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Chaotic Vibration of a Two-dimensional Non-strictly Hyperbolic Equation

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Abstract. The study of chaotic vibration for multidimensional PDEs due to nonlinear boundary conditions is challenging. In this paper, we mainly investigate the chaotic oscillation of a two-dimensional non-strictly hyperbolic equation due to an energy-injecting boundary condition and a distributed self-regulating boundary condition. By using the method of characteristics, we give a rigorous proof of the onset of the chaotic vibration phenomenon of the 2D non-strictly hyperbolic equation. We have also found a regime of the parameters when the chaotic vibration phenomenon occurs. Numerical simulations are also provided.

1 Introduction

The main objective of this research is to study chaotic vibration of the wave equation in multidimensional domains due to boundary nonlinearities. Here, the spatial dimension under consideration is $n = 2$.

For the wave equation in 1D on the unit interval:

$$\frac{1}{c^2}\frac{\partial^2 w(x,t)}{\partial t^2} - \frac{\partial^2 w(x,t)}{\partial x^2} = 0, \quad x \in (0,1),\ t > 0, \tag{1.1}$$

when the boundary conditions at the left end $x = 0$ and the right end $x = 1$ are, respectively, *energy-pumping* and *self-regulating* of the van der Pol type, many articles have already been published (*cf.* [2–6]) on the chaotic vibration of (1.1) (at the level of (w_x, w_t)) when the parameter (η) enters a certain regime. The main methodology in proving the *onset of chaos* is the determination and analysis of the nonlinear interval map between the two *Riemann invariants* through the *boundary reflection relation* and *ray tracing*.

For the wave equation in multidimensional setting, namely,

$$\frac{1}{c^2}\frac{\partial^2 w(x,t)}{\partial t^2} - \nabla^2 w(x,t) = 0, \quad x \in \Omega \subseteq \mathbb{R}^n,\ t > 0, \tag{1.2}$$

we have made repeated attempts to generalize the 1D theory and methodology to (1.2), but with extremely limited success. For example, we can think about the case $n = 2$

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in (1.2) with certain spherical geometry. Then the use of polar coordinates gives

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\Big(r^2\frac{\partial}{\partial r}\Big) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}. \tag{1.3}$$

Further, for simplicity, assume angular independence. Then one can drop the $\frac{\partial^2}{\partial\theta^2}$ term in (1.3). Thus,

$$\begin{aligned}\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 &= \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{1}{r^2}\frac{\partial}{\partial r}\Big(r^2\frac{\partial}{\partial r}\Big)\\ &= \Big[\frac{1}{c}\frac{\partial}{\partial t} - \Big(\frac{\partial}{\partial r} + \frac{1}{r}\Big)\Big]\Big[\frac{1}{c}\frac{\partial}{\partial t} + \Big(\frac{\partial}{\partial r} + \frac{1}{r}\Big)\Big];\end{aligned}$$

i.e., the wave operator factors into a product analogously as the factoring of

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} = \Big(\frac{1}{c}\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\Big)\Big(\frac{1}{c}\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\Big) \tag{1.4}$$

for (1.1). Therefore, for a 2D *annular domain* Ω bounded by two concentric circles:

$$\Omega = \{(r,\theta) | r_1 < r < r_2, 0 \le \theta < 2\pi\},$$

for given positive r_1, r_2, the wave equation is reduced to a 1D problem on the interval (r_1, r_2). Therefore, the 1D methodology can be extended to cover the case of the 2D angular-independent wave equation

$$\frac{1}{c^2}\frac{\partial^2 w(x,t)}{\partial t^2} - \frac{1}{r^2}\frac{\partial}{\partial r}\Big(r^2\frac{\partial w(x,t)}{\partial r}\Big) = 0, \quad r_1 < r < r_2,\ t > 0,$$

on an annular disk.

In reviewing the discussions in the preceding paragraph, several key elements have come to our attention:

(a) The operator-factoring method and the resulting Riemann invariants are useful ideas for possibly treating multidimensional problems.

(b) The geometry/shape of multidimensional domains poses a new challenge. For example, if the boundary of the annular domain Ω were formed by circles that are *not concentric*, then the method will not work.

(c) The dimension n of the domain strongly matters. For example, when $n = 3$, under the assumption of independence of the polar and azimuth angles θ and ϕ, the *Laplacian* in spherical coordinates is reduced to

$$\nabla^2 = \frac{1}{r^3}\frac{\partial}{\partial r}\Big(r^3\frac{\partial}{\partial r}\Big).$$

However, the wave operator

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{1}{r^3}\frac{\partial}{\partial r}\Big(r^3\frac{\partial}{\partial r}\Big)$$

does not admit a factoring as (1.4) such as the 2D case.

Motivated by the above understanding, especially item (a), the authors began to study a *second order factorizable* partial differential equation in 2D of the form

$$\Big(\frac{\partial}{\partial t} + a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}\Big)\Big(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x} + d\frac{\partial}{\partial y}\Big)w(x,y,t) = 0. \tag{1.5}$$

But, again, on a bounded domain we face challenging and complicated ray tracing and reflection situations as the coefficients a, b, c and d are somewhat arbitrary (as long as the two 3D vectors $(a, b, 1)$ and $(c, d, 1)$ are linearly independent). Equation (1.5) has two directions for its rays:

$$\ell_1 = (a, b, 1) \quad \text{and} \quad \ell_2 = (c, d, 1),$$

i.e., straight lines in the (x, y, t)-space satisfying, respectively,

$$\begin{aligned} \frac{x - x_1}{a} &= \frac{y - y_1}{b} = \frac{t - t_1}{1} = \sigma, \\ \frac{x - x_2}{c} &= \frac{y - y_2}{d} = \frac{t - t_2}{1} = \tau, \quad (\sigma, \tau) \in \mathbb{R}^2, \end{aligned} \tag{1.6}$$

for any given (x_1, y_1, t_1), (x_2, y_2, t_2). For a general 2D bounded domain Ω with boundary Γ, a ray can travel an *arbitrarily short time* before it hits the boundary $\Gamma \times \mathbb{R}_+$, while other rays may not have hit any boundary point on $\Gamma \times \mathbb{R}_+$ at all. Also, the formation of *foci* (where many rays converge) and also possibly that of *caustics* causing the development of *singularities* is a real concern of the involved technical complexity.

In view of the above, we consider a special case of (1.5), namely,

$$\Big(\frac{\partial}{\partial t} - \frac{\partial}{\partial x} - \frac{\partial}{\partial y}\Big)\Big(\frac{\partial}{\partial t} + \frac{\partial}{\partial x} + \frac{\partial}{\partial y}\Big) w(x, y, t) = 0, \quad (x, y) \in \Omega, \ t > 0,$$

as the governing equation. The above makes

$$w_{tt} - \nabla^2 w - 2w_{xy} = 0, \quad (\nabla^2 = \partial_x^2 + \partial_y^2). \tag{1.7}$$

It has the advantage that it somehow resembles the wave equation.

As noted in item (b) previously, in order to make our problem tractable, the choice of Ω is important. Here we choose

$$\Omega = \big\{ (x, y) \in \mathbb{R}^2 \mid -1 < x + y < 1, \ -1 < x - y < 1 \big\}. \tag{1.8}$$

Its boundary Γ, where $\Gamma = \partial\Omega$, consists of four parts:

$$\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4,$$

where

$$\begin{aligned} \Gamma_1 &= \big\{ (x, y) \in \Omega \mid x + y = -1 \big\}, & \Gamma_2 &= \big\{ (x, y) \in \Omega \mid x - y = -1 \big\}, \\ \Gamma_3 &= \big\{ (x, y) \in \Omega \mid x + y = 1 \big\}, & \Gamma_4 &= \big\{ (x, y) \in \Omega \mid x - y = 1 \big\}; \end{aligned}$$

see Figure 1.

Remark 1.1 In order to make the **method of characteristics** and **ray tracing** work for a generally given bounded convex domain Ω, the following assumption is needed. For any given ray satisfying (1.6), let $P_1 = (a_1, b_1, t^{(1)})$, $P_2 = (a_2, b_2, t^{(2)})$ be two points on the same ray such that $p_1 = (a_1, b_1)$, $p_2 = (a_2, b_2)$ are their respective projections on the (x, y)-plane. Consider

$$\mathcal{S} = \big\{ |\overrightarrow{p_1 p_2}| \;\big|\; p_1, p_2 \in \Gamma; \overrightarrow{p_1 p_2} \text{ is the (x,y)-plane projection of a ray } \overrightarrow{P_1 P_2} \big\}.$$

If

$$\inf \mathcal{S} = \sup \mathcal{S} > 0, \tag{1.9}$$

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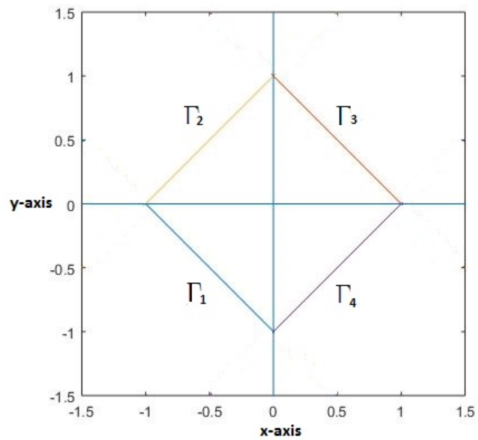

Figure 1. The domain Ω.

then every ray starting on a boundary point on $\Gamma \times \mathbb{R}_+$ will hit another boundary point on $\Gamma \times \mathbb{R}_+$ in exactly the same duration of time.

One can easily check that for the domain Ω in (1.8), the condition (1.9) is satisfied.

Remark 1.2 Equation of the type (1.7) is called a non-strictly hyperbolic equation, as its (negative) elliptic part

$$\nabla^2 + 2\frac{\partial^2}{\partial x \partial y}$$

satisfies the condition

$$\sum_{i=1}^{2} \xi_i^2 + 2\xi_1\xi_2 = (\xi_1 + \xi_2)^2 \geq 0 \quad \text{for } \xi = (\xi_1, \xi_2) \in \mathbb{R}^2,$$

but not

$$\sum_{i=1}^{2} \xi_i^2 + 2\xi_1\xi_2 > 0 \quad \text{for } \xi \in \mathbb{R}^2.$$

The reader can find some literature about relevant non-strictly hyperbolic equations in [8–10].

Equation (1.7) is invariant under translations

$$(x, y, t) \longmapsto (x - x_0, y - y_0, t - t_0)$$

and reflections

$$(x, y) \longmapsto (-x, -y).$$

However, it is not invariant under single axis reflections

$$x \longmapsto -x \quad \text{or} \quad y \longmapsto -y, \quad \text{(mutually exclusively)},$$

nor under rotations

$$(x, y) \longmapsto (x\cos\theta + y\sin\theta, -x\sin\theta + y\cos\theta).$$

Thus, the lack of such invariance makes (1.7) physically unnatural. Admittedly, this is a shortcoming of our model. Nevertheless, we hope that our work here can stimulate better, more physical multidimensional models in the future.

Now we describes the boundary conditions. On the boundary Γ_1, we have a linear boundary condition

$$w_t(x,y,t) = -\eta\big(w_x(x,y,t) + w_y(x,y,t)\big), \quad (x,y)\in\Gamma_1,\ t>0,\ 0<\eta\neq 1. \tag{1.10}$$

When $(x,y)\in\Gamma_3$, we have a nonlinear boundary condition

$$\begin{aligned} w_t(x,y,t) &= \alpha\big(w_x(x,y,t)+w_y(x,y,t)\big) \\ &\quad - \beta\big(w_x(x,y,t)+w_y(x,y,t)\big)^3, \quad t>0, \end{aligned} \tag{1.11}$$

where $0<\alpha<1$, $\beta>0$. On Γ_2 and Γ_4, we have the Dirichlet boundary conditions:

$$w(t,x,y) = 0, \quad (x,y)\in\Gamma_2\cup\Gamma_4,\ t>0.$$

Thus, the overall system is:

$$\begin{cases} w_{tt} = \Delta w + 2w_{xy}, & (x,y)\in\Omega,\ t>0, \\ w_t = -\eta(w_x+w_y), & (x,y)\in\Gamma_1,\ t>0, \\ w_t = \alpha(w_x+w_y) - \beta(w_x+w_y)^3, & (x,y)\in\Gamma_3,\ t>0, \\ w(x,y,t) = 0, & (x,y)\in\Gamma_2\cup\Gamma_4,\ t>0, \\ w(x,y,0) = w_0(x,y), \quad w_t(x,y,0) = w_1(x,y), & (x,y)\in\overline{\Omega}, \end{cases} \tag{1.12}$$

where $0<\eta\neq 1$, $0<\alpha<1$, $\beta>0$, and the initial data w_0 and w_1 satisfy

$$w_0\in C^2(\overline{\Omega}), \qquad w_1\in C^2(\overline{\Omega});$$

and

$$\begin{aligned} &w_0(x,y) = w_1(x,y) = 0,\ (x,y)\in\Gamma_2\cup\Gamma_4; \\ &w_1(x,y) = -\eta\Big(\frac{\partial w_0}{\partial x} + \frac{\partial w_0}{\partial y}\Big),\ (x,y)\in\Gamma_1; \\ &\qquad w_t = \alpha\Big(\frac{\partial w_0}{\partial x} + \frac{\partial w_0}{\partial y}\Big) - \beta\Big(\frac{\partial w_0}{\partial x} + \frac{\partial w_0}{\partial y}\Big)^3, \quad (x,y)\in\Gamma_3. \end{aligned}$$

Remark 1.3 When $\eta = 1$, the system is not well posed [13].

Remark 1.4 For a linear second order PDE of the form

$$a_{00}(x_1,x_2,t)w_{tt} - \sum_{i,j=1}^{2} a_{ij}(x_1,x_2,t)w_{x_ix_j} = 0, \quad \big((x_1,x_2)=(x,y)\big),$$

let $S(x_1,x_2,t)=c$ denote its characteristic surface [11]. Then S satisfies

$$a_{00}(x_1,x_2,t)\Big(\frac{\partial S}{\partial t}\Big)^2 - \sum_{i,j=1}^{2} a_{ij}(x_1,x_2,t)\frac{\partial S}{\partial x_i}\frac{\partial S}{\partial x_j} = 0. \tag{1.13}$$

It is well known that as a Cauchy problem, PDEs with initial data defined on characteristic surfaces may lack the existence and uniqueness of solutions, and solutions may have discontinuities across characteristic surfaces.

Here, for our PDE problem (1.12), the boundary condition $(1.12)_4$, namely,

$$w(x,y,t) = 0, \text{ on } \big(\Gamma_2 \times \{t > 0\}\big) \cup \big(\Gamma_4 \times \{t > 0\}\big),$$

the surfaces

$$S_1 \equiv \left\{ (x,y,t) \mid x - y = -1, t > 0, (x,y) \in \overline{\Omega} \right\} = \Gamma_2 \times \{t > 0\},$$

and

$$S_2 \equiv \left\{ (x,y,t) \mid x - y = 1, t > 0, (x,y) \in \overline{\Omega} \right\} = \Gamma_4 \times \{t > 0\},$$

satisfy

$$S_i(x,y,t) = x - y = 2i - 3 = \text{constant, for } i = 1, 2.$$

Thus,

$$\frac{\partial S_i}{\partial x} = 1, \ \frac{\partial S_i}{\partial y} = -1, \ \frac{\partial S_i}{\partial t} = 0, \ i = 1, 2.$$

We see, from (1.13) that for the non-strictly hyperbolic equation $(1.12)_1$ and $k = 1, 2$,

$$\begin{aligned} a_{00}(x_1,x_2,t)(S_{k,t})^2 - \sum_{i,j=1}^{2} a_{ij}(x_1,x_2,t)(S_{k,x_i})(S_{k,x_j}) &= 0 - [1\cdot 1 - 2\cdot 1\cdot 1 + 1\cdot 1] \\ &= 0. \end{aligned}$$

Therefore, S_1 and S_2 are actually characteristic surfaces, and in general, (1.12) has no solutions. This is indeed true. In what follows, the reader can clearly see that compatibility conditions need to be imposed on the functions at $(\Gamma_2 \cup \Gamma_4) \cap (\Gamma_1 \cup \Gamma_3)$. Then the initial and boundary data for (1.12) become consistent. Then under sufficient smoothness of the data w_0 and w_1 in $(1.12)_5$, existence and uniqueness of solutions become self-evident by Theorem 2.6 and Remark 2.7.

At time t, the energy of the system (1.12) is

$$E(t) = \frac{1}{2} \int_\Gamma w_t^2 + (w_x + w_y)^2 dS.$$

By applying the Green's formula, we see the rate of change of energy is

$$E'(t) = \sqrt{2}\eta \int_{\Gamma_1} (w_x + w_y)^2 d\sigma + \sqrt{2} \int_{\Gamma_3} (w_x + w_y)^2 \big(\alpha - \beta(w_x + w_y)^2\big)\, d\sigma. \tag{1.14}$$

More details about the derivation of (1.14) can be found in the Appendix. We can find that if $\eta > 0$, energy is injected to the system from Γ_1. For this reason, we refer to (1.10) as an *energy injecting (or pumping)* boundary condition. Note that the nonlinearities are distributed on the entire Γ_3, and the sign of the second term of the RHS of (A.1) is dependent of the integral; we can call (1.11) a *distributed self-regulating* boundary condition.

This paper is organized as follows. In Section 2, we provide preliminary analysis of the system (1.12). We transform the system to a new system in terms of two Riemann invariants. Then we find the explicit solutions of the new system in terms of two composite nonlinear operations of reflection relations. In Section 3, by applying the period-doubling bifurcation theorem and investigating the growth rate of total variation, we study the chaotic dynamics of the composite operations. In Section 4, we present a theorem proving the occurrence of the chaotic vibration phenomenon for the 2D non-strictly hyperbolic system. In Section 5, we provide numerical simulations to illustrate our theoretical results. An Appendix on the derivation of the energy function is provided at the end.

2 Preliminary Analysis

Recall that we have the system:

$$\begin{cases} w_{tt} = \Delta w + 2w_{xy}, & (x,y) \in \Omega,\ t>0, \\ w_t = -\eta(w_x + w_y), & (x,y) \in \Gamma_1,\ t>0, \\ w_t = \alpha(w_x + w_y) - \beta(w_x + w_y)^3, & (x,y) \in \Gamma_3,\ t>0, \\ w(x,y,t) = 0, & (x,y) \in \Gamma_2 \cup \Gamma_4,\ t>0, \\ w(x,y,0) = w_0(x,y) \in C^2(\overline{\Omega}), \\ w_t(x,y,0) = w_1(x,y) \in C^2(\overline{\Omega}), \end{cases} \tag{2.1}$$

where

$$0 < \eta \neq 1, \quad 0 < \alpha < 1, \quad \beta > 0.$$

Define two linear operators:

$$\mathcal{L}_1 = \frac{\partial}{\partial t} + \frac{\partial}{\partial x} + \frac{\partial}{\partial y}, \quad \mathcal{L}_2 = \frac{\partial}{\partial t} - \frac{\partial}{\partial x} - \frac{\partial}{\partial y}.$$

If w is a C^2 function, we have

$$\mathcal{L}_2(w) = w_t - w_x - w_y, \quad \mathcal{L}_1\mathcal{L}_2(w) = w_{tt} - w_{xx} - w_{yy} - 2w_{xy} = 0.$$

Similarly, we have $\mathcal{L}_2\mathcal{L}_1(w) = 0$.

Therefore, we can rewrite the first equation of system (2.1) as

$$\mathcal{L}_1\mathcal{L}_2(w) = \mathcal{L}_2\mathcal{L}_1(w) = 0.$$

Let u and v be the Riemann invariants of (2.1) defined by

$$u = \frac{1}{2}\mathcal{L}_1(w) = \frac{w_t + w_x + w_y}{2}, \quad v = \frac{1}{2}\mathcal{L}_2(w) = \frac{w_t - w_x - w_y}{2}. \tag{2.2}$$

Consequently,

$$w_x + w_y = u - v \quad \text{and} \quad w_t = u + v.$$

Therefore, the first equation of system (2.1) can be written as

$$\mathcal{L}_2 u = 0 \quad \text{or} \quad \mathcal{L}_1 v = 0.$$

For $t > 0$, the boundary condition on Γ_1 can be represented as a reflection relation between u and v:

$$v(x,y,t) = \frac{\eta+1}{\eta-1}u(x,y,t) := G_\eta\big(u(x,y,t)\big), \quad (x,y) \in \Gamma_1. \tag{2.3}$$

Moreover, the nonlinear condition on Γ_3 is equivalent to the relation of u and v:

$$\beta\big(u(x,y,t) - v(x,y,t)\big)^3 + (1-\alpha)\big(u(x,y,t) - v(x,y,t)\big) + 2v(x,y,t) = 0, \quad (x,y) \in \Gamma_3.$$

Since $\alpha < 1$ and $\beta > 0$, let $f = u(x, y, t) - v(x, y, t)$; then $f = p(v)$ satisfies the cubic equation

$$\beta f^3 + (1-\alpha) f + 2v = 0. \tag{2.4}$$

Remark 2.1 Since we have $\beta > 0$ and $0 < \alpha < 1$, the real solution f is uniquely defined by *Cardano's formula*

$$f = \Big(-\frac{v}{\beta} + \sqrt{D}\Big)^{1/3} + \Big(-\frac{v}{\beta} - \sqrt{D}\Big)^{1/3}, \quad D = \frac{(1-\alpha)^3}{27\beta^3} + \frac{v^2}{\beta^2} > 0.$$

Therefore, the reflection relation between u and v on Γ_3 takes the form:

$$u(x, y, t) = v(x, y, t) + p\big(v(x, y, t)\big) := F_{\alpha,\beta}\big(v(x, y, t)\big), \quad (x, y) \in \Gamma_3. \tag{2.5}$$

On $\Gamma_2 \cup \Gamma_4$, we have $w_t = 0,\ w_x + w_y = 0$, implying

$$u(x, y, t) = v(x, y, t) = 0, \quad (x, y) \in \Gamma_2 \cup \Gamma_4,\ t > 0.$$

Consequently, for given smooth initial data $w_0 \in C^2(\overline{\Omega})$ and $w_1 \in C^1(\overline{\Omega})$, the system (2.1) is equivalent to a system of two coupled first order equations as follows:

$$\begin{cases} \mathcal{L}_1(v) = \mathcal{L}_2(u) = 0, & (x, y) \in \Omega,\ t > 0, \\ v(x, y, t) = G_\eta(u(x, y, t)), & (x, y) \in \Gamma_1,\ t > 0, \\ u(x, y, t) = F_{\alpha,\beta}(v(x, y, t)), & (x, y) \in \Gamma_3,\ t > 0, \\ u(x, y, t) = v(x, y, t) = 0, & (x, y) \in \Gamma_2 \cup \Gamma_4,\ t > 0, \\ u(x, y, 0) = u_0(x, y) \in C^1(\overline{\Omega}), \\ v(x, y, 0) = v_0(x, y) \in C^1(\overline{\Omega}), \end{cases} \tag{2.6}$$

where the initial data u_0 and v_0 are now in the form

$$u_0 = \frac{w_1 + \frac{\partial w_0}{\partial x} + \frac{\partial w_0}{\partial y}}{2}, \quad v_0 = \frac{w_1 - \frac{\partial w_0}{\partial x} - \frac{\partial w_0}{\partial y}}{2}.$$

In order to ensure u and v are C^1 functions, we need u_0 and v_0 to be in C^1, and also satisfy some *compatibility conditions*:

$$\begin{aligned} v_0(\tilde{x}, \tilde{y}) &= G_\eta(u_0(\tilde{x}, \tilde{y})) && \text{for } (\tilde{x}, \tilde{y}) \in \Gamma_1, \\ u_0(\tilde{x}, \tilde{y}) &= F_{\alpha,\beta}(v_0(\tilde{x}, \tilde{y})) && \text{for } (\tilde{x}, \tilde{y}) \in \Gamma_3, \\ u_0(\tilde{x}, \tilde{y}) &= v_0(\tilde{x}, \tilde{y}) = 0 && \text{for } (\tilde{x}, \tilde{y}) \in \Gamma_2 \cup \Gamma_4; \\ D_{\vec{v}} v_0(x, y)|_{(\tilde{x}, \tilde{y})} &= D_{\vec{v}} G_\eta(u_0(x, y))|_{(\tilde{x}, \tilde{y})} && \text{for all } (\tilde{x}, \tilde{y}) \in \Gamma_1,\ \vec{v} \in \mathbb{R}^2; \\ D_{\vec{u}} u_0(x, y)|_{(\tilde{x}, \tilde{y})} &= D_{\vec{u}} F_{\alpha,\beta}(v_0(x, y))|_{(\tilde{x}, \tilde{y})} && \text{for all } (\tilde{x}, \tilde{y}) \in \Gamma_3,\ \vec{u} \in \mathbb{R}^2, \end{aligned}$$

where for a vector $\vec{\alpha} = (\alpha_1, \alpha_2)$ $(\vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3))$,

$$D_{\vec{\alpha}} = \alpha_1 \frac{\partial}{\partial x} + \alpha_2 \frac{\partial}{\partial y} \Big(D_{\vec{\alpha}} = \alpha_1 \frac{\partial}{\partial x} + \alpha_2 \frac{\partial}{\partial y} + \alpha_3 \frac{\partial}{\partial t} \Big).$$

Remark 2.2 Note that the two boundary conditions in (2.6) are "reflection" boundary conditions that result from wave reflection on the boundaries.

Therefore, proving the well-posedness of the main system (2.1) is equivalent to proving the well-posedness of system (2.6). From now on, we will focus on system (2.6).

Lemma 2.3 *Let u and v be given by (2.2). Then, u is constant along the direction $\overrightarrow{l_1}$ and v is constant along the direction $\overrightarrow{l_2}$, where*

$$\overrightarrow{l_1} = (-1,-1,1), \quad \overrightarrow{l_2} = (1,1,1).$$

Proof The proof follows from the fact that

$$D_{\overrightarrow{l_1}} u = \Big(\frac{\partial}{\partial t} - \frac{\partial}{\partial x} - \frac{\partial}{\partial y} \Big) u = \mathcal{L}_2 u = 0.$$

Similarly, we have $D_{\overrightarrow{l_2}} v = 0$. ∎

Next, we show the existence of solutions of system (2.6) on the set $\overline{\Omega} \times [0,2]$.

Lemma 2.4 *Let $t \in [0,2]$ and $(x,y) \in \overline{\Omega}$,* i.e., $-1 \le x+y \le 1$ *and* $-1 \le x-y \le 1$. *Then $u(x,y,t)$ and $v(x,y,t)$ can be uniquely solved.*

Proof First, recall that for $t > 0$,

$$\begin{aligned} v(x,y,t) &= G_\eta(u(x,y,t)) && \text{for } (x,y) \in \Gamma_1, \\ u(x,y,t) &= F_{\alpha,\beta}(v(x,y,t)) && \text{for } (x,y) \in \Gamma_3. \end{aligned}$$

Note that the points (x,y,t) and $(x+t,y+t,0)$ are on the same characteristics along $\overrightarrow{l_1}$, by applying Lemma 2.3, we have

$$u(x,y,t) = u(x+t,y+t,0) = u_0(x+t,y+t), \quad t \le \frac{1-x-y}{2}.$$

When $\frac{1-x-y}{2} < t \le \frac{1-x-y}{2} + 1$, we have

$$\Big(x + \frac{1-x-y}{2}, y + \frac{1-x-y}{2} \Big) \in \Gamma_3.$$

Also, (x,y,t) and $\big(x + \frac{1-x-y}{2}, y + \frac{1-x-y}{2}, t - \frac{1-x-y}{2} \big)$ are on the same characteristics along $\overrightarrow{l_1}$. $\big(x + \frac{1-x-y}{2}, y + \frac{1-x-y}{2}, t - \frac{1-x-y}{2} \big)$ and $(1-y-t, 1-x-t, 0)$ are on the same characteristics along $\overrightarrow{l_2}$. By applying the reflection relation (2.5) and Lemma 2.3, we have

$$\begin{aligned} u(x,y,t) &= u\Big(x + \frac{1-x-y}{2}, y + \frac{1-x-y}{2}, t - \frac{1-x-y}{2} \Big) \\ &= F_{\alpha,\beta}\Big(v\Big(x + \frac{1-x-y}{2}, y + \frac{1-x-y}{2}, t - \frac{1-x-y}{2} \Big)\Big) \\ &= F_{\alpha,\beta}\big(v(1-y-t, 1-x-t, 0)\big) \\ &= F_{\alpha,\beta}\big(v_0(1-y-t, 1-x-t)\big). \end{aligned}$$

When $\frac{1-x-y}{2}+1<t\le 2$, note that

$$\Big(x+\frac{1-x-y}{2}-1, y+\frac{1-x-y}{2}-1\Big)\in\Gamma_1,$$
$$\Big(x+\frac{1-x-y}{2}, y+\frac{1-x-y}{2}\Big)\in\Gamma_3.$$

From the reflection relations (2.3), (2.5) and Lemma 2.3, we have

$$\begin{aligned}
u(x,y,t) &= u\Big(x+\frac{1-x-y}{2}, y+\frac{1-x-y}{2}, t-\frac{1-x-y}{2}\Big)\\
&= F_{\alpha,\beta}\Big(v\Big(x+\frac{1-x-y}{2}, y+\frac{1-x-y}{2}, t-\frac{1-x-y}{2}\Big)\Big)\\
&= F_{\alpha,\beta}\Big(v\Big(x+\frac{1-x-y}{2}-1, y+\frac{1-x-y}{2}-1, t-\frac{1-x-y}{2}-1\Big)\Big)\\
&= F_{\alpha,\beta}\circ G_\eta\Big(u\Big(x+\frac{1-x-y}{2}-1, y+\frac{1-x-y}{2}-1, t-\frac{1-x-y}{2}-1\Big)\Big)\\
&= F_{\alpha,\beta}\circ G_\eta\big(u(x+t-2, y+t-2, 0)\big)\\
&= F_{\alpha,\beta}\circ G_\eta\big(u_0(x+t-2, y+t-2)\big).
\end{aligned}$$

So u can be solved as:

$$u(t,x,y)=\begin{cases} u_0(x+t, y+t), & 0\le t\le \dfrac{1-x-y}{2},\\[2mm] F_{\alpha,\beta}(v_0(1-y-t, 1-x-t)), & \dfrac{1-x-y}{2}<t\le\dfrac{1-x-y}{2}+1,\\[2mm] F_{\alpha,\beta}\circ G_\eta(u_0(x+t-2, y+t-2)), & \dfrac{1-x-y}{2}+1<t\le 2.\end{cases}$$

Similarly, v can be solved as:

$$v(t,x,y)=\begin{cases} v_0(x-t, y-t), & 0\le t\le \dfrac{x+y+1}{2},\\[2mm] G_\eta(u_0(t-y-1, t-x-1)), & \dfrac{x+y+1}{2}<t\le\dfrac{x+y+1}{2}+1,\\[2mm] G_\eta\circ F_{\alpha,\beta}(v_0(x+2-t, y+2-t)), & \dfrac{x+y+1}{2}+1<t\le 2.\end{cases}$$

For the uniqueness, suppose there is another pair of solution (u',v'), and we set $(r,s)=(u'-u, v'-v)$. Then (r,s) will satisfy (2.6) with zero initial data. From the explicit solution formulas we obtained above, we have $(r,s)=(0,0)$. So the solution is unique. ■

Lemma 2.5 *For $t\ge 0$ and $(x,y)\in\overline{\Omega}$, we have $u(x,y,t+2)=F_{\alpha,\beta}\circ G_\eta(u(x,y,t))$, $v(x,y,t)=G_\eta\circ F_{\alpha,\beta}(v(x,y,t))$.*

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Proof We have

$$\begin{aligned} u(x,y,t+2) =& u\Big(x+\frac{1-x-y}{2}, y+\frac{1-x-y}{2}, t+2-\frac{1-x-y}{2}\Big) \\ =& F_{\alpha,\beta}\Big(v\Big(x+\frac{1-x-y}{2}, y+\frac{1-x-y}{2}, t+2-\frac{1-x-y}{2}\Big)\Big) \\ =& F\Big(v\Big(x+\frac{1-x-y}{2}-1, y+\frac{1-x-y}{2}-1, t+2-\frac{1-x-y}{2}-1\Big)\Big) \\ =& F\circ G_\eta\Big(u\Big(x+\frac{1-x-y}{2}-1, y+\frac{1-x-y}{2}-1, t+1-\frac{1-x-y}{2}\Big)\Big) \\ =& F\circ G\big(u(x,y,t)\big). \end{aligned}$$

Similarly, we have $v(x,y,t+2) = G_\eta \circ F_{\alpha,\beta}(v(x,y,t))$. ■

Theorem 2.6 *The system (2.6) is uniquely solvable on $\overline{\Omega}\times[0,+\infty)$. Moreover, for any $t \ge 0$, we can write $t = 2n+\tau$ where $n \in \mathbb{N}$ and $\tau \in [0,2)$. Then the solution of (2.6) is given by*

$$(2.7)\quad u(x,y,t) = \begin{cases} (F_{\alpha,\beta}\circ G_\eta)^n(u_0(x+\tau,y+\tau)), & 0 \le \tau \le \frac{1-x-y}{2}, \\ (F_{\alpha,\beta}\circ G_\eta)^n(F_{\alpha,\beta}(v_0(1-y-\tau,1-x-\tau))), & \frac{1-x-y}{2} < \tau \le \frac{1-x-y}{2}+1, \\ (F_{\alpha,\beta}\circ G_\eta)^n(F_{\alpha,\beta}\circ G_\eta(u_0(x+\tau-2,y+\tau-2))), & \frac{1-x-y}{2}+1 < \tau \le 2, \end{cases}$$

and

$$(2.8)\quad v(x,y,t) = \begin{cases} (G_\eta\circ F_{\alpha,\beta})^n(v_0(x-\tau,y-\tau)), & 0 \le \tau \le \frac{x+y+1}{2}, \\ (G_\eta\circ F_{\alpha,\beta})^n(G_\eta(u_0(\tau-y-1,\tau-x-1))), & \frac{x+y+1}{2} < \tau \le \frac{x+y+1}{2}+1, \\ (G_\eta\circ F_{\alpha,\beta})^n(G_\eta\circ F_{\alpha,\beta}(v_0(x+2-\tau,y+2-\tau))), & \frac{x+y+1}{2}+1 < \tau \le 2, \end{cases}$$

where $(F_{\alpha,\beta}\circ G_\eta)^n$ represents the n-times iterative composition of $F_{\alpha,\beta}\circ G_\eta$ and $(G_\eta\circ F_{\alpha,\beta})^n$ represents the n-times iterative composition of $G_\eta\circ F_{\alpha,\beta}$.

Proof Let $t \ge 0$, there exist unique $\tau \in [0,2)$ and an integer $n \in \mathbb{N}$ such that $t = 2n+\tau$. For $(x,y) \in \Omega$, by applying Lemmas 2.3–2.5 and by induction, we have

$$u(x,y,t) = u(x,y,\tau+2n) = (F_{\alpha,\beta}\circ G_\eta)^n\big(u(x,y,\tau)\big),$$

and

$$v(x,y,t) = v(x,y,\tau+2n) = (G_\eta\circ F_{\alpha,\beta})^n\big(v(x,y,\tau)\big).$$

Proof of the uniqueness is similar to the proof in Lemma 2.4. ■

Remark 2.7 (i) From (2.7) and (2.8), u and v are chaotic if $F\circ G$ or $G\circ F$ are chaotic. (ii) After we have obtained the explicit formulas of (u,v), (w_x,w_y,w_t) can be computed by

$$w_x + w_y = u - v \quad \text{and} \quad w_t = u + v.$$

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Together with the initial data, we can solve for w by the formula

$$w(x,y,t) = \int_0^t (u+v)dt + w_0(x,y).$$

From this, we then also obtain w_x and w_t.

To summarize, we find that the solution (u,v) is fully determined by the maps $G \circ F(\cdot)$ and $F \circ G(\cdot)$. Before introducing the properties of the composite function $H_\eta(\cdot)$, we display the graphics of the composite functions $G_\eta \circ F_{\alpha,\beta}(\cdot)$ and $F_{\alpha,\beta} \circ G_\eta(\cdot)$ for certain values of η, α and β. See Figures 2 and 3. Since $F \circ G = G^{-1} \circ (G \circ F) \circ G$, these two maps are topologically conjugate. So we only need to study one of them. Let us focus on $G \circ F(\cdot)$; from now on, we fix α and β. So $G \circ F(\cdot)$ is a family of maps with a varying parameter η, denoted as

$$H_\eta(\cdot) \triangleq G_\eta \circ F_{\alpha,\beta}(\cdot). \tag{2.9}$$

Moreover, for the case $\eta > 1$, we can apply the transformation $H_{1/\eta}(\cdot) = -H_\eta(\cdot)$. For this reason, from now on we will only study the map $H_\eta(\cdot)$ for the case $\eta \in (0,1)$.

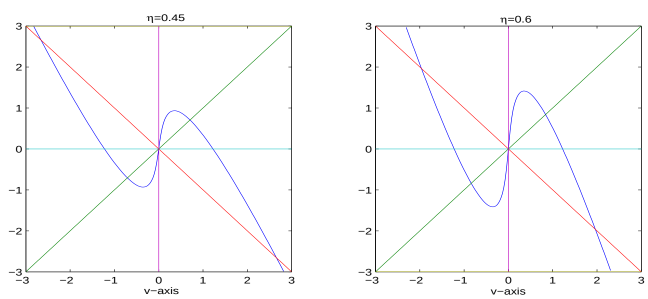

Figure 2. The graphs of $G \circ F(v)$, when $\alpha = 0.5$, $\beta = 1$ and (left) $\eta = 0.45$, (right) $\eta = 0.6$.

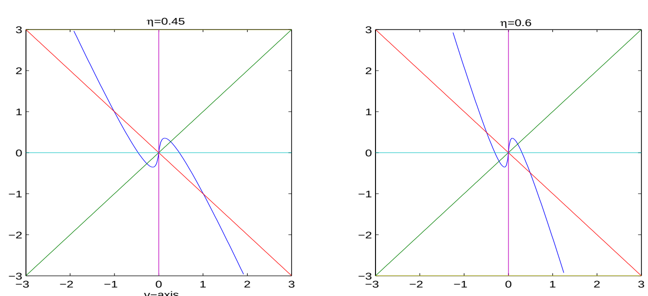

Figure 3. The graphs of $G \circ F(v)$, when $\alpha = 0.5$, $\beta = 1$ and (left) $\eta = 0.45$, (right) $\eta = 0.6$.

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3 Chaotic Dynamics of the Composite Maps

Recall from (2.3), (2.4), (2.5), and (2.9), the definition of the composite reflection map H_η. We now study the basic properties of H_η. Much of the analysis of H_η is already available in [3]. Our work in what follows is somewhat more concise, provided here for the purpose of easier referencing and self-containedness.

Lemma 3.1 *Let $0 < \alpha < 1$, $\beta > 0$. Assume η is varying on the interval $(0,1)$. Then $H_\eta(\cdot)$ is odd, and*

(i) *$H_\eta(\cdot)$ has three fixed points: 0, x_0 and $-x_0$, where*

$$x_0 = \frac{\eta+1}{2}\sqrt{\frac{\eta+\alpha}{\beta}};$$

(ii) *$-H_\eta(\cdot)$ has three fixed points: 0, x_1 and $-x_1$, where*

$$x_1 = \frac{\eta+1}{2\eta}\sqrt{\frac{1+\alpha\eta}{\beta\eta}};$$

(iii) *the equation $H_\eta(x) = 0$ has three roots: 0, x_2 and $-x_2$, where*

$$x_2 = \sqrt{\frac{1+\alpha}{\beta}};$$

(iv) *the equation $\frac{\partial H_\eta(x)}{\partial x} = 0$ has two roots: x_3 and $-x_3$, where*

$$x_3 = \frac{2-\alpha}{3}\sqrt{\frac{1+\alpha}{3\beta}};$$

(v) *$H_\eta(\cdot)$ has two local extremal values M and m :*

$$M = H_\eta(x_3) = \frac{1+\alpha}{3}\cdot\frac{1+\eta}{1-\eta}\cdot\sqrt{\frac{1+\alpha}{3\beta}},$$
$$m = H_\eta(-x_3) = -H_\eta(x_3) = -M,$$

and $H_\eta(\cdot)$ is strictly increasing on $(-x_3, x_3)$, but strictly decreasing on $(-\infty, -x_3]$ and $[x_3, +\infty)$.

Proof All of the proofs are straightforward. ∎

Remark 3.2 From Figure 2, we find that $0 < x_3 < x_0 < x_2 < x_1$.

Fix $0 < \alpha < 1$, $\beta > 0$. Consider the equations

$$(M =)\frac{1+\alpha}{3}\cdot\frac{1+\eta}{1-\eta}\cdot\sqrt{\frac{1+\alpha}{3\beta}} = \sqrt{\frac{1+\alpha}{\beta}}(= x_2), \tag{3.1}$$
$$(M =)\frac{1+\alpha}{3}\cdot\frac{1+\eta}{1-\eta}\cdot\sqrt{\frac{1+\alpha}{3\beta}} = \frac{\eta+1}{2\eta}\sqrt{\frac{1+\alpha\eta}{\beta\eta}}(= x_1).$$

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The first and second equations in (3.1) determine two critical values: η_1 and η_2, respectively. More specifically, we have

$$\eta_1 = \frac{3\sqrt{3} - (1+\alpha)}{3\sqrt{3} + 1 + \alpha}, \tag{3.2}$$

whereas η_2 satisfies the equation

$$\Big(\frac{1}{\eta_2} - 1\Big)\sqrt{\frac{1}{\eta_2} + \alpha} = 2\Big(\frac{1+\alpha}{3}\Big)^{\frac{3}{2}}. \tag{3.3}$$

Since

$$\Big(\frac{1}{\eta_1} - 1\Big)\sqrt{\frac{1}{\eta_1} + \alpha} = \Big(\frac{1}{\eta_1} + 1\Big)\frac{1+\alpha}{3\sqrt{3}}\sqrt{\frac{1}{\eta_1} + \alpha} > 2\frac{1+\alpha}{3\sqrt{3}}\sqrt{1+\alpha} = 2\Big(\frac{1+\alpha}{3}\Big)^{\frac{3}{2}},$$

we have $0 < \eta_1 < \eta_2 < 1$.

Lemma 3.3 *Let $0 < \alpha < 1$, $\beta > 0$ and $\eta \in (0,1)$. The following hold:*

(i) *If $0 < \eta \le \eta_2$, i.e., $M \le x_1$, then the iterates of every point in the set $V = (-\infty, -x_1) \bigcup (x_1, \infty)$ escape to $\pm\infty$, while those of any point in $R \smallsetminus \overline{V}$ are attracted to the bounded invariant interval $I = [-M, M]$ of $H_\eta(\cdot)$.*

(ii) *If $\eta_2 < \eta < 1$, then there is no bounded invariant interval for the map H_η.*

Proof The results follow from Lemma 3.1 and the piecewise monotonic properties of $G_\eta \circ F_{\alpha,\beta}$, as can be confirmed from Fig. 2. ∎

We recall some properties about periodic points and total variations from the literature for use in the next section (in particular, as the prerequisites for Theorem 4.2).

Lemma 3.4 ([7, Main Theorem 8]) *Let $f(\cdot) \in C^0(I, I)$, where I is a bounded interval, and V is the total variation. Assume that $f(\cdot)$ has two distinct fixed points and a periodic point with period 2. Then $V_{[x_0,p]}(\cdot)$; the total variation on the interval $[x_0, p]$ satisfies*

$$\lim_{n\to\infty} V_{[x_0,p]}(f^n(\cdot)) = \infty, \tag{3.4}$$

where x_0 is the smaller fixed point and p is the periodic point with period 2.

Note that the rate of growth with respect to n in (3.4) is not exponential.

Lemma 3.5 ([1, Section II, Lemma 3]) *Let $f(\cdot) \in C^0(I, I)$. If $f(\cdot)$ is turbulent, then $f(\cdot)$ has periodic points of all periods.*

Lemma 3.6 ([7, Lemma 7.4]) *Let $f(\cdot) \in C^0(I, I)$ and f be piecewise monotone. Then the following conditions are equivalent:*

(i) *$f(\cdot)$ has a periodic point whose period is not a power of 2 ($1 = 2^0$ is regarded as a power of 2);*

(ii) *the growth rate of the total variation of $f^n(\cdot)$ is exponential w.r.t. n.*

Now, return to the dynamics of the map $H_\eta(\cdot)$ defined by (2.9). Recall the notation used in Lemmas 3.1 and 3.3 about x_i, $i = 0, 1, 2, 3$, M and m.

Proposition 3.7 *Let $0 < \alpha < 1$, $\beta > 0$ be fixed, and let $\eta \in (0,1)$ be a varying variable. Given η_1 and η_2, defined by (3.2) and (3.3), respectively. Then, if $\eta \in [\eta_1, \eta_2]$, $H_\eta(\cdot)$ has periodic points in $[-M, M]$ with periods which are not a power of 2.*

Proof The proof follows from [12, Theorem 3.1]. ∎

From Lemmas 3.4–3.6 and Proposition 3.7, we have the following result.

Proposition 3.8 *Let $0 < \alpha < 1$, $\beta > 0$ be fixed, and let $\eta \in (0,1)$ be a varying variable. Then, for every $\eta \in [\eta_1, \eta_2]$, there are positive constants ε, c_1 and c_2 such that*

$$V_{I_\varepsilon}(H_\eta^n) \geq c_1(\exp(c_2 n)), \text{ as } n \to \infty,$$

where $I_\varepsilon = [0, \varepsilon]$ or $[-\varepsilon, 0]$. Thus, the rate of growth is exponential.

4 Chaotic Vibration Phenomenon of the PDE System

Recall the PDE system considered in Section 1:

$$(4.1)\quad \begin{cases} w_{tt} = \Delta w + 2w_{xy}, & (x,y) \in \Omega,\ t > 0, \\ w_t = -\eta(w_x + w_y), & (x,y) \in \Gamma_1,\ t > 0, \\ w_t = \alpha(w_x + w_y) - \beta(w_x + w_y)^3, & (x,y) \in \Gamma_3,\ t > 0, \\ w(x,y,t) = 0, & (x,y) \in \Gamma_2 \cup \Gamma_4,\ t > 0, \\ w(x,y,0) = w_0(x,y), \quad w_t(x,y,0) = w_1(x,y), & (x,y) \in \overline{\Omega}. \end{cases}$$

To our knowledge, there is no universally accepted definition of chaos for PDEs in 2D. Following [7], where those authors characterized the chaotic behavior by the growth rate of the total variation, we give a suitable definition of chaos for system (4.1).

First, recall that a *simple curve* $\mathcal{C}$ in a 2D domain Ω is defined through a continuous function g from a real number interval $I = [a, b]$ to Ω. The image $g(I)$ is called a *curve*. The adjective "simple" here means that g is injective. More specifically, $\mathcal{C}$ is the set of all $g(s)$ when $s \in [a, b]$, where

$$g(s) = (x_{\mathcal{C}}(s), y_{\mathcal{C}}(s)) \in \Omega.$$

Definition 4.1 We say that a PDE system of w on the 2D domain Ω is chaotic or has chaotic vibration phenomenon, if there exists at least one direction $\overrightarrow{l}$ in $\mathbb{R}^3$, such that for any simple curve $\mathcal{C}$ with $g(a), g(b) \in \Gamma$ and $g(\xi) \in \Omega$, for any $\xi \in (a, b)$, the directional derivative $D_{\overrightarrow{l}} w$ satisfies

(i) $D_{\overrightarrow{l}} w(x_{\mathcal{C}}(s), y_{\mathcal{C}}(s), t)$ is uniformly bounded;

(ii) $V_{[a,b]}(D_{\overrightarrow{l}} w(x_{\mathcal{C}}(\cdot), y_{\mathcal{C}}(\cdot), t))$ is exponentially increasing as time t increases.

With the above prerequisites ready, we are now in a position to state the final main theorem of this section.

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Theorem 4.2 *Consider the system (2.1). Let* $0 < \alpha < 1$*,* $\beta > 0$ *be fixed, for* $\eta \in [\eta_1, \eta_2]$*, where* η_1 *and* η_2 *are given by (3.2) and (3.3), respectively. Then, for a certain class of initial conditions, the system (4.1) is chaotic.*

Proof Let $\eta \in [\eta_1, \eta_2]$. From Lemma 3.3, $G_\eta \circ F_{\alpha,\beta}$ has an invariant interval $[-M, M]$, where M is a local maximum of $G_\eta \circ F_{\alpha,\beta}$ given by

$$M = \frac{1+\alpha}{3} \cdot \frac{1+\eta}{1-\eta} \cdot \sqrt{\frac{1+\alpha}{3\beta}}.$$

Choose the initial data $w_0 = 0$ and $w_1 \in C^2(\overline{\Omega})$ satisfying

$$w_1(x,y) \begin{cases} > 0 & \text{for } (x,y) \in \Gamma, \\ = 0 & \text{for } (x,y) \in \Omega. \end{cases} \tag{4.2}$$

Furthermore, assume that

$$\text{Range}(w_1) \cup \text{Range}\Big(\frac{\eta+1}{\eta-1} \cdot w_1\Big) \cup \text{Range}\Big(\frac{\eta+1}{\eta-1}(F_{\alpha,\beta}(w_1))\Big) \subset [-M, M]. \tag{4.3}$$

Consider the direction vector $\overrightarrow{l} = (-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$, and let $v = D_{\overrightarrow{l}} w$. In fact, v is the same definition as (2.5).

Consider any simple curve $\mathcal{C}$ in Ω with

$$g(a), g(b) \in \Gamma, \quad g(\xi) \in \Omega \text{ for any } \xi \in (a,b). \tag{4.4}$$

Under assumption (4.3), from Lemmas 2.4 and 3.3, we have

$$|v(x_{\mathcal{C}}(s), y_{\mathcal{C}}(s), t)| \le M \quad \text{for } t \ge 0,\ s \in [a,b],$$

which is to say that $D_{\overrightarrow{l}} w(x_{\mathcal{C}}(s), y_{\mathcal{C}}(s), t)$ is uniformly bounded.

Moreover, given any $t \ge 0$, let $t = 2n + \tau$ where $\tau \in [0,2)$ and $n \in \mathbb{N}$. From Theorem 2.6, we have

$$v\big(x_{\mathcal{C}}(\xi), y_{\mathcal{C}}(\xi), t\big) = (G_\eta \circ F_{\alpha,\beta})^n\big(v(x_{\mathcal{C}}(\xi), y_{\mathcal{C}}(\xi), \tau)\big) \quad \text{for } \xi \in [a,b].$$

It follows from Proposition 3.8(iii) that there exist constants $c_1 > 0$ and $c_2 > 0$ such that for any $\epsilon > 0$,

$$V_{[0,\epsilon]}\big(G_\eta \circ F_{\alpha,\beta}\big)^n \ge c_1 e^{c_2 n}, \quad n \in \mathbb{N}. \tag{4.5}$$

Under assumptions (4.2) and (4.4), we have an $\epsilon_0 > 0$ such that

$$[0, \epsilon_0] \subset \text{Range}\big(v\big(x_{\mathcal{C}}(\cdot), y_{\mathcal{C}}(\cdot), \tau\big)\big).$$

Take $\epsilon = \epsilon_0$ in (4.5). Consequently, we have

$$\begin{aligned} V_{[a,b]}\big(D_{\overrightarrow{l}} w\big(x_{\mathcal{C}}(\cdot), y_{\mathcal{C}}(\cdot), t\big)\big) &= V_{[a,b]}\big(v\big(x_{\mathcal{C}}(\cdot), y_{\mathcal{C}}(\cdot), t\big)\big) \\ &= V_{[a,b]}\big((G_\eta \circ F_{\alpha,\beta})^n\big(v\big(x_{\mathcal{C}}(\cdot), y_{\mathcal{C}}(\cdot), \tau\big)\big)\big) \\ &\ge c_1 e^{c_2 n} \ge c_1 e^{c_2 \frac{t-\tau}{2}}. \end{aligned}$$

Thus, system (4.1) is chaotic. ∎

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5 Numerical Simulations

In this section, we present numerical simulations for system (1.12) to illustrate the theoretical results.

Throughout, we fix $\alpha = 0.5$, $\beta = 1$, and let η be a varying parameter. The initial conditions are chosen to be

$$w_0(x,y) = 0, \quad w_1(x,y) = \frac{1}{10}\cdot\left(\left((x+y)^2-1\right)^3\cdot\left((x-y)^2-1\right)^3\right),$$

for all $(x,y) \in \overline{\Omega}$, and these satisfy the conditions in the proof of Theorem 4.2.

We can obtain the three critical parameter values by following our established recipes:

$$\eta_1 \approx 0.552, \qquad \eta_2 \approx 0.667.$$

Theorem 4.2 shows that when $\eta \in [0.552, 0.667]$ the system (1.12) is chaotic. To verify this, in our numerical simulation, we compare two cases: $\eta = 0.45$ and $\eta = 0.6$.

Numerical simulations for w_t are provided in Figures 4 and 5.

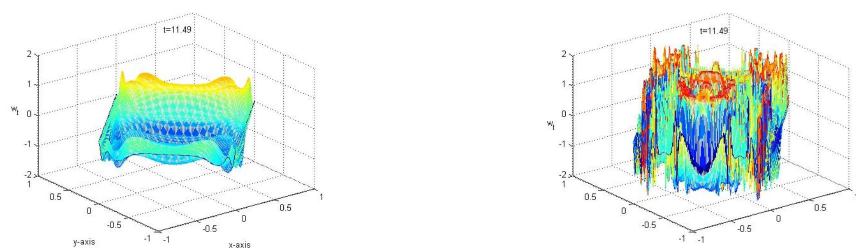

Figure 4. The profile of w_t at t=11.49 for $\eta = 0.45$(left) and $\eta = 0.6$(right).

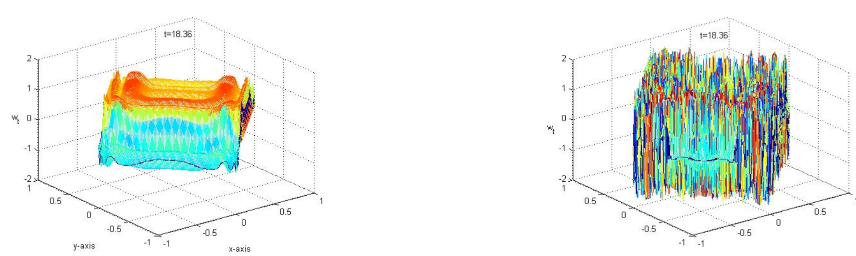

Figure 5. The profile of w_t at t=18.36 for $\eta = 0.45$(left) and $\eta = 0.6$(right).

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Since chaotic vibration is a *dynamics* process, we provide video animations for visualization, viewable at the following URLs :
for $\eta = 0.45$ in time duration $[0, 20]$,
https://www.dropbox.com/s/m9o7zffmj39ewd7/nonchaotic.mp4?dl=0;
for $\eta = 0.6$ in time duration $[0, 20]$,
https://www.dropbox.com/s/4exocjirixmre7g/chaotic2.mp4?dl=0.
These numerical simulations are also consistent with Theorem 4.2.

Concluding Remarks

Chaos in multidimensional dynamic processes manifests complex behavior and phenomena. Here we get to see some of these through rigorously justified simulations and video animations. But there is too much we have not been able to rigorously treat or to even just simulate computationally. The authors hope to continue this investigation of multidimensional chaos by improving and generalizing the methodology here as well as by developing new, more powerful techniques.

A Appendix: Derivation of $E'(t)$

Theorem A.1 *Consider the system (1.12) and the energy function*

$$E(t) = \frac{1}{2}\int_\Omega w_t^2 + (w_x + w_y)^2 dS, \quad t > 0.$$

Then for sufficiently smooth data, the derivative of the energy functional has the following form:

$$E'(t) = \sqrt{2}\eta \int_{\Gamma_1} (w_x + w_y)^2 d\sigma + \sqrt{2}\int_{\Gamma_3} (w_x + w_y)^2 \big(\alpha - \beta (w_x + w_y)^2\big)\, d\sigma, \quad t > 0.$$

Proof Let w be up to C^2-smooth. Consider the vector field

$$\mathbb{H} = (w_x + w_y, w_x + w_y).$$

Then

$$\operatorname{div}(\mathbb{H}) = \nabla^2 w + 2w_{xy}.$$

For $t > 0$, we have

$$E'(t) = \int_\Omega w_t \operatorname{div}(\mathbb{H}) + \mathbb{H} \cdot \nabla w_t dS,$$

since

$$\operatorname{div}(w_t \mathbb{H}) = w_t \operatorname{div}(\mathbb{H}) + \mathbb{H} \cdot \nabla w_t.$$

Applying Green's formula, we have

$$E'(t) = \int_\Omega w_t \operatorname{div}(\mathbb{H}) + \mathbb{H} \cdot \nabla w_t dS = \int_\Omega \operatorname{div}(w_t \mathbb{H}) dS = \int_\Gamma (w_t \mathbb{H} \cdot \vec{n}) d\sigma.$$

Note that $\mathbb{H} \cdot \vec{n} = 0$ on the boundaries $\Gamma \setminus (\Gamma_1 \cup \Gamma_3)$. By applying the boundary conditions on Γ_1 and Γ_3, respectively, we have for all $t > 0$,

$$E'(t) = \sqrt{2}\eta \int_{\Gamma_1} (w_x + w_y)^2 d\sigma + \sqrt{2}\int_{\Gamma_3} (w_x + w_y)^2 \big(\alpha - \beta (w_x + w_y)^2\big)\, d\sigma. \tag{A.1}$$ ∎

References

[1] L. S. Block and W. A. Coppel, *Dynamics in one dimension*. Lecture Notes in Mathematics, 1513, Springer-Verlag, Berlin, 1992. http://dx.doi.org/10.1007/BFb0084762

[2] G. Chen, S. B. Hsu, J. Zhou, *Chaotic vibrations of the one-dimensional wave equation due to a self-excitation boundary condition. Part I: Controlled hysteresis*. Appendix C. by G. Chen amd G. Crosta. Trans. Amer. Math. Soc. **350**(1998), no. 11, 4265–4311. http://dx.doi.org/10.1090/S0002-9947-98-02022-4

[3] ______, *Chaotic vibration the one-dimensional wave equation due to a self-excitation boundary condition PartII: Energy Pumping, Period doubling and homoclinic orbits*. Internat. J. Bifur. Chaos Appl. Sci. Engrg. **8**(1998), no. 3, 423-445. http://dx.doi.org/10.1142/S0218127498000280

[4] ______, *Chaotic vibrations of the one-dimensional wave equation due to a self-excitation boundary condition. III. Natural hysteresis memory effects*. Internat. J. Bifur. Chaos Appl. Sci. Engrg. **8**(1998), no. 3, 447–470. http://dx.doi.org/10.1142/S0218127498000292

[5] ______, *Snapback repellers as a cause of chaotic vibration of the wave equation with a van der Pol boundary condition and energy injection at the middle of the span*. J. Math. Phys. **39**(1998), 6459–6489. http://dx.doi.org/10.1063/1.532670

[6] ______, *Nonisotropic spatiotemporal chaotic vibration of the wave equation due to mixing energy transport and a van der Pol boundary condition*. Internat. J. Bifur. Chaos Appl. Sci. Engrg. **12**(2002), no. 3, 535–559. http://dx.doi.org/10.1142/S0218127402004504

[7] G. Chen, T. Huang, and Y. Huang, *Chaotic behavior of interval maps and total variations of iterates*. Internat. J. Bifur. Chaos Appl. Sci. Engrg. **14**(2004), no. 7, 2161–2186. http://dx.doi.org/10.1142/S0218127404010540

[8] F. Columbini, E. De Giorgi, and S. Spagnolo, *Sur les équations hyperboliques avec des coefficients qui ne dépendent que du temps*. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **6**(1979), no. 3, 511–559.

[9] F. Columbini, E. Jannelli, and S. Spagnolo, *Well-posedness in the Gevrey classes of the Cauchy problem for a non-strictly hyperbolic equation with coefficients depending on time*. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **10**(1983), no. 2, 291–312.

[10] F. Columbini and S. Spagnolo, *An example of a weakly hyperbolic Cauchy problem not well posed in* C^∞. Acta Math. **148**(1982), no. 1, 243–253. http://dx.doi.org/10.1007/BF02392730

[11] R. Courant and D. Hilbert, *Methods of mathematical physics, Vol. II: Partial differential equations*. Wiley-Interscience, New York-London, 1962.

[12] L. Li, Y. Chen, and Y. Huang, *Nonisotropic spatiotemporal chaotic vibrations of the one-dimensional wave equation with a mixing transport term and general nonlinear boundary condition*. J. Math. Phys. **51**(2010), no. 10, 102703. http://dx.doi.org/10.1063/1.3486070

[13] J. Liu, Y. Huang, H. Sun, and M. Xiao, *Numerical methods for weak solution of wave equation with van der Pol type nonlinear boundary conditions*. Numer. Methods Partial Differential Equations **32**(2016), no. 2, 373–398. http://dx.doi.org/10.1002/num.21997

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