to a from p to q we evaluate

$$\int_0^\infty \frac{\log^k x}{(x+p)(x+q)}\,dx, \; p \neq q, \; p \text{ and } q > 0.$$

In conclusion, we mention the well-known theorem of Hoelder which states that the Γ-function satisfies no polynomial differential equation of the type

$$P(x, y, y', \ldots, y^{(n)}) = 0.$$

Thus any formulas, recursive or otherwise, for its n-th derivative are apt to be complicated.

Reference

1. N. Nielsen, *Handbuch der Theorie der Gammafunktion*, Chelsea, 1965.

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