The expansions of sin$x$ and cos$x$.—Dr Stokes has already shown (*Mathematical Notes*, No. 13, p. 143) how the summation of series may be put to its proper use in elementary mathematics, namely, to provide a sequence of approximations to a limit. The method may be used to give a sequence of approximations to sin$x$ and cos$x$.

It is unnecessary to show in detail how, by means of summation of series of the type

$$\frac{\sin}{\cos}\alpha+\frac{\sin}{\cos}(\alpha+\beta)+\frac{\sin}{\cos}(\alpha+2\beta)+\ldots,$$

approximations are found to the areas under the graphs of sin$x$ and cos$x$, giving finally

area under graph of sin$x$ from origin to abscissa $x = 1-\cos x$,

area under graph of cos$x$ from origin to abscissa $x=\sin x$.

From the well-known inequality

$$\sin x<x \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

we see that

area under graph of sin$x$ < area under graph of x

$$\therefore\quad 1-\cos x<\frac{x^2}{2}$$

$$\therefore\quad \cos x>1-\frac{x^2}{2}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)'$$

Hence

area under graph of cos$x$ > area under graph of $\left(1-\dfrac{x^2}{2}\right)$,

$$\therefore\quad \sin x>x-\frac{x^3}{\underline{|3}}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Hence

area under graph of sin$x$ > area under graph of $\left(x-\dfrac{x^3}{\underline{|3}}\right)$

$$\therefore\quad 1-\cos x>\frac{x^2}{\underline{|2}}-\frac{x^4}{\underline{|4}}$$

$$\therefore\quad \cos x<1-\frac{x^2}{\underline{|2}}+\frac{x^4}{\underline{|4}}\ldots\ldots\ldots\ldots\ldots\ldots(2)'$$

Proceeding in this way, we get in order

$$\sin x < x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} \quad \ldots\ldots\ldots\ldots (3)$$

$$\cos x > 1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} - \frac{x^6}{\lfloor 6} \quad \ldots\ldots\ldots\ldots (3)'$$

$$\sin x > x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} - \frac{x^7}{\lfloor 7} \quad \ldots\ldots\ldots\ldots (4)$$

$$\cos x < 1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} - \frac{x^6}{\lfloor 6} + \frac{x^8}{\lfloor 8} \quad \ldots\ldots\ldots\ldots (4)'$$

and so on.

These inequalities are true for all positive values of x, since (1) is true for all positive values of x. If x is negative, the signs of inequality are reversed. Hence follow the infinite series for sin$x$ and cos$x$.

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https://doi.org/10.1017/S1757748900001237 Published online by Cambridge University Press