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EDGE WEIGHTING FUNCTIONS ON THE SEMITOTAL DOMINATING SET OF CLAW-FREE GRAPHS

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Abstract

In an isolate-free graph G, a subset S of vertices is a *semitotal dominating set* of G if it is a dominating set of G and every vertex in S is within distance 2 of another vertex of S. The *semitotal domination number* of G, denoted by $\gamma_{t2}(G)$, is the minimum cardinality of a semitotal dominating set in G. Using edge weighting functions on semitotal dominating sets, we prove that if $G \neq N_2$ is a connected claw-free graph of order $n \geq 6$ with minimum degree $\delta(G) \geq 3$, then $\gamma_{t2}(G) \leq \frac{4}{11}n$ and this bound is sharp, disproving the conjecture proposed by Zhu *et al.* ['Semitotal domination in claw-free cubic graphs', *Graphs Combin.* **33**(5) (2017), 1119–1130].

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1. Introduction

Domination and its variations have been extensively studied (see, for example, [1, 2, 4, 6, 11]). A subset D of vertices in a graph G is a *dominating set* of G if every vertex of $V(G) \setminus D$ is adjacent to a vertex in D. The minimum cardinality $\gamma(G)$ of a dominating set is called the *dominating number* of G. A subset D of vertices in a graph G is a *total dominating set* of G if every vertex of $V(G)$ is adjacent to a vertex in D. The minimum cardinality $\gamma_t(G)$ of a total dominating set is called the *total dominating number* of G. It is worth noting that the study of total dominating sets is meaningful only on an isolate-free graph.

Semitotal domination, introduced by Goddard *et al.* [3] in 2014, is a relaxed form of total domination. A subset D of vertices in an isolate-free graph G is a *semitotal dominating set*, abbreviated semi-TD-set, of G if it is a dominating set of G and every vertex in D is within distance 2 of another vertex of D. The *semitotal domination number* of G, denoted by $\gamma_{t2}(G)$, is the minimum cardinality of a semi-TD-set in G.

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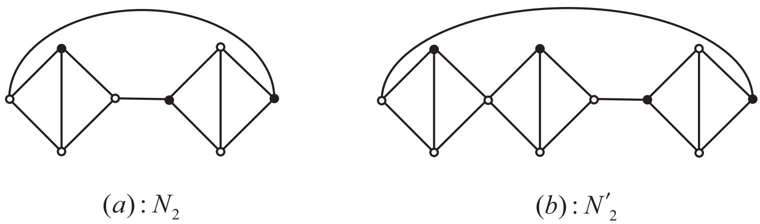

FIGURE 1. Two graphs: N_2 and N'_2, where the black vertices form a minimum semi-TD-set of their respective graphs.

We refer to a minimum semi-TD-set of G as a $\gamma_{t2}(G)$-*set*. Since every total dominating set is a semi-TD-set and every semi-TD-set is a dominating set, $\gamma(G) \leqslant \gamma_{t2}(G) \leqslant \gamma_t(G)$. However, the semitotal domination number is very different from the domination and total domination number. For example, the total domination number cannot be compared with the matching number, while the semitotal domination number is comparable with the matching number and cannot be greater than the matching number plus one (see [7, 8]). That makes the study of semitotal domination interesting.

There is much interest in bounds for the semitotal domination number of graphs. For example, Goddard *et al.* [3] proved that if G is a connected graph of order $n \geqslant 4$, then $\gamma_{t2}(G) \leqslant \frac{1}{2}n$ and proposed Conjecture 1.1 below. Henning and Marcon [9] proved that if G is a connected claw-free cubic graph of order $n \geqslant 10$, then $\gamma_{t2}(G) \leq \frac{4}{11}n$, and conjectured that this bound can be improved to $\frac{1}{3}n$ if $G \notin \{K_4, N_2\}$, where N_2 is a graph shown in Figure 1(a). This conjecture was solved by Zhu *et al.* [13] and they proposed Conjecture 1.2 below. Zhu and Liu [12] proved that Conjectures 1.1 and 1.2 hold for line graphs with minimum degree 3 and 4, respectively. In [5], Henning established the tight upper bounds on the upper semitotal domination number of a regular graph using edge weighting functions. For algorithmic aspects of semitotal domination in graphs, Henning and Pandey [10] showed the semitotal domination problem is NP-complete for planar graphs, chordal bipartite graphs and split graphs.

CONJECTURE 1.1 [3]. If $G \neq K_4$ is a graph of order n with minimum degree $\delta(G) \geqslant 3$, then $\gamma_{t2}(G) \leqslant \frac{2}{5}n$.

CONJECTURE 1.2 [13]. If $G \neq N_2$ is a connected claw-free graph of order $n \geq 6$ with minimum degree $\delta(G) \geq 3$, then $\gamma_{t2}(G) \leq \frac{1}{3}n$.

Inspired by [5], using edge weighting functions, we establish the tight upper bound on the semitotal domination number of a connected claw-free graph with minimum degree at least 3. In Section 2, we give some basic definitions and a lemma as preliminaries. In Section 3, we prove that if $G \neq N_2$ is a connected claw-free graph of order $n \geq 6$ with minimum degree $\delta(G) \geq 3$, then $\gamma_{t2}(G) \leq \frac{4}{11}n$. Also, we construct a graph attaining this bound and thus disprove Conjecture 1.2.

2. Preliminaries

In this section, we introduce some basic definitions and a useful lemma.

Let $G = (V(G), E(G))$ be a connected finite simple undirected graph with vertex set $V(G)$ and edge set $E(G)$ of order $n = |V(G)|$. For a vertex $v \in V(G)$, we denote by $N_G(v) = \{u \in V(G) \mid uv \in E(G)\}$ the *neighbourhood* of v and by $N_G[v] = N_G(v) \cup \{v\}$ the *closed neighbourhood* of v. The *degree* of v is $d_G(v) = |N_G(v)|$ and the number $\delta(G) = \min\{d_G(v) \mid v \in V(G)\}$ is the *minimum degree* of G. We call a path connecting vertices u and v a (u, v)-*path*. The *distance* $d_G(u, v)$ between u and v is the length of a shortest (u, v)-path in G. For a subset S of $V(G)$, we denote by $N_S(v)$ the neighbourhood of v restricted on S and by $G[S]$ the subgraph of G induced by S, while the graph $G - S$ is the graph obtained from G by deleting the vertices in S and all edges incident with S. A graph is *claw-free* if it does not contain the complete bipartite graph $K_{1,3}$ as an induced subgraph. If there is no confusion, then the subscript G is omitted in the notation, such as $N(v), d(v), d(u, v)$ and so on.

Now consider S_1 and S_2 which are two disjoint subsets of $V(G)$. Let $E[S_1, S_2] = \{u_1u_2 \mid u_1 \in S_1 \text{ and } u_2 \in S_2\}$. For a vertex v of S, the S-*external private neighbourhood* of v, denoted by $epn(v, S)$, is the set of all vertices in $V(G) \setminus S$ that are adjacent to v but to no other vertex of S. In other words, if $u \in epn(v, S)$, then $u \in V(G) \setminus S$ and $N_G(u) \cap S = \{v\}$. The S-*internal private* 2-*neighbourhood* of v, denoted by $ipn_2(v, S)$, is the set of all vertices in $S \setminus \{v\}$ that are within distance 2 of v in G but at a distance greater than 2 from every other vertex of S. In other words, if $u \in ipn_2(v, S)$, then $u \in S \setminus \{v\}$, $d(v, u) \leqslant 2$ and $d(u, w) > 2$ for any vertex $w \in S \setminus \{u, v\}$.

A semi-TD-set in a graph G is a *minimal semi-TD-set* if it contains no semi-TD-set of G as a proper subset. The following result in [8] provides a characterisation of minimal semi-TD-sets.

LEMMA 2.1 [8]. *Let S be a semi-TD-set in a graph G. Then, S is a minimal semi-TD-set of G if and only if every vertex $v \in S$ satisfies at least one of the following three properties:*

(a) *the vertex v is isolated in $G[S]$;*
(b) *$ipn_2(v, S) \neq \emptyset$;*
(c) *$epn(v, S) \neq \emptyset$.*

3. Main result

In this section, we establish the tight upper bound on the semitotal domination number of a connected claw-free graph with minimum degree at least 3 using edge weighting functions. Before that, we define two graphs N_2 and N_2' as in Figure 1. Note that N_2' is a graph attaining the bound of Theorem 3.1. This shows that Conjecture 1.2 is not true.

THEOREM 3.1. *If $G \neq N_2$ is a connected claw-free graph of order $n \geq 6$ with minimum degree $\delta(G) \geq 3$, then $\gamma_{t2}(G) \leq \frac{4}{11}n$, and this bound is sharp.*

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PROOF. Suppose that the theorem is false. Let G be a counterexample such that $|V(G)|$ is as small as possible. By the choice of G, $G \neq N_2$ is a connected claw-free graph of order $n \geq 6$ with $\delta(G) \geq 3$ such that $\gamma_{t2}(G) > \frac{4}{11}n$, and any connected claw-free graph $G' \neq N_2$ of order $n' < n$ with $\delta(G') \geq 3$ has $\gamma_{t2}(G') \leq \frac{4}{11}n'$, where $n' \geq 6$.

For a $\gamma_{t2}(G)$-set S, set $\overline{S} = V(G) \setminus S$. Define sets $A^S = \{v \in S \mid ipn_2(v, S) \neq \emptyset\}$, $A_1^S = \{v \in A^S \mid v \in ipn_2(v', S)$ for some vertex $v' \in S\}$ and $A_2^S = A^S \setminus A_1^S$. Let $v \in A_1^S$ and $v \in ipn_2(v', S)$. Then v' is the only vertex of S within distance 2 from v in G. Since $ipn_2(v, S) \neq \emptyset$, $v' \in ipn_2(v, S)$ and $v' \in A_1^S$. Further, $ipn_2(v, S) = \{v'\}$ and $ipn_2(v', S) = \{v\}$. This implies that the vertices in A_1^S are paired off. For each vertex $u \in A_2^S$, let $S_u = ipn_2(u, S) \cup \{u\}$. If $u' \in ipn_2(u, S)$, then $u' \notin A^S$. Otherwise, $u' \in A^S$ and $u \in ipn_2(u', S)$, for u is the only vertex of S within distance 2 from u' in G, which contradicts the fact that $u \in A_2^S$. We note that if u_1 and u_2 are two distinct vertices in A_2^S, then $ipn_2(u_1, S) \cap ipn_2(u_2, S) = \emptyset$. Hence, $S_{u_1} \cap S_{u_2} = \emptyset$ for each pair of different vertices $u_1, u_2 \in A_2^S$. Let $B^S = \bigcup_{u \in A_2^S} S_u$ and $C^S = S \setminus (A_1^S \cup B^S)$. Further, we partition C^S into three subsets: $C_0^S = \{z \mid z \in C^S$ and $|epn(z, S)| = 0\}$, $C_1^S = \{z \mid z \in C^S$ and $|epn(z, S)| = 1\}$, and $C_2^S = \{z \mid z \in C^S$ and $|epn(z, S)| \geq 2\}$. Then $S = A_1^S \cup B^S \cup C_0^S \cup C_1^S \cup C_2^S$.

In particular, a vertex u of A_2^S is special if $|S_u| = 2$, $d(u') = 3$ and $|N_{\overline{S}}(u) \setminus N_{\overline{S}}(u')| = 1$, where $\{u'\} = S_v \setminus \{u\}$. Further, we define sets $A^S_{\underset{\sim}{2}} = \{u \mid u \in A_2^S$ and u is special$\}$ and $C^S_{\underset{\sim}{i}} = \{z \mid z \in C_i^S$ and $d(z) = 3\}$ for $i \in \{0, 1\}$.

A diamond in G is an induced graph of G isomorphic to $K_4 - e$. We call a diamond of G a special diamond if each of its vertices has degree 3 in G. Let $\mathcal{D}$ be the set of vertices in a special diamond. Among all $\gamma_{t2}(G)$-set, we choose a $\gamma_{t2}(G)$-set S satisfying the following conditions:

(1) the number of edges in $G[S]$, denoted by $\lambda(S)$, is minimised;
(2) subject to condition (1), $|\mathcal{D} \cap S|$ is minimised;
(3) subject to condition (2), $|C^S_{\underset{\sim}{0}}|$ is minimised;
(4) subject to condition (3), $|C_0^S|$ is minimised;
(5) subject to condition (4), $|C^S_{\underset{\sim}{1}}|$ is minimised.

We prove the following claim about the set S. □

Claim 1. S is an independent set of G.

Suppose to the contrary that there exist two adjacent vertices v_1 and v_2 in S. If $epn(v_1, S) \neq \emptyset$ or $epn(v_2, S) \neq \emptyset$, then without loss of generality, consider $epn(v_1, S) \neq \emptyset$. Let x_1 be a vertex in $epn(v_1, S)$. Since G is claw-free, each vertex of $N(v_1) \setminus \{x_1, v_2\}$ is adjacent to either x_1 or v_2. Thus, $S_1 = (S \setminus \{v_1\}) \cup \{x_1\}$ is a $\gamma_{t2}(G)$-set. However, $\lambda(S_1) < \lambda(S)$, for x_1 is adjacent to no vertex of $S \setminus \{v_1\}$, which contradicts the choice of S. Hence, $epn(v_1, S) = \emptyset$ and $epn(v_2, S) = \emptyset$. By Lemma 2.1, $ipn_2(v_1, S) \neq \emptyset$ and $ipn_2(v_2, S) \neq \emptyset$.

If $ipn_2(v_1, S) \neq \{v_2\}$, then there exists a vertex $v_3 \in ipn_2(v_1, S) \setminus \{v_2\}$. Combined with $v_1v_2 \in E(G)$, $d(v_1, v_3) = 2$. As $v_3 \in ipn_2(v_1, S)$, any vertex of $N(v_3)$ belongs to $\overline{S}$ and is adjacent to no vertex of $S \setminus \{v_1, v_3\}$. Let x_2 be a vertex connecting v_1 and v_3. Then $x_2v_2 \notin E(G)$. Since G is claw-free, each vertex of $N(v_1) \setminus \{x_2, v_2\}$ is adjacent to either x_2 or v_2. When all vertices of $N(v_3) \setminus \{x_2\}$ are adjacent to x_2, $(S \setminus \{v_1, v_3\}) \cup \{x_2\}$ is a semi-TD-set of G, which contradicts the minimality of S. However, when there exists a vertex $x_3 \in N(v_3) \setminus \{x_2\}$ such that $x_2x_3 \notin E(G)$, each vertex of $N(v_3) \setminus \{x_2, x_3\}$ is adjacent to either x_2 or x_3 as G is claw-free. Then $S_1 = (S \setminus \{v_1, v_3\}) \cup \{x_2, x_3\}$ is a $\gamma_{t2}(G)$-set. But, $\lambda(S_1) < \lambda(S)$, which is a contradiction.

Hence, $ipn_2(v_1, S) = \{v_2\}$. Similarly, $ipn_2(v_2, S) = \{v_1\}$. Further, all vertices in $N_{\overline{S}}(v_1) \cup N_{\overline{S}}(v_2)$ are adjacent to no vertex of $S \setminus \{v_1, v_2\}$. Recall that $epn(v_1, S) = \emptyset$ and $epn(v_2, S) = \emptyset$. Thus, $N_{\overline{S}}(v_1) = N_{\overline{S}}(v_2)$. Since $n \geq 6$, $\gamma_{t2}(G) \geq \frac{4}{11}n > 2$. This implies that $\{v_1, v_2\}$ is not a $\gamma_{t2}(G)$-set. As G is connected, there exists a vertex x_4 in $\overline{S} \setminus N_{\overline{S}}(v_1)$ such that x_4 is adjacent to a vertex x_5 in $N_{\overline{S}}(v_1)$. We note that x_4 has a neighbour in $S \setminus \{v_1, v_2\}$. When all vertices of $N_{\overline{S}}(v_1)$ are adjacent to x_5, $S_1 = (S \setminus \{v_1, v_2\}) \cup \{x_5\}$ is a semi-TD-set of G with $|S_1| < |S|$, which is a contradiction. When there exists a vertex $x_6 \in N_{\overline{S}}(v_1)$ such that $x_5x_6 \notin E(G)$, each vertex of $N_{\overline{S}}(v_1) \setminus \{x_5, x_6\}$ is adjacent to either x_5 or x_6 as G is claw-free. Then $S_1 = (S \setminus \{v_1, v_2\}) \cup \{x_5, x_6\}$ is a $\gamma_{t2}(G)$-set with $\lambda(S_1) < \lambda(S)$, which is a contradiction. This completes the proof of Claim 1.

Combining Claim 1 and the claw-freeness of G, we see that x has at most two neighbours in S for any vertex x of $\overline{S}$. We define an edge weighting function w on G: $[\overline{S}, S] \to [0, 1]$. For each vertex $x \in \overline{S}$, the function w assigns weight for each edge $e \in [\{x\}, S]$ as follows.

- If x is an S-external private neighbour, then for the unique edge $e \in [\{x\}, S]$, $w(e) = 1$.
- If x is not an S-external private neighbour and $N_{C^S_{\tilde{0}}}(x) = \emptyset$, then $w(e) = \frac{1}{2}$ for each edge $e \in [\{x\}, S]$.
- Assume that x is not an S-external private neighbour and $N_{C^S_{\tilde{0}}}(x) \neq \emptyset$. Let $N_{\overline{S}}(x) = \{y_1, y_2\}$, where $y_1 \in N_{C^S_{\tilde{0}}}(x)$. It follows from the partition of S that $y_2 \in A^S_2 \cup C^S_0 \cup C^S_1 \cup C^S_2$.
  - If $y_2 \in A^S_{\tilde{2}} \cup C^S_{\tilde{0}}$, then $w(xy_1) = w(xy_2) = \frac{1}{2}$.
  - If either $y_2 \in (A^S_2 \setminus A^S_{\tilde{2}}) \cup (C^S_1 \setminus C^S_{\tilde{1}})$, or $y_2 \in C^S_{\tilde{1}}$ and $|\{u \mid u \in N_{\overline{S}}(y_2) \text{ and } N_{C^S_{\tilde{0}}}(u) \neq \emptyset\}| = 1$, then $w(xy_1) = \frac{3}{4}$ and $w(xy_2) = \frac{1}{4}$.
  - If either $y_2 \in C^S_0 \setminus C^S_{\tilde{0}}$ and $|\{u \mid u \in N_{\overline{S}}(y_2) \text{ and } N_{C^S_{\tilde{0}}}(u) \neq \emptyset\}| \leq 2$, or $y_2 \in C^S_{\tilde{1}}$ and $|\{u \mid u \in N_{\overline{S}}(y_2) \text{ and } N_{C^S_{\tilde{0}}}(u) \neq \emptyset\}| = 2$, then $w(xy_1) = \frac{5}{8}$ and $w(xy_2) = \frac{3}{8}$.
  - If $y_2 \in C^S_0 \setminus C^S_{\tilde{0}}$ and $|\{u \mid u \in N_{\overline{S}}(y_2) \text{ and } N_{C^S_{\tilde{0}}}(u) \neq \emptyset\}| \geq 3$, then $w(xy_1) = \frac{9}{16}$ and $w(xy_2) = \frac{7}{16}$.
  - If $y_2 \in C^S_2$, then $w(xy_1) = 1$ and $w(xy_2) = 0$.

From the definition of the edge weighting functions, the sum of the weights assigned to the edges joining x to S is 1. For any subset S_1 of S, we define a weighting function f on S_1 with $f(S_1) = \sum_{e \in [\overline{S}, S_1]} w(e)$. We prove the following claims.

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Claim 2. $f(A_1^S) > \frac{7}{4}|A_1^S|$.

Recall that the vertices in A_1^S are paired off. Let v_1 and v_2 be a pair of vertices in A_1^S. Then $ipn_2(v_1, S) = \{v_2\}$ and $ipn_2(v_2, S) = \{v_1\}$. This implies that all vertices in $N_{\overline{S}}(v_1) \cup N_{\overline{S}}(v_2)$ are adjacent to no vertex of $S \setminus \{v_1, v_2\}$. Further, we have $f(\{v_1, v_2\}) = |N_{\overline{S}}(v_1) \cup N_{\overline{S}}(v_2)|$. Combining Claim 1 and $\delta(G) \geq 3$, we have $|N_{\overline{S}}(v_1) \cup N_{\overline{S}}(v_2)| \geq 3$. If $|N_{\overline{S}}(v_1) \cup N_{\overline{S}}(v_2)| = 3$, then $N_{\overline{S}}(v_1) = N_{\overline{S}}(v_2)$ and $|N_{\overline{S}}(v_1)| = 3$. In this case, $n = 5$ as G is claw-free and G is connected, which is a contradiction. Thus, $|N_{\overline{S}}(v_1) \cup N_{\overline{S}}(v_2)| \geq 4$. Hence $f(\{v_1, v_2\}) \geq 4 > \frac{7}{2}$ and then $f(A_1^S) > \frac{7}{4}|A_1^S|$.

Claim 3. $f(B^S) \geq \frac{7}{4}|B^S|$.

Note that $B^S = \bigcup_{u \in A_2^S} S_u$ and $S_u \cap S_{u'} = \emptyset$ for any two different vertices $u, u' \in A_2^S$. We show that for any vertex u_1 of A_2^S, $f(S_{u_1}) \geq \frac{7}{4}|S_{u_1}|$. Let $S_{u_1} = \{u_1, \ldots, u_r\}$, where $r = |S_{u_1}| \geq 2$. Since $\{u_2, \ldots, u_r\} \subseteq ipn_2(u_1)$, all neighbours of u_i in $\overline{S}$ are adjacent to no vertex of $S \setminus \{u_1, u_i\}$, where $i \in \{2, \ldots, r\}$. Combined with Claim 1, $f(S_{u_1}) \geq \sum_{i \in \{2,\ldots,r\}} d(u_i)$. If $u_1 \in A_{\overline{2}}^S$, then $f(S_{u_1}) = f(\{u_1, u_2\}) = w(x_1u_1) + d(u_2) = w(x_1u_1) + 3$, where $\{x_1\} = N_{\overline{S}}(u_1) \setminus N_{\overline{S}}(u_2)$. Since $u_1 \notin ipn_2(u_2, S)$, x_1 has a neighbour in S other than u_1. Thus, $w(x_1u_1) = \frac{1}{2}$. Further, $f(\{u_1, u_2\}) = \frac{7}{2}$ and $f(S_{u_1}) = \frac{7}{4}|S_{u_1}|$, as desired. Thus, we may assume that $u_1 \in A_2^S \setminus A_{\overline{2}}^S$. Then either $r \geq 3$, or $r = 2$ and $d(u_2) \geq 4$, or $r = 2$ and $d(u_2) = 3$ and $|N_{\overline{S}}(u_1) \setminus N_{\overline{S}}(u_2)| \geq 2$.

If $r \geq 3$, then $3r - 3 > \frac{7}{4}r$. Since $\delta(G) \geq 3$, $f(S_{u_1}) \geq \sum_{i \in \{2,\ldots,r\}} d(u_i) \geq 3(r-1) = 3r - 3$. Further, $f(S_{u_1}) > \frac{7}{4}r$. When $r = 2$ and $d(u_2) \geq 4$, $f(S_{u_1}) = f(\{u_1, u_2\}) \geq d(u_2) \geq 4 > \frac{7}{4}r$. When $r = 2$, $d(u_2) = 3$ and $|N_{\overline{S}}(u_1) \setminus N_{\overline{S}}(u_2)| \geq 2$, let x_1 be a vertex in $N_{\overline{S}}(u_1) \setminus N_{\overline{S}}(u_2)$. From the definition of the edge weighting functions, we have $w(x_1u_1) \geq \frac{1}{4}$. Thus, $f(S_{u_1}) = f(\{u_1, u_2\}) \geq 2w(x_1u_1) + d(u_2) \geq \frac{1}{2} + 3 = \frac{7}{2} \geq \frac{7}{4}r$. This completes the proof of Claim 3.

Claim 4. $f(C_0^S \setminus C_{\overline{0}}^S) \geq \frac{7}{4}|C_0^S \setminus C_{\overline{0}}^S|$.

Let z_1 be a vertex in $C_0^S \setminus C_{\overline{0}}^S$ and let $N_{\overline{S}}(z_1) = \{x_1, \ldots, x_r\}$, where $r \geq 4$. If we have $|\{x \mid x \in N_{\overline{S}}(z_1) \text{ and } N_{C_{\overline{0}}^S}(x) \neq \emptyset\}| \leq 2$, then $|\{x \mid x \in N_{\overline{S}}(z_1) \text{ and } N_{C_{\overline{0}}^S}(x) = \emptyset\}| \geq r - 2 \geq 2$. Without loss of generality, consider $N_{C_{\overline{0}}^S}(x_1) = \emptyset$ and $N_{C_{\overline{0}}^S}(x_2) = \emptyset$. By the definition of the edge weighting functions, $w(x_1z_1) = \frac{1}{2}$, $w(x_2z_1) = \frac{1}{2}$ and $w(x_iz_1) \geq \frac{3}{8}$ for any $i \in \{3, \ldots, r\}$. Hence, $f(\{z_1\}) \geq w(x_1z_1) + w(x_2z_1) + \sum_{i \in \{3,\ldots,r\}} w(x_iz_1) \geq 1 + \frac{3}{8}(r-2) \geq \frac{7}{4}$. When $|\{x \mid x \in N_{\overline{S}}(z_1) \text{ and } N_{C_{\overline{0}}^S}(x) \neq \emptyset\}| \geq 3$, $w(x_iz_1) \geq \frac{7}{16}$ for any $i \in \{1, \ldots, r\}$. Then $f(\{z_1\}) \geq \frac{7}{16}r \geq \frac{7}{4}$. In all cases, we have $f(\{z_1\}) \geq \frac{7}{4}$. Therefore, $f(C_0^S \setminus C_{\overline{0}}^S) \geq \frac{7}{4}|C_0^S \setminus C_{\overline{0}}^S|$.

Claim 5. $f(C_1^S) \geq \frac{7}{4}|C_1^S|$.

Let z_1 be a vertex in C_1^S and $N_{\overline{S}}(z_1) = \{x_1, x_2, \ldots, x_r\}$, where $\{x_1\} = epn(z_1, S)$ and $r \geq 3$. According to the definition of the edge weighting functions, $w(x_1z_1) = 1$. When

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$z_1 \in C_1^S \setminus C_{\bar{1}}^S$, we have $r \geq 4$ and $w(x_i z_1) \geq \frac{1}{4}$ for any $i \in \{2, \ldots, r\}$. Thus, $f(\{z_1\}) = w(x_1 z_1) + \sum_{i \in \{2,\ldots,r\}} w(x_i z_1) \geq 1 + \frac{1}{4}(r-1) \geq \frac{7}{4}$. When $z_1 \in C_{\bar{1}}^S$ and either $N_{C_{\bar{0}}^S}(x_2) = \emptyset$ or $N_{C_{\bar{0}}^S}(x_3) = \emptyset$, without loss of generality, we can take $N_{C_{\bar{0}}^S}(x_2) = \emptyset$. Then $w(x_2 z_1) = \frac{1}{2}$ and $w(x_3 z_1) \geq \frac{1}{4}$. Further, $f(z_1) = w(x_1 z_1) + w(x_2 z_1) + w(x_3 z_1) \geq 1 + \frac{1}{2} + \frac{1}{4} = \frac{7}{4}$. When both $z_1 \in C_{\bar{1}}^S$ and $N_{C_{\bar{0}}^S}(x_2) \neq \emptyset$ and $N_{C_{\bar{0}}^S}(x_3) \neq \emptyset$, we have $w(x_2 z_1) = w(x_3 z_1) = \frac{3}{8}$. Further, $f(\{z_1\}) = w(x_1 z_1) + w(x_2 z_1) + w(x_3 z_1) = \frac{7}{4}$. In both cases, $f(\{z_1\}) \geq \frac{7}{4}$. Therefore, $f(C_1^S) \geq \frac{7}{4}|C_1^S|$.

Claim 6. $f(C_2^S) > \frac{7}{4}|C_2^S|$.

Let z_1 be a vertex in C_2^S and x_1, x_2 be two vertices in $epn(z_1, S)$. Then $w(x_1 z_1) = w(x_2 z_1) = 1$ and further $f(\{z_1\}) \geq 2$. Hence, $f(C_2^S) \geq 2|C_2^S| > \frac{7}{4}|C_2^S|$.

If $f(C_{\bar{0}}^S) \geq \frac{7}{4}|C_{\bar{0}}^S|$, then $f(S) \geq \frac{7}{4}|S|$ by Claims 2–6. From the definition of the edge weighting functions, $f(S) = n - |S|$. It follows that $|S| \leq \frac{4}{11}n$, which is a contradiction. Thus, $f(C_{\bar{0}}^S) < \frac{7}{4}|C_{\bar{0}}^S|$ and there exists a vertex $y_1 \in C_{\bar{0}}^S$ such that $f(\{y_1\}) < \frac{7}{4}$. Suppose that $N_{\bar{S}}(y_1) = \{x_1, x_2, x_3\}$ and $N_S(x_i) = \{y_1, y_{i+1}\}$ for any $i \in \{1, 2, 3\}$. If $\{y_2, y_3, y_4\} \cap ((A_2^S \setminus A_{\bar{2}}^S) \cup (C_1^S \setminus C_{\bar{1}}^S) \cup C_2^S) \neq \emptyset$, then at least one edge of $\{x_1 y_1, x_2 y_1, x_3 y_1\}$ has a weight of at least $\frac{3}{4}$. Further, $f(\{y_1\}) = w(x_1 y_1) + w(x_2 y_1) + w(x_3 y_1) \geq \frac{3}{4} + \frac{1}{2} + \frac{1}{2} \geq \frac{7}{4}$, which is a contradiction. Thus, $\{y_2, y_3, y_4\} \cap ((A_2^S \setminus A_{\bar{2}}^S) \cup (C_1^S \setminus C_{\bar{1}}^S) \cup C_2^S) = \emptyset$. Next, we prove two claims about the set $\{y_2, y_3, y_4\}$.

Claim 7. $\{y_2, y_3, y_4\} \cap A_{\bar{2}}^S = \emptyset$.

In contrast, we may assume that $y_2 \in A_{\bar{2}}^S$. Let $S_{y_2} = \{y_2, y_5\}$ and $N(y_5) = \{x_4, x_5, x_6\}$, where x_4 is a vertex connecting y_2 and y_5. According to the definition of $A_{\bar{2}}^S$, $N(y_2) \subseteq \{x_1, x_4, x_5, x_6\}$. Note that $ipn_2(y_1, S) = \emptyset$ and $epn(y_1, S) = \emptyset$. If $d(x_1, y_5) \leq 2$, then $(S \setminus \{y_1, y_2\}) \cup \{x_1\}$ is a semi-TD-set of G, which contradicts the minimality of S. Thus, $d(x_1, y_5) \geq 3$ which implies that $x_1 x_i \notin E(G)$ for any $i \in \{4, 5, 6\}$. Combining $\delta(G) \geq 3$ and the claw-freeness of G, we have $x_1 x_2 \in E(G)$ or $x_1 x_3 \in E(G)$ and $x_4 x_5 \in E(G)$ or $x_4 x_6 \in E(G)$. Without loss of generality, consider $x_1 x_2 \in E(G)$ and $x_4 x_5 \in E(G)$.

If $x_4 x_6 \in E(G)$, then $S_1 = (S \setminus \{y_1, y_2, y_5\}) \cup \{x_1, x_4\}$ is a semi-TD-set of G, which is a contradiction. Thus, $x_4 x_6 \notin E(G)$. Since G is claw-free, $y_2 x_6 \notin E(G)$. Then $y_2 x_5 \in E(G)$ as $N(y_2) \subseteq \{x_1, x_4, x_5, x_6\}$ and $d(y_2) \geq 3$. Further, $d(y_2) = 3$. If $x_5 x_6 \in E(G)$, then $S_1 = (S \setminus \{y_1, y_2, y_5\}) \cup \{x_1, x_5\}$ is a semi-TD-set of G, which is a contradiction. Thus, $x_5 x_6 \notin E(G)$. Since G is claw-free, $N(x_4) = \{y_2, y_5, x_5\}$ and $N(x_5) = \{y_2, y_5, x_4\}$. We note that $d(x_4) = d(x_5) = 3$ and then $G[\{y_2, y_5, x_4, x_5\}]$ is a special diamond.

Since $y_5 \in ipn_2(y_2, S)$ and $x_6 y_2 \notin E(G)$, x_6 is adjacent to no vertex of $S \setminus \{y_5\}$. If x_6 is not in a special diamond, then $S_1 = (S \setminus \{y_2, y_5\}) \cup \{x_4, x_6\}$ is a $\gamma_{t2}(G)$-set with $\lambda(S_1) = \lambda(S)$ but with $|\mathcal{D} \cap S_1| < |\mathcal{D} \cap S|$, which contradicts our choice of S. Thus, x_6 is in a special diamond D. Observe that $x_6 x_2 \notin E(G)$ and $x_6 x_3 \notin E(G)$. Let $V(D) = \{x_6, x_7, x_8, y_6\}$, where $x_6 y_6$ is the missing edge in the special diamond D.

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Then $\{x_7, x_8\} \subseteq \overline{S}$. To dominate x_7 and x_8, we must have $y_6 \in S$. If $y_6x_2 \in E(G)$, then $y_6 \in ipn_2(y_1, S)$ which contradicts $y_1 \in C^S_{\widetilde{0}}$. Thus, $y_6x_2 \notin E(G)$. Similarly, $y_6x_3 \notin E(G)$.

Let $N(y_6) = \{x_7, x_8, x_9\}$. We note that $y_6 \in ipn_2(v, S)$ for some vertex $v \in S$ so call $v = y_7$. Then $y_7x_9 \in E(G)$. Recall that $\{y_3, y_4\} \cap (A^S_2 \setminus A^S_{\widetilde{2}}) = \emptyset$. If $y_7x_2 \in E(G)$ or $y_7x_3 \in E(G)$, then $y_7 = y_3$ or y_4 and $y_7 \in A^S_2 \setminus A^S_{\widetilde{2}}$, which is a contradiction. Thus, $y_7x_2 \notin E(G)$ and $y_7x_3 \notin E(G)$. If $y_7 \notin ipn_2(y_6, S)$, then $S_1 = (S \setminus \{y_1, y_2, y_6\}) \cup \{x_1, x_7\}$ is a semi-TD-set of G, which is a contradiction. Hence, $y_7 \in ipn_2(y_6, S)$. Let $S_1 = (S \setminus \{y_5\}) \cup \{x_6\}$. Clearly, S_1 is a $\gamma_{t2}(G)$-set, $\lambda(S_1) = \lambda(S)$ and $|\mathcal{D} \cap S_1| = |\mathcal{D} \cap S|$. We note that $y_2 \in ipn_2(y_1, S_1)$ and $\{x_6, y_7\} \subset ipn_2(y_6, S_1)$. Further, $\{y_1, y_2, x_6, y_6, y_7\} \subseteq B^{S_1}$ and $|C^{S_1}_{\widetilde{0}}| < |C^S_{\widetilde{0}}|$, which contradicts the choice of S.

By Claim 7, each vertex of $\{y_2, y_3, y_4\}$ belongs to $C^S_0 \cup C^S_{\widetilde{1}}$.

Claim 8. $\{y_2, y_3, y_4\} \cap C^S_{\widetilde{1}} \neq \emptyset$.

Suppose to the contrary that $\{y_2, y_3, y_4\} \cap C^S_{\widetilde{1}} = \emptyset$. Then $\{y_2, y_3, y_4\} \subseteq C^S_0$. Since G is claw-free, there exists an edge in $G[\{x_1, x_2, x_3\}]$, say x_1x_2. If $y_2 \neq y_3$, then $x_3y_2 \notin E(G)$ or $x_3y_3 \notin E(G)$, as x_3 has only one neighbour in $S \setminus \{y_1\}$. By symmetry, consider $y_2x_3 \notin E(G)$ (that is, $y_2 \neq y_4$). Then $S_1 = (S \setminus \{y_1, y_2\}) \cup \{x_1\}$ is a dominating set of G as $y_2 \in C^S_0$ and $y_1 \in C^S_{\widetilde{0}}$. Since $|S_1| < |S|$, S_1 cannot be a semi-TD-set of G. Combined with $\{y_1, y_2\} \subseteq C^S_0$, y_1 and y_2 are the only two vertices of S within distance 2 from y_4 in G, and $y_4 \neq y_3$. Thus, all vertices of $N_{\overline{S}}(y_4) \setminus \{x_3\}$ are adjacent to y_2. Further, $S_1 = (S \setminus \{y_1, y_3\}) \cup \{x_2\}$ is a semi-TD-set of G as $y_3 \in C^S_0$ and $y_1 \in C^S_{\widetilde{0}}$, which contradicts the minimality of S. Hence, $y_2 = y_3$.

Since $y_1 \notin ipn_2(y_2, S)$, $y_2x_3 \notin E(G)$ (that is, $y_2 \neq y_4$). Let $S_2 = (S \setminus \{y_1, y_4\}) \cup \{x_3\}$. As $y_4 \in C^S_0$, S_2 is a dominating set of G. If $d(y_2, x_3) \leq 2$, then S_2 is a semi-TD-set of G, which is a contradiction. Thus, $d(y_2, x_3) > 2$. Further, $x_1x_3 \notin E(G)$ and $x_2x_3 \notin E(G)$. Combining $d(x_3) \geq 3$ and the claw-freeness of G, there exists a vertex x_4 such that $x_4x_3 \in E(G)$ and $x_4y_4 \in E(G)$. By Claim 1, $x_4 \in \overline{S}$. We note that $y_2x_4 \notin E(G)$ as $d(y_2, x_3) > 2$. Since $y_4 \in C^S_0$, $x_4 \notin epn(y_4, S)$ and x_4 has a neighbour y_5 in S other than y_4. As S_2 cannot be a semi-TD-set of G, y_1 and y_4 are the only two vertices of S within distance 2 from y_2 in G. Thus, all vertices of $N_{\overline{S}}(y_2) \setminus \{x_1, x_2\}$ are adjacent to y_4.

Let x_5 be a vertex in $N_{\overline{S}}(y_2) \setminus \{x_1, x_2\}$. Then $x_5y_4 \in E(G)$. If $x_1x_5 \in E(G)$, then $(S \setminus \{y_1, y_2\}) \cup \{x_1\}$ is a semi-TD-set of G, which is a contradiction. Thus, $x_1x_5 \notin E(G)$. Recall that $x_1x_3 \notin E(G)$. This implies that $d(x_1) = 3$. Similarly, $d(x_2) = 3$. Note that $x_5x_3 \notin E(G)$ as $d(y_2, x_3) > 2$. Since G is claw-free, $x_5x_4 \notin E(G)$ and each vertex of $N(y_4) \setminus \{x_3, x_5\}$ is adjacent to either x_3 or x_5. Let $S_3 = (S \setminus \{y_1, y_2, y_4\}) \cup \{x_1, x_3, x_5\}$. Then S_3 is a $\gamma_{t2}(G)$-set and $\lambda(S_3) = \lambda(S)$. If $d(y_2) = 3$, then $G[\{y_1, y_2, x_1, x_3\}]$ is a special diamond. Further, $|\mathcal{D} \cap S_2| < |\mathcal{D} \cap S|$, which contradicts the choice of S. Thus, $d(y_2) \geq 4$. Let x_6 be a vertex in $N_{\overline{S}}(y_2) \setminus \{x_1, x_2, x_5\}$. Then $x_6y_4 \in E(G)$. We note that $\{y_2, y_4\} \in C^S_0 \setminus C^S_{\widetilde{0}}$ and $|\{x \mid x \in N_{\overline{S}}(y_2) \text{ and } N_{C^S_{\widetilde{0}}}(x) \neq \emptyset\}| \leq 2$. From the definition

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of the edge weighting functions, we have $w(x_1y_1) = w(x_2y_1) = \frac{5}{8}$ and $w(x_3y_1) \geq \frac{9}{16}$. Further, $f(\{y_1\}) = w(x_1y_1) + w(x_2y_1) + w(x_3y_1) > \frac{7}{4}$, which contradicts the choice of y_1.

By Claim 8, we may assume that $y_2 \in C_{\tilde{1}}^S$. Then $w(x_1y_1) \geq \frac{5}{8}$. If $y_3 \in C_{\tilde{1}}^S$, then $w(x_2y_1) \geq \frac{5}{8}$ and further $f(\{y_1\}) = w(x_1y_2) + w(x_2y_2) + w(x_3y_2) \geq \frac{5}{8} + \frac{5}{8} + \frac{1}{2} \geq \frac{7}{4}$, which is a contradiction. Thus, $y_3 \in C_0^S$. Similarly, $y_4 \in C_0^S$. This implies that $y_2 \neq y_3$ and $y_2 \neq y_4$. Let $N(y_2) = \{x_1, x_4, x_5\}$, where $\{x_4\} = epn(y_2, S)$. If $N_{C_{\tilde{0}}^S}(x_5) = \emptyset$, then by the definition of the edge weighting functions, we have $w(x_1y_1) = \frac{3}{4}$. Further, $f(\{y_1\}) = w(x_1y_2) + w(x_2y_2) + w(x_3y_2) \geq \frac{3}{4} + \frac{1}{2} + \frac{1}{2} \geq \frac{7}{4}$, which is a contradiction. Hence, $N_{C_{\tilde{0}}^S}(x_5) \neq \emptyset$. We proceed with a series of claims that culminate in a contradiction.

Claim 9. $|E(G[\{x_1, x_2, x_3\}])| = 1$.

Since G is claw-free, $|E(G[\{x_1, x_2, x_3\}])| \geq 1$. If $x_2x_1 \in E(G)$ and $x_2x_3 \in E(G)$, then $(S \setminus \{y_1, y_3\}) \cup \{x_2\}$ is a semi-TD-set of G as $y_1 \in C_{\tilde{0}}^S$ and $y_3 \in C_0^S$, which contradicts the minimality of S. Thus, $x_2x_1 \notin E(G)$ or $x_2x_3 \notin E(G)$. Similarly, $x_3x_1 \notin E(G)$ or $x_3x_2 \notin E(G)$. Suppose that $|E(G[\{x_1, x_2, x_3\}])| \geq 2$. Then $x_1x_2 \in E(G)$, $x_1x_3 \in E(G)$ and $x_2x_3 \notin E(G)$. This means G has a claw, which contradicts the claw-freeness of G. Hence, $|E(G[\{x_1, x_2, x_3\}])| = 1$.

Claim 10. $y_3 = y_4$.

Assume, to the contrary, that $y_3 \neq y_4$. By Claim 9, without loss of generality, we consider $E(G[\{x_1, x_2, x_3\}]) = \{x_1x_2\}$ or $\{x_2x_3\}$. Let $S_1 = (S \setminus \{y_1, y_3\}) \cup \{x_2\}$. Since $y_3 \in C_0^S$, S_1 is a dominating set of G. Note that S_1 cannot be a semi-TD-set of G. Thus, there exists a vertex y such that y_1 and y_3 are the only two vertices of S within distance 2 from y in G and $d(y, x_2) > 2$. If $E(G[\{x_1, x_2, x_3\}]) = \{x_2x_3\}$, then $d(x_2, y_4) \leq 2$ and $y = y_2$. This implies that $x_5y_3 \in E(G)$. However, then $(S \setminus \{y_1, y_4\}) \cup \{x_3\}$ is a semi-TD-set of G as $y_4 \in C_0^S$, which contradicts the minimality of S. Hence, $E(G[\{x_1, x_2, x_3\}]) = \{x_1x_2\}$. In this case, $d(y_2, x_2) \leq 2$ and $y = y_4$. Thus, all vertices of $N_{\overline{S}}(y_4) \setminus \{x_3\}$ are adjacent to y_3. Let x be a vertex in $N_{\overline{S}}(y_4) \setminus \{x_3\}$. Then $xy_3 \in E(G)$. Recall that $d(y, x_2) > 2$. Thus, $d(y_4, x_2) > 2$ and $x_2x \notin E(G)$. Since G is claw-free, $N_{\overline{S}}(y_4) \setminus \{x_3\}$ is a clique of G. Combining $d(x_3) \geq 3$ and the claw-freeness of G, there exists a vertex x' in $N_{\overline{S}}(y_4) \setminus \{x_3\}$ such that $x_3x' \in E(G)$. Then $(S \setminus \{y_3, y_4\}) \cup \{x'\}$ is a semi-TD-set of G as $y_3 \in C_0^S$, which is a contradiction.

If $y_3 (= y_4) \in C_0^S \setminus C_{\tilde{0}}^S$, then $w(x_2y_1) \geq \frac{9}{16}$ and $w(x_3y_1) \geq \frac{9}{16}$. Further, $f(\{y_1\}) = w(x_1y_2) + w(x_2y_1) + w(x_3y_1) \geq \frac{5}{8} + \frac{9}{16} + \frac{9}{16} \geq \frac{7}{4}$, which contradicts the choice of y_1. Thus, $y_3 \in C_{\tilde{0}}^S$.

Claim 11. $x_2x_3 \notin E(G)$.

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For the sake of contradiction, suppose that $x_2x_3 \in E(G)$. If $d(y_2, x_2) \leq 2$, then $(S \setminus \{y_1, y_3\}) \cup \{x_2\}$ is a semi-TD-set of G as $y_3 \in C_{\tilde{0}}^S$, which contradicts the minimality of S. Thus, $d(y_2, x_2) \geq 3$. Similarly, $d(y_2, x_3) \geq 3$. This implies that $E[\{x_1, x_4, x_5\}, \{x_2, x_3\}] = \emptyset$. If $d(x_2) > 3$, then there exists a vertex x in $\overline{S}$ adjacent to x_2. Since G is claw-free, $xy_3 \in E(G)$. Thus, x has a neighbour in S different from y_3 as $y_3 \in C_{\tilde{0}}^S$. Combined with $N_{C_{\tilde{0}}^S}(x_5) \neq \emptyset$, $(S \setminus \{y_1, y_3\}) \cup \{x_2\}$ is a semi-TD-set of G, which is a contradiction. Thus, $d(x_2) = 3$. Similarly, $d(x_3) = 3$. Observe that y_1 and y_3 are in the same special diamond of G.

If $x_1x_4 \in E(G)$, then $S_1 = (S \setminus \{y_1, y_2, y_3\}) \cup \{x_1, x_2\}$ is a dominating set of G. Otherwise, x_5 is not dominated by S_1, and further $x_5x_1 \notin E(G)$ and $x_5y_3 \in E(G)$ as $y_2 \in C_{\tilde{1}}^S$ and $y_3 \in C_{\tilde{0}}^S$. Since $d(x_5) \geq 3$ and G is claw-free, $N(x_5) = \{y_2, y_3, x_4\}$. In this case, $G = N_2$, which is a contradiction. As $N_S(x_5) \setminus \{y_2\} \subseteq C_{\tilde{0}}^S$ and $y_3 \in C_{\tilde{0}}^S$, there does not exist a vertex in $S \setminus \{y_1\}$ such that y_2 and y_3 are the only two vertices of S within distance 2 from it in G. Thus, S_1 is a semi-TD-set of G which contradicts the minimality of S. Hence, $x_1x_4 \notin E(G)$. Since $d(x_1) \geq 3$ and G is claw-free, $x_1x_5 \in E(G)$. Note that $d(x_1) = 3$. If $x_4x_5 \in E(G)$, then G has a claw, for x_5 has a neighbour in S other than y_2, which is a contradiction. Thus, $x_4x_5 \notin E(G)$. This implies that x_1 is not in a special diamond.

Let $S_2 = (S \setminus \{y_1, y_2, y_3\}) \cup \{x_1, x_2, x_4\}$. Observe that S_2 is a $\gamma_{t2}(G)$-set and $\lambda(S_2) = \lambda(S)$. If x_4 is not in a special diamond, then $|\mathcal{D} \cap S_2| < |\mathcal{D} \cap S|$, which contradicts the choice of S. Thus, x_4 is in a special diamond D of G. Let $V(D) = \{x_4, x_6, x_7, y_5\}$, where x_4y_5 is the missing edge in the special diamond D. Clearly, $\{x_6, x_7\} \subseteq \overline{S}$ and $y_5 \in S$. We note that y_5 is an S-internal private neighbour. Let $y_5 \in ipn_2(y_6, S)$ and x_8 be the vertex of $\overline{S}$ connecting y_5 and y_6.

If $epn(y_6, S) = \emptyset$, then $S_3 = (S \setminus \{y_1, y_2, y_5, y_6\}) \cup \{x_1, x_4, x_8\}$ is a dominating set of G. Since $d(x_8) \geq 3$, there exists a vertex x adjacent to $xx_8 \in E(G)$. Further, $xy_6 \in E(G)$ as G is claw-free. Combined with $epn(y_6, S) = \emptyset$, x has a neighbour in $S \setminus \{y_1, y_5, y_6\}$. Thus, there exists a vertex in S_3 within distance 2 from x_8. It follows from the minimality of S that S_3 cannot be a semi-TD-set of G. Thus, y_3 is at a distance greater than 2 from every other vertex of S_3. This implies that $y_3x_5 \notin E(G)$, $x'y_6 \in E(G)$ and $x'x_8 \notin E(G)$, where $\{x'\} = N(y_3) \setminus \{x_2, x_3\}$. We note that neither x_8 nor x' are in a special diamond. Otherwise, $epn(y_6, S) \neq \emptyset$ or G has a claw, which is a contradiction. Further, $(S \setminus \{y_1, y_2, y_3, y_5, y_6\}) \cup \{x_1, x_2, x_6, x_8, x'\}$ is a semi-TD-set of G with $\lambda(S_3) = \lambda(S)$ but with $|\mathcal{D} \cap S_3| < |\mathcal{D} \cap S|$, which contradicts the choice of S. Hence, $epn(y_6, S) \neq \emptyset$.

Let x_9 be a vertex in $epn(y_6, S)$. Then $|\mathcal{D} \cap S_2| = |\mathcal{D} \cap S|$, $\{x_1, x_2\} \cap C_{\tilde{0}}^{S_2} = \emptyset$ and $y_6 \notin C_0^{S_2}$. Further, $|C_{\tilde{0}}^{S_2}| \leq |C_{\tilde{0}}^S|$ and $|C_0^{S_2}| \leq |C_0^S|$. If $y_6 \notin C_{\tilde{1}}^{S_2}$, then $|C_{\tilde{1}}^{S_2}| < |C_{\tilde{1}}^S|$, which contradicts the choice of S. Thus, $y_6 \in C_{\tilde{1}}^{S_2}$. Let $N(y_6) = \{x_8, x_9, x_{10}\}$ (possibly, $x_{10} = x_5$). Then x_{10} has a neighbour in $S_2 \setminus \{y_6, y_5, x_4, x_2\}$. Hence, x_{10} has a neighbour in $S \setminus \{y_5, y_6\}$. Let $S_4 = (S \setminus \{y_5\}) \cup \{x_6\}$. Then S_4 is a $\gamma_{t2}(G)$-set. Now, $\lambda(S_4) = \lambda(S)$, $|\mathcal{D} \cap S_4| = |\mathcal{D} \cap S|$, $x_6 \in epn_2(y_2, S_4)$ and $y_6 \in ipn_2(y, S_4)$ for some vertex y of S_4. Thus, $|C_{\tilde{0}}^{S_4}| \leq |C_{\tilde{0}}^S|$, $|C_0^{S_4}| \leq |C_0^S|$ and $|C_{\tilde{1}}^{S_4}| < |C_{\tilde{1}}^S|$, which contradicts the choice of S.

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By Claims 9–11, we may assume that $E(G[\{x_1, x_2, x_3\}]) = \{x_1x_2\}$. If $x_1x_4 \in E(G)$, then $(S \setminus \{y_1, y_2\}) \cup \{x_1\}$ is a semi-TD-set of G as $y_2 \in C^S_{\tilde{1}}$, which contradicts the minimality of S. Thus, $x_1x_4 \notin E(G)$.

Suppose first that $x_5y_3 \in E(G)$. Since $d(x_3) \geq 3$ and G is claw-free, $x_5x_3 \in E(G)$. If $x_5x_4 \in E(G)$, then $(S \setminus \{y_1, y_2, y_3\}) \cup \{x_1, x_5\}$ is a semi-TD-set of G, which contradicts the choice of S. Thus, $x_5x_4 \notin E(G)$. Combining the claw-freeness of G and $x_1x_4 \notin E(G)$, we have $x_5x_1 \in E(G)$. Since G is claw-free, $N(x_1) = \{y_1, y_2, x_2, x_5\}$, $N(x_2) = \{y_1, y_3, x_1\}$, $N(x_3) = \{y_1, y_3, x_5\}$, $N(x_5) = \{y_2, y_3, x_1, x_3\}$ and $X_1 := N(x_4) \setminus \{y_2\}$ is a clique of G. We construct G' from G by removing all vertices of $\{y_1, y_3, x_1, x_2, x_3, x_5\}$ and adding the edges between $\{y_2\}$ and X_1 such that $\{y_2\} \cup X_1$ is a clique of G'. Since $d(x_4) \geq 3$, we have $|X_1| \geq 2$. Thus, $G' \neq N_2$ is a connected claw-free graph of order $n' = n - 6$ with $\delta(G') \geq 3$. Note that $X_1 \subseteq \overline{S}$ as $x_4 \in epn(y_2, S)$. Since S is a semi-TD-set of G, there exist a vertex y of $S \setminus \{y_1, y_2, y_3\}$ adjacent to some vertices of X_1 and a vertex y' of $S \setminus \{y_1, y_2, y_3, y\}$ within distance 2 from y in S. Hence, $n' \geq 6$. By the minimality of G, $\gamma_{t2}(G') \leq \frac{4}{11}n'$. Let S' be a $\gamma_{t2}(G')$-set. When $y_2 \notin S'$, $S' \cup \{y_1, x_5\}$ is a semi-TD-set of G. When $y_2 \in S'$, $(S' \setminus \{y_2\}) \cup \{y_1, x_5, x_4\}$ is a semi-TD-set of G. In both cases, $\gamma_{t2}(G) \leq \frac{4}{11}n' + 2 = \frac{4}{11}(n - 6) + 2 < \frac{4}{11}n$, which contradicts the choice of G.

Suppose next that $x_5y_3 \notin E(G)$. Let $N_S(x_5) \setminus \{y_2\} = \{y_5\}$ and $N(y_3) \setminus \{x_2, x_3\} = \{x_6\}$. Recall that $N_{C^S_{\tilde{0}}}(x_5) \neq \emptyset$. Thus, $y_5 \in C^S_{\tilde{0}}$. Since $d(x_3) \geq 3$ and G is claw-free, $x_3x_6 \in E(G)$. If $y_5x_6 \in E(G)$, then $(S \setminus \{y_3, y_5\}) \cup \{x_6\}$ is a semi-TD-set of G, which is a contradiction. Thus, $y_5x_6 \notin E(G)$. Let $N(y_5) = \{x_5, x_7, x_8\}$ and $N_S(x_6) \setminus \{y_3\} = \{y_6\}$. If $x_5x_1 \notin E(G)$, then $x_4x_5 \in E(G)$ as G is claw-free and $x_1x_4 \notin E(G)$. In this case, $(S \setminus \{y_1, y_2, y_5\}) \cup \{x_1, x_5\}$ is a semi-TD-set of G, which contradicts the minimality of S. Thus, $x_5x_1 \in E(G)$. If $x_5x_4 \in E(G)$, then $(S \setminus \{y_2, y_5\}) \cup \{x_5\}$ is a semi-TD-set of G, which is a contradiction. Thus, $x_5x_4 \notin E(G)$. Since G is claw-free, $d(x_1) = 4$, $d(x_2) = 3$, $d(x_3) = 3$, $N(x_5) \subseteq \{y_2, y_5, x_1, x_7, x_8\}$ and $X_2 := N(x_6) \setminus \{y_3, x_3\}$ is a clique of G. Let G' be the graph obtained from G by removing all vertices of $\{x_1, x_2, x_3, x_6, y_1, y_3\}$ and adding the edges between $\{y_2, x_5\}$ and X_2 such that $\{y_2, x_5\} \cup X_2$ is a clique of G'. We observe that $G' \neq N_2$ is a connected claw-free graph of order $n' = n - 6 \geq 6$ with $\delta(G') \geq 3$. Then G' has a $\gamma_{t2}(G')$-set S' with at most $\frac{4}{11}n'$ vertices by the minimality of G. When $X_2 \cap S' \neq \emptyset$, $S' \cup \{x_1, y_3\}$ is a semi-TD-set of G. When $X_2 \cap S' = \emptyset$, $S' \cup \{y_1, x_6\}$ is a semi-TD-set of G. In either case, $\gamma_{t2}(G) \leq \frac{4}{11}n' + 2 = \frac{4}{11}(n - 6) + 2 < \frac{4}{11}n$, which is a contradiction.

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