J. Austral. Math. Soc. (Series A) 26 (1978), 46–52

MINKOWSKI'S FUNDAMENTAL INEQUALITY FOR REDUCED POSITIVE QUADRATIC FORMS

E. S. BARNES

To Kurt Mahler for his seventy-fifth birthday

(Received 11 November 1977)

Communicated by J. H. Coates

Abstract

Forms which are reduced in the sense of Minkowski satisfy the "fundamental inequality" $a_{11} a_{22} \dots a_{nn} \leqslant \lambda_n D$; the best possible value of λ_n is known for $n \leqslant 5$. A more precise result for the minimum value of D in terms of the diagonal coefficients has been stated by Oppenheim for ternary forms. The corresponding precise result for quaternary forms is established here by considering a convex polytope $\mathscr{D}(\boldsymbol{\alpha})$, defined as the intersection of the cone of reduced forms with the hyperplanes $a_{ii} = \alpha_i$ $(i = 1, \dots, n)$.

Subject classification (Amer. Math. Soc. (MOS) 1970): primary 10 E 25; secondary 10 E 20.

1. Introduction

Minkowski established the existence of a number λ_n with the property that, if $f(\mathbf{x}) = \sum_1^n a_{ij} x_i x_j$ is positive definite and reduced (in the sense of Minkowski), with determinant $D = \det(a_{ij})$, then

$$a_{11} a_{22} \dots a_{nn} \leqslant \lambda_n D. \tag{1.1}$$

Lekkerkerker (1969, Section 10) and Van der Waerden (1956) give detailed accounts of reduction theory and the best estimates for λ_n in this "fundamental inequality".

Mahler has made several contributions to the theory of Minkowski reduction. In particular, he obtained in (1938) an estimate for λ_n for all n, applicable to general convex bodies; and in (1940) and (1946) he gave proofs of the best possible results for $n = 3$ and $n = 4$. Best possible results are now known for $n \leqslant 5$; these are

$$\lambda_2 = \tfrac{4}{3}, \quad \lambda_3 = 2, \quad \lambda_4 = 4, \quad \lambda_5 = 8 \tag{1.2}$$

(so that in fact for all $n \leqslant 5$, $\lambda_n = \gamma_n^n$); for $n = 5$, see Van der Waerden (1969) and Nelson (1974).

Oppenheim (1946, p. 257) made the laconic comment, in a different but obvious notation, for the case $n = 3$: "It does not appear to have been observed that this inequality may be replaced by the sharper inequality

$$abc + \tfrac{1}{2}ab(c-b) + \tfrac{1}{2}ac(b-a) \leqslant 2\Delta.\text{"} \tag{1.3}$$

This observation suggests a different way of approaching the inequality (1.1), namely the determination of the least value of D for positive reduced forms f with given values of the diagonal coefficients $a_{11}, a_{22}, \ldots, a_{nn}$ (necessarily satisfying $a_{11} \leqslant a_{22} \leqslant \ldots \leqslant a_{nn}$).

The main purpose of this article is to carry through this determination for $n = 4$. We prove

THEOREM. *Suppose that $f(\mathbf{x}) = \sum_1^n a_{ij} x_i x_j$ is positive definite and reduced, with determinant D; and set*

$$a_{11} = a, \quad a_{22} = b, \quad a_{33} = c, \quad a_{44} = d, \quad \ldots \tag{1.4}$$

where necessarily

$$0 < a \leqslant b \leqslant c \leqslant d \leqslant \ldots \tag{1.5}$$

Then

(i) *if $n = 2$,*

$$4D \geqslant 3ab + a(b-a); \tag{1.6}$$

(ii) *if $n = 3$,*

$$4D \geqslant 2abc + ab(c-b) + ac(b-a); \tag{1.7}$$

(iii) *if $n = 4$,*

$$4D \geqslant abcd + acd(b-a) + abd(c-b) + abc(d-c) + \tfrac{1}{4}a^2(b-c)^2.$$

These inequalities are all best possible for all a, b, c, d and they imply (1.1), (1.2) *for $n \leqslant 4$.*

2. Minkowski reduction, the cones $\mathscr{M}$, $\mathscr{M}^+$ and the polytopes $\mathscr{D}$, $\mathscr{D}^+$

The condition for f to be reduced is that, for all $i = 1, \ldots, n$ and for all intgeral $\mathbf{x} = (x_1, x_2, \ldots, x_n)$,

$$\text{if g.c.d.}\,(x_i, x_{i+1}, \ldots, x_n) = 1, \text{ then } f(\mathbf{x}) \geqslant a_{ii}. \tag{2.1}$$

In the $\frac{1}{2}n(n+1)$-dimensional space $\mathscr{P}$ of non-negative definite forms, the set $\mathscr{M}$ of reduced forms is a polyhedral cone, since in fact finitely many inequalities (2.1) suffice to define it. We denote by $\mathcal{M}^+$ the subset of $\mathcal{M}$ consisting of "properly reduced" forms satisfying

$$a_{i,i+1} \geqslant 0 \quad (i = 1, \ldots, n-1). \tag{2.2}$$

$\mathcal{M}^+$ is also a polyhedral cone; and every $f \in \mathcal{M}$ is equivalent to an $f^+ \in \mathcal{M}^+$ under a suitable change of sign of the variables.

For real $a, b, c, \ldots$ satisfying (1.5), we define $\mathcal{D}(\boldsymbol{\alpha}) = \mathcal{D}(a, b, c, \ldots)$ as the intersection of $\mathcal{M}$ with the hyperplanes (1.4). Thus $\mathcal{D}(\boldsymbol{\alpha})$ is the set of positive reduced forms with prescribed diagonal coefficients $a, b, c, \ldots$. We define $\mathcal{D}^+(\boldsymbol{\alpha})$ similarly in relation to $\mathcal{M}^+$. Since the reduction conditions (2.1) include the inequalities

$$|2a_{ij}| \leqslant a_{ii} \quad (1 \leqslant i < j \leqslant n),$$

it follows that $\mathcal{D}(\boldsymbol{\alpha})$ and $\mathcal{D}^+(\boldsymbol{\alpha})$ are bounded and are therefore convex polytopes.

Finally, define

$$\Delta(\boldsymbol{\alpha}) = \min_{f \in \mathcal{D}(\boldsymbol{\alpha})} D(f) = \min_{f \in \mathcal{D}^+(\boldsymbol{\alpha})} D(f). \tag{2.4}$$

Since the region $D(f) \geqslant \text{const}$, for $f \in \mathcal{P}$, is strictly convex, we have immediately

LEMMA. $\Delta(\boldsymbol{\alpha})$ *is attained at a vertex of* $\mathcal{D}(\boldsymbol{\alpha})$.

In order to establish the theorem, it now suffices to specify $\mathcal{D}(\boldsymbol{\alpha})$ for $n \leqslant 4$, determine its vertices and evaluate D at the vertices. This is a feasible programme for $n \leqslant 4$, since a complete description of $\mathcal{M}$ and $\mathcal{M}^+$ is then known. However, even with the assistance of a computer, the computation may not be practicable for $n \geqslant 5$. In Section 5 I shall indicate a classification of the vertices of $\mathcal{D}(\boldsymbol{\alpha})$ which may be of assistance in examining the problem for $n \geqslant 5$.

3. Two- and three-dimensional forms

For $n = 2$, the reduction conditions are

$$a_{11} \leqslant a_{22}, \quad |2a_{12}| \leqslant a_{11},$$

so that $\mathcal{D}(a, b)$ is the line segment $\{a_{12} \mid |2a_{12}| \leqslant a\}$. Hence trivially, since $D = a_{11} a_{22} - \frac{1}{4} a_{12}^2$,

$$\Delta(\boldsymbol{\alpha}) = ab - \tfrac{1}{4}a^2 = \tfrac{3}{4}ab + \tfrac{1}{4}a(b-a),$$

giving (1.5).

For $n = 3$, it is well known that $f \in \mathcal{M}^+$ if and only if, in addition to (2.2) and the inequalities $a_{11} \leqslant a_{22} \leqslant a_{33}$, (2.1) is satisfied for $\mathbf{x} = (1, -1, 0), (1, 0, -1), (1, 0, 1), (0, 1, -1)$ and $(1, -1, 1)$. Hence, writing for convenience $f_{ij} = 2a_{ij}$ $(i \neq j)$, a form

$$f(\mathbf{x}) = ax_1^2 + bx_2^2 + cx_3^2 + f_{12} x_1 x_2 + f_{13} x_1 x_3 + f_{23} x_2 x_3$$

belongs to $\mathscr{D}^+ = \mathscr{D}^+(a,b,c)$ if and only if

$$0 \leqslant f_{12} \leqslant a, \quad |f_{13}| \leqslant a, \quad 0 \leqslant f_{23} \leqslant b, \quad f_{12} - f_{13} + f_{23} \leqslant a+b.$$

In the three-dimensional space of the coefficients f_{12}, f_{13} and f_{23}, $\mathscr{D}^+$ thus has 7 facets and is easily found to have the 9 vertices

$$(f_{12}, f_{13}, f_{23}) = (a,a,b),\ (a,0,b),\ (a,-a,-a+b),\ (0,a,b),\ (0,-a,b),$$
$$(a,a,0),\ (a,-a,0),\ (0,a,0),\ (0,-a,0).$$

Denoting the 9 vertices by $v_1, \ldots, v_9$ respectively, it is easily checked that v_6, v_7, v_8 and v_9 are not vertices of $\mathscr{D}$; $v_4 \sim v_5$ trivially; $v_1 \sim v_2$ under $x_2 \mapsto x_2 + x_3$, $x_3 \mapsto -x_3$; $v_1 \sim v_3$ under $x_1 \mapsto x_1 - x_3$. Hence

$$\begin{aligned}\Delta(a,b,c) &= \min(D(v_1), D(v_4)) \\ &= \min(abc - \tfrac{1}{4}ab^2 - \tfrac{1}{4}a^2c,\ abc - \tfrac{1}{4}a^2b - \tfrac{1}{4}ab^2) \\ &= abc - \tfrac{1}{4}ab^2 - \tfrac{1}{4}a^2c.\end{aligned}$$

This confirms Oppenheim's result (1.6), and shows that, apart from forms equivalent trivially by change of sign of variables, equality holds for all a, b, c for precisely the three reduced forms

$$\begin{aligned}v_1(\mathbf{x}) &= ax_1^2 + ax_1x_2 + ax_1x_3 + bx_2^2 + bx_2x_3 + cx_3^2, \\ v_2(\mathbf{x}) &= ax_1^2 + ax_1x_2 \qquad\qquad + bx_2^2 + bx_2x_3 + cx_3^2, \\ v_3(\mathbf{x}) &= ax_1^2 + ax_1x_2 - ax_1x_3 + bx_2^2 + (-a+b)x_2x_3 + cx_3^2.\end{aligned}$$

4. Quaternary forms

For $n = 4$, it is shown in Barnes and Cohn (1976) that $\mathscr{M}$ has 39 facets, which correspond to the 3 inequalities

(4.1) $$a_{11} \leqslant a_{22} \leqslant a_{33} \leqslant a_{44}$$

and all 36 inequalities of the form (2.1) for which $x_i = 1$, $x_j = 0$ if $j > i$, and the other $x_j = 0$ or ± 1 (excluding the 4 unit vectors). It appears to be computationally more economical to use $\mathscr{M}^+$ and then reject those notices of $\mathscr{D}^+(\boldsymbol{\alpha})$ which are not vertices of $\mathscr{D}(\boldsymbol{\alpha})$. $\mathscr{M}^+$ has, in addition to the 6 arising from the inequalities (4.1) and (2.2), 20 facets corresponding to the inequalities (2.1) for the following 20 vectors $\mathbf{x}$:

$$\begin{aligned}&(1,0,1,0),\ (1,0,0,1),\ (0,1,0,1),\ (-1,1,0,0),\ (-1,0,1,0),\ (-1,0,0,1),\\ &(0,-1,1,0),\ (0,-1,0,1),\ (0,0,-1,1),\ (0,1,-1,1),\ (1,-1,0,1),\\ &(-1,1,0,1),\ (1,0,-1,1),\ (-1,0,-1,1),\ (1,-1,1,0),\ (1,-1,1,1),\\ &(1,1,-1,1),\ (-1,-1,1,1),\ (-1,1,-1,1),\ (1,-1,-1,1).\end{aligned}$$

Hence $\mathscr{D}^+(\boldsymbol{\alpha})$ is specified minimally by the following system of 23 inequalities, where for convenience we again write $f_{ij} = 2a_{ij}\ (i \neq j)$:

$$f_{12} \geqslant 0, \quad f_{23} \geqslant 0, \quad f_{34} \geqslant 0,$$

$$f_{12} \leqslant a, \quad \pm f_{13} \leqslant a, \quad \pm f_{14} \leqslant a, \quad f_{23} \leqslant b, \quad \pm f_{24} \leqslant b, \quad f_{34} \leqslant c,$$

$$f_{12} - f_{13} + f_{23} \leqslant a + b,$$

$$f_{12} - f_{14} + f_{24} \leqslant a + b,$$

$$f_{12} + f_{14} - f_{24} \leqslant a + b,$$

$$f_{13} - f_{14} + f_{34} \leqslant a + c,$$

$$-f_{13} + f_{14} + f_{34} \leqslant a + c,$$

$$f_{23} - f_{24} + f_{34} \leqslant b + c,$$

$$f_{12} - f_{13} - f_{14} + f_{23} + f_{24} - f_{34} \leqslant a + b + c,$$

$$-f_{12} + f_{13} - f_{14} + f_{23} - f_{24} + f_{34} \leqslant a + b + c,$$

$$-f_{12} + f_{13} + f_{14} + f_{23} + f_{24} - f_{34} \leqslant a + b + c,$$

$$f_{12} - f_{13} + f_{14} + f_{23} - f_{24} + f_{34} \leqslant a + b + c,$$

$$f_{12} + f_{13} - f_{14} - f_{23} + f_{24} + f_{34} \leqslant a + b + c.$$

Because of the very simple form of the first 12 inequalities, bounding the 6 variables f_{ij}, it is not difficult to determine the vertices of $\mathscr{D}^+(\boldsymbol{\alpha})$ by considering all possible sets of 6 linearly independent equations that yield a solution of the inequalities. In this way it is found that $\mathscr{D}^+(\boldsymbol{\alpha})$ has 81 vertices that are also vertices of $\mathscr{D}(\boldsymbol{\alpha})$. Denoting each vertex by the corresponding vector $(f_{12}, f_{13}, f_{14}, f_{23}, f_{24}, f_{34})$, these fall into 9 classes of equivalent vertices, as follows:

14 vertices equivalent to $v_1 = (a, 0, a, b, b, c)$,

4 vertices equivalent to $v_2 = (0, 0, a, 0, b, c)$,

9 vertices equivalent to $v_3 = (a, a, a, 0, b, c)$,

10 vertices equivalent to $v_4 = (0, a, a, b, b, c)$,

12 vertices equivalent to $v_5 = (a, a, a, b, b, c)$,

12 vertices equivalent to $v_6 = (0, a, a, 0, b, c)$,

6 vertices equivalent to $v_7 = (a, 0, a, 0, b, c)$,

6 vertices equivalent to $v_8 = (0, 0, a, b, 0, c)$,

8 vertices equivalent to $v_9 = (0, a, a, b, 0, c)$.

https://doi.org/10.1017/S1446788700011484 Published online by Cambridge University Press

It is now easily verified that, for all a, b, c, d satisfying (1.5),

$$D(v_1) = \tfrac{1}{16}[16abcd - 4a^2cd - 4ab^2d - 4abc^2 + a^2(b-c)^2]$$
$$= \min_{1 \leq k \leq 4} D(v_k);$$
$$D(v_5) = \tfrac{1}{16}[16abcd - 4a^2cd - 4ab^2d - 4abc^2 + a^2c^2]$$
$$= \min_{5 \leq k \leq 9} D(v_k);$$

and that $D(v_1) < D(v_5)$. It follows that $\Delta(\boldsymbol{\alpha}) = D(v_1)$, which establishes part (iii) of the theorem. Equality holds for general values of a, b, c, d only for the 14 vertices equivalent to v_1, although other listed vertices may have equal determinant or even be identical for particular values of a, b, c, d. Indeed if $a = b = c = d$, all forms v_1, v_2, v_3, v_4 are equivalent to the absolutely extreme form; then and only then, $4D = abcd$.

For completeness we list all 14 vertices of $\mathscr{D}^+(\boldsymbol{\alpha})$ with $D = \Delta(\boldsymbol{\alpha})$; all reduced forms for which equality holds in (1.8) are trivially equivalent to one of these by change of sign of variables. It suffices to specify the coefficient vectors $(f_{12}, f_{13}, f_{14}, f_{23}, f_{24}, f_{34})$:

$$(a,0,a,b,b,c),\ (a,0,-a,b,0,c),\ (a,0,a,b,a,c),\ (a,0,-a,b,-a+b,c),$$
$$(a,a,0,b,b,c),\ (a,a,a,b,a-b,a-b+c),\ (a,a,-a,b,-b,-b+c),$$
$$(a,a,-a,b,-a,-a+c),\ (a,a,a,b,0,c),\ (a,-a,0,-a+b,b,c),$$
$$(a,-a,-a,-a+b,-b,a-b+c),\ (a,-a,a,-a+b,a-b,-b+c),$$
$$(a,0,0,b,-b,-b+c), (a,-a,-a,-a+b,-a,c).$$

It is noteworthy that the whole analysis may be carried through at once for all a, b, c, d satisfying (1.5), with the single exception that, of the 4 vertices of $\mathscr{D}(\boldsymbol{\alpha})$ trivially equivalent to $(0, a, a, b, b, a+b-c)$ and having $f_{23} = +b$, two are in $\mathscr{D}^+(\boldsymbol{\alpha})$ if $c < a+b$, the other two are if $c > a+b$, while all four are in $\mathscr{D}^+(\boldsymbol{\alpha})$ if $c = a+b$.

5. Forms extreme with respect to $\mathscr{D}(\boldsymbol{\alpha})$

In establishing the lemma of Section 2, we have already observed that a form belonging to $\mathscr{D}(\boldsymbol{\alpha})$ must be a vertex of $\mathscr{D}(\boldsymbol{\alpha})$ if it provides a local minimum of the determinant $D(f)$ for $f \in \mathscr{D}(\boldsymbol{\alpha})$. The converse statement is, however, false. Consider, for example, the quaternary form

$$v(x) = ax_1^2 + ax_1x_2 - ax_1x_3 - ax_1x_4 + bx_2^2 - bx_2x_4 + cx_3^2 + cx_3x_4 + dx_4^2 \tag{5.1}$$

https://doi.org/10.1017/S1446788700011484 Published online by Cambridge University Press

subject to (1.5); v is a vertex of $\mathscr{D}(\boldsymbol{\alpha})$, trivially equivalent to v_3 of Section 4. It is easy to verify that

$$f_\varepsilon(x) = v(x) + \varepsilon x_2 x_3 + \varepsilon x_2 x_4 \tag{5.2}$$

is reduced for $0 \leqslant \varepsilon \leqslant b-a$ and hence $\in \mathscr{D}(\boldsymbol{\alpha})$; and that

$$D(f_\varepsilon) = D(v) - \tfrac{1}{4}a(ad-bc)\,\varepsilon - \tfrac{1}{4}ad\varepsilon^2. \tag{5.3}$$

Hence, if $a<b$ and $ad \geqslant bc$, $D(f_\varepsilon) < D(v)$ for all sufficiently small $\varepsilon > 0$; thus for such values of a, b, c, d the vertex v does not provide a local minimum of $\mathscr{D}(f)$ for $f \in \mathscr{D}(\boldsymbol{\alpha})$.

It may therefore be useful to extend the classical concept of an extreme form to that of "extremeness with respect to $\mathscr{D}(\boldsymbol{\alpha})$". Clearly a perfect form, in the classical sense, corresponds here to a vertex of $\mathscr{D}(\boldsymbol{\alpha})$, and Voronoi's criterion of eutaxy can be adapted to obtain a necessary and sufficient condition for a vertex of $\mathscr{D}(\boldsymbol{\alpha})$ to be extreme. These ideas will be taken up in a subsequent article.

References

E. S. Barnes and M. J. Cohn (1976), "On Minkowski reduction of positive quaternary quadratic forms", *Mathematika* **23**, 156–158.

C. G. Lekkerkerker (1969), *Geometry of Numbers* (Wolters-Noordhoff, Groningen, 1969).

K. Mahler (1938), "On Minkowski's theory of reduction of positive quadratic forms", *Quart. J. Math.* **9**, 259–262.

K. Mahler (1940), "On reduced positive definite ternary quadratic forms", *J. London Math. Soc.* **15**, 193–195.

K. Mahler (1946), "On reduced positive definite quaternary quadratic forms", *Nieuw Arch. Wiskunde* (2) **22**, 207–212.

C. E. Nelson (1974), "The reduction of positive definite quinary quadratic forms", *Aequationes Math.* **11**, 163–168.

A. Oppenheim (1946), "A positive definite quadratic form as the sum of two positive definite quadratic forms (I)", *J. London Math. Soc.* **21**, 252–257.

B. L. Van der Waerden (1956), "Die Reduktionstheorie der positiven quadratischen Formen", *Acta Math.* **96**, 265–309.

B. L. Van der Waerden (1969), "Das Minimum von $D/f_{11} f_{12} \dots f_{55}$ für reduzierte positive quinäre quadratische Formen", *Aequationes Math.* **2**, 233–247.

The University of Adelaide
Adelaide, 5001
Australia