18

Change and Persistence

18.1 Introduction

Our interest in this chapter is in *change*, which is another way of discussing *persistence*, or the apparent endurance over intervals of time of spatially extended structures.

Persistence is the phenomenon that underpins all human activity. Given the rule that we may think of as the *first law of time*, the dictum of Heraclitus that *everything changes*, persistence is our name for those remarkable processes that appear to circumvent that law and give stability to our lives. We should be interested, for instance, in the fact that when we wake up each morning, we feel that we are the same individuals who went to sleep the previous night. Indeed, without persistence in one form or another, nothing would make sense, including logic, rational thought, and mathematics, for these depend on comparisons of standards and rules that persist in memory with information that we acquire in process time.

Persistence is necessary for physics to make sense. It is emphatically *not* a metaphysical topic but of the greatest relevance to science, including quantum mechanics (QM). Persistence is implicit in Wheeler's dictum that only acts of observation are meaningful in physics, because observers and their apparatus have to endure long enough to make observations. In quantized detector networks (QDN), the persistence of observers and their apparatus long enough to perform experiments should be regarded as axiomatic.

The subject of persistence is a deep and complex one, and will probably never be fully understood. Several reasons contribute to this.

Observers Themselves are Subject to Change

Contrary to the general implicit belief that the laws of physics transcend the subjective behavior of observers, it is our thesis that this is a vacuous proposition. It cannot be proved; it is just an assertion, albeit a very useful one. A more

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pragmatic view is the one held throughout this book, that the laws of physics are contextual to observers. Without observers, science means nothing in a literal sense. But according to the first law of time, observers themselves change over time. This includes their memories and the conditioning that they have; these are carried over from one stage to another. If the processes transporting those memories and belief structures change those memories and belief structures, then the laws of physics *as understood by the observers at the time* change as well. Five hundred years ago, science knew little of Newton's laws of motion; two hundred years ago, science did not know that there was such a thing as quantum mechanics.

We deal with the variability of observers by recognizing that truth values of propositions, including the laws of physics, are contextual to observers at the time of observation only. That the laws of physics themselves may change as the Universe expands is not a fanciful idea: the possibility that the so-called constants of nature have changed has been looked at by theorists such as Dirac (1938b) and Magueijo (2003). A complete theory of observation would take into account dynamical interaction between systems under observation, observers, and the Universe that contains those observers.

Time Scales Are Important

Give the contextuality of persistence, what are the factors that contribute to that contextuality? One of them is *time scale*, for the time scales over which objects are said to persist is a crucial factor in creating the illusion. Indeed, persistence may be thought of as a comparison of two time scales: the first is associated with the SUOs being investigated, and the second is the time scale that the observer regards as significant *for their purposes*.

Example 18.1 A scientist investigating the flight of insects needs to employ video equipment that can adequately capture the motion of insect wings. There will be no point in using a video frame speed of 25 frames per second if an insect under observation flaps its wings 100 times per second.

Sheer Complexity

Another crucial factor associated with persistence is that observers and SUOs are hideously complex phenomena. Indeed, the concepts of science are attempts to organize that complexity into sufficiently simple forms that the brain can interpret relatively quickly and easily.

Example 18.2 The human eye has a detecting screen known as the retina, over which are detectors known as rods and cones. There are perhaps 120 million rods and 6 million cones in the typical human eye. When light from suitable sources strikes the retina, many of these detectors will not register

a signal. Of those that do, complex chemical processes in the rods and cones occur, resulting in signals being passed into specialized nerve layers below the retina. These raw signals from the rods and cones are then processed in those nerve layers before more complex signals are passed further on into the brain, where much more complex processes of comparison and pattern recognition must be occurring. The templates that the brain uses in these comparisons appear to be relatively stable themselves, so much so that the brain does its best to match the incoming signals to what it, the brain, has already prepared and expected for. When this process fails in some way, optical illusions can occur, such as a failure to recognize individuals whom we have met before. The simplicity that we think we see around us is a total illusion.

Quantum State Fragility

A critical factor contributing to persistence is atomic stability: atoms are generally stable, at least long enough to create the illusion of persistence. This brings us to perhaps *the* greatest enigma in physics, one that is necessary for persistence in the first place and one that could not be resolved by classical mechanics (CM). According to CM principles, an accelerating electric charge should dissipate energy by electromagnetic radiation. This led to the classical prediction that hydrogen atoms should be unstable, contrary to empirical evidence.

In order to bypass this prediction, Bohr constructed a model of hydrogen based on an unexplained veto of such radiation (Bohr, 1913). More refined QM, such as wave mechanics and quantum field theory, account for many structural aspects of atoms, but a completely satisfactory, reductionist explanation of atomic stability does not exist. For instance, the postulate of normalized wave functions in Schrödinger mechanics that leads to quantized atomic energy levels in hydrogen is manifestly an emergent one based on an implicit assumption of a persistent exophysical observer dealing only with normalized states. In interacting quantum field theories too, there is generally no proof that the vacuum, or state of lowest energy, exists; such a state is normally postulated.

The stability of atoms helps create the illusions of persistence. But below that classical layer of illusion is a seething mass of change. Of relevance here is the idea that the classical world view is good when quantum phases are so random and disorganized that a classical average approach can be relied on. The greatest developments in QM have occurred principally in the theoretical understanding and empirical control of quantum phase. For example, Feynman's path integral formulation of QM is based on a specific mathematical recipe for adding quantum phases associated with different dynamical paths. On the empirical side, the various quantum optics experiments discussed in the present book demonstrate the point excellently.

Our ambition to understand persistence will be limited. We shall discuss the persistence of labstates in a carefully controlled environment. Specifically, given

an initial labstate $\boldsymbol{\Psi}_m$ at stage Σ_m, suppose it is allowed to evolve through a quantized detector network to labstate $\boldsymbol{\Psi}_n$ at a later[1] stage Σ_n, under the evolution given by $\boldsymbol{\Psi}_n = \mathbb{U}_{n,m}\boldsymbol{\Psi}_m$. A natural question to ask is: *how different are states* $\boldsymbol{\Psi}_m$ *and* $\boldsymbol{\Psi}_n$?

This question is not simple to answer, for several reasons.

The Parallel Transport Problem

An analogous problem arises in general relativity (GR), where there is a need to make a comparison of directions at different places. Specifically, given two points P and Q in a GR spacetime manifold, there is a tangent space associated with each point, denoted T_P for the tangent space at P, and T_Q for that at Q. Suppose we pick a vector $\boldsymbol{v}_P$ in T_P and vector $\boldsymbol{v}_Q$ in T_Q. The question is: how can we compare $\boldsymbol{v}_P$ and $\boldsymbol{v}_Q$? The problem is that these are vectors in two different vector spaces. Somehow, a method of "bringing them together" has to be found, so that a proper comparison can be made.

A solution to this problem was found by the mathematicians Christoffel, Levi-Civita, and others, who developed Riemannian (differential) geometry in the nineteenth and early twentieth centuries. Their approach was to introduce the notion of a *connection*, a rule for relating basis vectors in T_P to those in T_Q. Such a rule involves *parallel transport*, which means defining what it means to transport a vector from one tangent space to another in a manifold "without changing direction."

There is no unique connection over a manifold. Fortunately for Einstein, who from about 1911 onward was developing GR, there is a natural and unique connection for any manifold that has a metric, or distance, rule. The so-called *metric connection* can be derived from the metric tensor of the spacetime concerned, and GR deals specifically with manifolds with Lorentzian signature metrics. The metric connection in GR has been found over the last century to give excellent empirical predictions, such as the precession of the orbit of the planet Mercury, the gravitational deflection of light, and galactic lensing. Accordingly, it is the connection in general use to this day, although alternatives such as those with torsion are occasionally proposed.

In QDN, an important feature that assists us here is that empirical context gives a preferred basis B_m for the quantum register $\mathcal{Q}_m$ at stage Σ_m and likewise for stage Σ_n. Therefore, the parallel transport problem seems to have a natural resolution.

However, QDN does not insist on the constancy of quantum register dimensions from stage to stage. When discussing persistence, we need to ensure that the dimensions of the registers concerned are the same, that is, $\dim \mathcal{Q}_{n+1} = \dim \mathcal{Q}_n$. Otherwise, we would be trying to relate labstates in vector spaces that were

[1] In this context, stage Σ_n *later* than stage Σ_m always means $n > m$.

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inherently dissimilar. This dimensionality issue does not arise in the case of Riemannian geometry because the dimension of each tangent space in a manifold is precisely the same as the dimension of the manifold over which they are defined.

We shall henceforth assume that when an observer is measuring the change in a state of an SUO, it will be under carefully controlled conditions such that $\dim \mathcal{Q}_n = \dim \mathcal{Q}_m \equiv r$. For instance, we might be interested in some branch of macroscopic quantum mechanics, such as superfluidity or superconductivity. Changes in quantum register dimension in that context would correspond to fluctuations in the numbers of electrons or other particles in the SUO, which would be a manifest breakdown of persistence.

Do We Know What We Are Doing?

One possible reason for the difficulty we have in understanding persistence is that perhaps we really don't understand the problem. It may be the case that QM is the wrong branch of physics needed to understand persistence. A comparison of $\mathbf{\Psi}_m$ and $\mathbf{\Psi}_n$ is a comparison of quantum states: these do not represent a reality that "exists" in a classical sense. As emphasized before in this book, quantum states represent the statistics of empirical context and are not snapshots of reality, while the persistence that we see around us is an illusion based on a near instantaneous comparison of signals received on our retinas and patterns in our memories. Whether QM is right way to discuss that process of comparison is debatable. The assumption that it is leads to vacuous concepts such as "wave functions for the Universe."

This issue has long been discussed in QM, examples being Ehrenfest's theorem (Ehrenfest, 1927) and decoherence theory (Zurek, 2002). These are viewed by us as attempts to bridge the gap known as the Heisenberg cut, from the relative internal contextual (RIC) side where QM is good and the relative external context (REC) where a classical description is good. We view these attempts as misguided, because REC is the domain proper of emergent physics, and we should not expect any reductionist approach to "explain" emergence. We note that despite initial hopes that decoherence would "explain" why the world around us appears classical, these expectations have not been met and have been criticized (Kastner, 2016).

As in other situations, the resolution of our problem comes from looking at what goes on in the laboratory, not in our theories. When all is said and done, the only things that matter in physics are the signals received in our detectors. That is where we can understand persistence and change. We shall continue our investigation into persistence, therefore, along the lines of comparing changes in labstate outcomes.

Correlation of Preferred Basis States

Given a computational basis representation (CBR) $B_m \equiv \{\boldsymbol{i}_m : 0 \leqslant i < 2^r\}$ for $\mathcal{Q}_m$ and a CBR $B_n \equiv \{\boldsymbol{i}_n : 0 \leqslant i < 2^r\}$ for $\mathcal{Q}_n$, we need to relate the individual

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elements of each preferred basis. Clearly, there will be no use attempting to relate these basis states if the quantum registers have unrelated contexts. For instance, if the stage Σ_m apparatus is observing photon signals and stage Σ_n is observing electron signals, then there may be no natural relationships between them. Therefore, the discussion requires us to correlate contexts.

Assuming this is the case, then since the labeling of CBR elements is arbitrary, we can assume that the ordering of elements in each preferred basis is correlated contextually. This means for instance that $\mathbf{0}_m$ is correlated to $\mathbf{0}_n$, that $\mathbf{1}_m$ is correlated to $\mathbf{1}_n$, and so on.

The Null Evolution Operator

Given the above provisos, an important operator will be the *null evolution operator* $\mathbb{N}_{n,m}$, defined by

$$\mathbb{N}_{n,m} \equiv \sum_{i=0}^{2^r-1} \boldsymbol{i}_n \overline{\boldsymbol{i}_m}. \tag{18.1}$$

This will have the critical role, rather like the metric connection in GR, of defining a concept of parallel transport in QDN, as we see from the following argument. Given an initial labstate

$$\boldsymbol{\Psi}_m \equiv \sum_{i=0}^{2^r-1} \Psi^i_{\ m} \boldsymbol{i}_m, \tag{18.2}$$

we find

$$\tilde{\boldsymbol{\Psi}}_n \equiv \mathbb{N}_{n,m} \boldsymbol{\Psi}_m = \sum_{i=0}^{2^r-1} \Psi^i_{\ m} \boldsymbol{i}_n, \tag{18.3}$$

which is a carbon copy of $\boldsymbol{\Psi}_m$ but at stage $\boldsymbol{\Sigma}_n$.

18.2 Comparisons

The discussion now reduces to a comparison of the naturally evolved labstate $\boldsymbol{\Psi}_n \equiv \mathbb{U}_{n,m} \boldsymbol{\Psi}_m$ and the persistent image state $\tilde{\boldsymbol{\Psi}}_n \equiv \mathbb{N}_{n,m} \boldsymbol{\Psi}_m$. The problem is that there is no natural measure of difference between two quantum register states in the same register, no measure of *distance*, that survives all criticism. The following are some possibilities.

The Born Measure of Similarity

In standard QM Hilbert space theory, the inner product (Φ, Ψ) of two normalized state vectors in the same Hilbert space has an empirical significance: if $(\Phi, \Psi) = 0$, the two states are regarded as totally different. More generally, the square modulus $P(\Phi|\Psi) \equiv |(\Phi|\Psi)|^2$ of this amplitude has the Born interpretation as the conditional probability of a positive answer if tested for state Φ given prepared state Ψ.

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We may, with some justification, refer to $P(\Phi|\Psi)$ as the *Born measure of similarity*.

The problem with the Born measure of similarity is that it can be far too crude for quantum register states, as the following example illustrates.

Example 18.3 Consider a rank-billion quantum register $\mathcal{Q}^{[10^9]}$. The states $\Psi \equiv \widehat{\mathbb{A}}^1\widehat{\mathbb{A}}^2\ldots\widehat{\mathbb{A}}^{999999999}\widehat{\mathbb{A}}^{10^9}\mathbf{0}$ and $\Phi \equiv \widehat{\mathbb{A}}^1\widehat{\mathbb{A}}^2\ldots\widehat{\mathbb{A}}^{999999999}\mathbf{0}$ have zero inner product, but differ in only one signal out of a billion. By any conventional, heuristic measure of similarity, these two states would be regarded as very similar though not identical, but the Born measure of similarity gives them as totally different.

We conclude that the Born measure of similarity is not good enough, in the context of this chapter and the next, as a measure of similarity.

The Hamming Measure of Dissimilarity

The problem of comparing two quantum register states in the same quantum register has an interesting parallel in the world of computing, cryptography, and information science. In those subjects, a frequent problem is to compare two strings of symbols, such as $S^1 \equiv \underline{\alpha Q55\{7}$ and $S^2 \equiv \underline{aP56[7}$. There are two possible cases: either the strings have the same number of elements (six in this case), or they have different numbers of elements. We shall discuss only the equal case.

Equal-Length Strings

In his study of the transmission of information over telephone networks, Hamming discussed the problem of identifying and then correcting errors in transmission (Hamming, 1950). This scenario is very much like a quantum experiment, with essentially the same architecture, including an information void. There are observers (the *speaker* and the *listener*) and their preparation devices and final state detectors (the telephones). The information void here consists of extensive transmission lines and modules, such as telephone exchanges, signal amplifiers, and so on.

Suppose $S \equiv \underline{s^1 s^2 \ldots s^r}$ is the message that is sent (the prepared state) and $T \equiv \underline{t^1 t^2 \ldots t^r}$ is the message that is actually received (the outcome state). A natural question is: *how different are these two strings?* Hamming devised a geometrical method of quantifying the difference between two character strings of equal length. He defined a distance, or metric, $d_H(S,T)$ between S and T according to the rule

$$d_H(S,T) \equiv \text{number of matched pairs } (s^i,t^i), i = 1,2,\ldots,r, \text{ for which } s^i \neq t^i. \tag{18.4}$$

Example 18.4 For $S \equiv \underline{\alpha Q55\{7}$ and $T \equiv \underline{aP56[7}$ we find $d_H(S,T) = 4$, as these strings of length six differ everywhere except in their third and sixth elements.

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The interpretation of the Hamming distance between two equal length strings is that it is the minimum number of single character replacements in one string that would convert it into the other. In Example 18.4, we can convert S into T by the four replacements $\alpha \to a$, $Q \to P$, $5 \to 6$, and $\{ \to [$.

Exercise 18.5 Prove that the Hamming metric is a true metric in the sense of a metric space.

Suppose we prepare a labstate $\mathbf{\Psi}_m$ in a rank-r quantum register $\mathcal{Q}_m$ at stage Σ_m, and pass it through an information void, until it is received as outcome $\mathbf{\Psi}_n$ in an identical rank quantum register $\mathcal{Q}_n$ at stage Σ_n, where $n > m$.[2] The two registers are copies of each other, having the same rank. Moreover, their preferred bases are contextually correlated: computational basis element $\boldsymbol{i}_m$ means the same to the speaker (the observer at stage $\mathbf{\Sigma}_m$) as $\boldsymbol{i}_n$ means to the listener (the observer at stage Σ_n). In other words, speaker and listener understand the same language.

Taking account of the parallel transport problem discussed above, we need to compare like with like. We choose to compare the persistent image $\tilde{\mathbf{\Psi}}_n \equiv \mathbb{N}_{n,m}\mathbf{\Psi}_m$ in $\mathcal{Q}_n$ of the original state with the transmitted state $\mathbf{\Psi}_n \equiv \mathbb{U}_{n,m}\mathbf{\Psi}_m$, also in $\mathcal{Q}_n$.

The problem is compounded here by *superposition*: neither $\mathbf{\Psi}_n$ nor $\tilde{\mathbf{\Psi}}_n$ are classical, but thay are complex superpositions of classical information (which is essentially what preferred basis elements are). This is a factor that Hamming did not face.

In the following, we drop the temporal index, as it does not play an essential role.

Hamming Distance between Preferred Basis Elements

In the simplest case, case, suppose $\mathbf{\Psi} = \boldsymbol{i}$ and $\tilde{\mathbf{\Psi}} = \boldsymbol{j}$, where $\boldsymbol{i}$ and $\boldsymbol{j}$ are elements of the computational basis representation (CBR) for $\mathcal{Q}$. Each basis element carries classical information, that is, corresponds to a signal state that could actually be observed in a run. We calculate the Hamming distance $d_H(\boldsymbol{i}, \boldsymbol{j})$ between these two elements by first converting each integer into its associated binary string and then working out the Hamming distance using the rule given in (18.4).

That means applying the process of binary decomposition. Specifically, we write

$$\begin{aligned} i &= i^{[1]} + i^{[2]}2 + i^{[3]}2^2 + \cdots + i^{[r]}2^{r-1}, \\ j &= j^{[1]} + j^{[2]}2 + j^{[3]}2^2 + \cdots + j^{[r]}2^{r-1}, \end{aligned} \tag{18.5}$$

[2] Note that the transmitted state is *not* an actual outcome (which is *classical information*), but a quantum state immediately before it is looked at for an outcome, and is therefore best thought of as a different form of information, referred to by us as *quantum information*.

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Table 18.1 *The Hamming distance between elements of a rank-three quantum register basis*

	$\underline{000}$	$\underline{100}$	$\underline{010}$	$\underline{110}$	$\underline{001}$	$\underline{101}$	$\underline{011}$	$\underline{111}$
$\mathbf{0} \equiv \underline{000}$	0	1	1	2	1	2	2	3
$\mathbf{1} \equiv \underline{100}$	1	0	2	1	2	1	3	2
$\mathbf{2} \equiv \underline{010}$	1	2	0	1	2	3	1	2
$\mathbf{3} \equiv \underline{110}$	2	1	1	0	3	2	2	1
$\mathbf{4} \equiv \underline{001}$	1	2	2	3	0	1	1	2
$\mathbf{5} \equiv \underline{101}$	2	1	3	2	1	0	2	1
$\mathbf{6} \equiv \underline{011}$	2	3	1	2	1	2	0	1
$\mathbf{7} \equiv \underline{111}$	3	2	2	1	2	1	1	0

where the coefficients $i^{[k]}, j^{[k]}$, $k = 1, 2, \ldots, r$, are each either zero or one. Next, we form the strings $S^i \equiv \underline{i^{[1]} i^{[2]} \ldots i^{[r]}}$, $S^j \equiv \underline{j^{[1]} j^{[2]} \ldots j^{[r]}}$. Finally, we calculate the Hamming distance $d_H(S^i, S^j)$.

Table 18.1 gives the Hamming distance between pairs of elements in the preferred basis for a rank-three quantum register.

We note the following:

1. The maximally saturated state $\mathbf{7} \equiv \underline{111}$ is furthest away in Hamming distance terms from the ground state $\mathbf{0} \equiv \underline{111}$,
2. All elements of the same signality σ are a Hamming distance σ from the signal ground state.
3. For any two register basis elements, the Hamming distance between them defines what can be thought of as a *relative signality*: if any one of these elements were chosen to be the new ground state, then the other state would have signality equal to its Hamming distance from that new ground state. This underlines the fact that the ground state in a QDN register is not intrinsic to the apparatus but *defined* contextually by the observer. The choice of signal ground state is *not* dictated by lowest energy state.

The power of Hamming's approach is that his distance rule is a genuine metric, so that all the theorems of metric spaces can be applied. Lest this appear trivial, we should consider that quantum registers modeling real situations may have immense rank, such as that modeling a superconducting quantum interferometer. In such situations, the Hamming metric distance approach may well prove useful.

The above approach is classical, in that it covers classical register states completely. QM, however, involves superpositions of register basis states, so we are faced with the problem of defining the equivalent of a Hamming distance between arbitrary, normalized quantum register states in the same register.

The Hamming Operator

For a given rank-r quantum register $\mathcal{Q}$ with CBR $\{\boldsymbol{i} : i = 0, 1, 2, \ldots, 2^r - 1\}$ we define the *Hamming operator* $\mathbb{H}$ by

$$\bar{\boldsymbol{i}} \mathbb{H} \boldsymbol{j} \equiv d_H(i, j), \tag{18.6}$$

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where $d_H(\boldsymbol{i}, \boldsymbol{j})$ is the Hamming distance between elements $\boldsymbol{i}$ and $\boldsymbol{j}$. Then using completeness, we have the representation

$$\mathbb{H} = \sum_{i,j=0}^{2^r-1} \boldsymbol{i} d_H(i,j) \overline{\boldsymbol{j}}. \tag{18.7}$$

We would in the first instance like to use this operator to attempt several definitions of "distance" between more general quantum register states. Examples are the following.

Quantum Hamming Measure of Dissimilarity
Given the CBR expansions $\boldsymbol{\Phi} \equiv \sum_{i=0}^{2^r-1} \Phi^i \boldsymbol{i}$ and $\boldsymbol{\Psi} \equiv \sum_{i=0}^{2^r-1} \Psi^i \boldsymbol{i}$ we define the quantum Hamming measure of dissimilarity $H_1(\boldsymbol{\Phi}, \boldsymbol{\Psi})$ as the magnitude of the matrix element $\overline{\boldsymbol{\Phi}} \mathbb{H} \boldsymbol{\Psi}$, that is,

$$H_1(\boldsymbol{\Phi}, \boldsymbol{\Psi}) \equiv \left| \sum_{i,j=0}^{2^r-1} \Phi^{i*} d_H(i,j) \Psi^j \right|. \tag{18.8}$$

A variant proposal is to define the quantum Hamming measure of dissimilarity H_2 as

$$H_2(\boldsymbol{\Phi}, \boldsymbol{\Psi}) \equiv \sum_{i,j=0}^{2^r-1} \Pr(\boldsymbol{i}|\boldsymbol{\Phi}) d_H(i,j) \Pr(\boldsymbol{j}|\boldsymbol{\Psi}) = \sum_{i,j=0}^{2^r-1} |\Phi^i|^2 d_H(i,j) |\Psi^j|^2. \tag{18.9}$$

Neither (18.8) nor (18.9) is a genuine metric, because they do not return zero in general when $\boldsymbol{\Phi} = \boldsymbol{\Psi}$. However, that is not necessarily a bad thing. After all, quantum states are not objective things. The fact that neither $H_1(\boldsymbol{\Psi}, \boldsymbol{\Psi})$ nor $H_2(\boldsymbol{\Psi}, \boldsymbol{\Psi})$ is not zero in general reflects the fact that labstates are not classical. If they were, they would be represented by a single element of the CBR, and in that case, both measures of dissimilarity vanish.

18.3 Signal Correlation Measure of Change

Suppose an observer is able to prepare and investigate, separately, two labstates $\boldsymbol{\Psi}$ and $\boldsymbol{\Phi}$ in a rank-r quantum register $\mathcal{Q}$. This means that the observer can examine each of these labstates separately and measure the answer to any maximal or partial question about either of these states. Another way of saying this is that the observer can determine, empirically, the frequencies of finding either labstate in a given preferred basis state. In reality, that is *all* that an observer can do.

With such information, the observer can meaningfully discuss differences in signal outcome probabilities, and it is on that basis that we now proceed. First we review briefly the concept of *correlation*.

Correlations in Statistics
The problem of comparing two data sets occurs frequently in statistics. Suppose we have two sets of data, X and Y, each consisting of n real numbers, such that

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the ith element of X is x^i and the ith element of Y is y^i. The question often arises: how "close" are the two sequences $\{x^1, x^2, \ldots, x^n\}$ and $\{y^1, y^2, \ldots, y^n\}$?

This question is a variant of the one originally addressed by Hamming. Statisticians have developed the concept of *correlation* in a variety of forms that attempt to give a number, usually between -1 and $+1$, that gives the degree of similarity, dependence, or correspondence between the two data sets (or sequences). A well-known correlation coefficient is the Pearson correlation coefficient, which in the case we are discussing is known as a *sample correlation coefficient* $C_{X,Y}$ defined by

$$C_{X,Y} \equiv \frac{\sum_{i=1}^{n}(x^i - \overline{x})(y^i - \overline{y})}{\sqrt{\sum_{j=1}^{n}(x^j - \overline{x})^2 \sum_{k=1}^{n}(y^k - \overline{y})^2}}, \tag{18.10}$$

where $\overline{x}$ and $\overline{y}$ are the respective sample averages. If a sample correlation coefficient has value $+1$, 0, or -1, the two samples are said to be perfectly correlated, uncorrelated, or perfectly anticorrelated, respectively.

Detector Correlations

In our case, our interest is directly related to detector physics. The "data" values in which we are interested, that is, the analogues of the x^i and y^i values in the above, are *truth* values, suitably modified to give correlations. What greatly helps here is that the values are binary. Suppose we look at two detectors and compare their signal status. If they both register a signal, or if they both register ground (no signal), then we can say they are perfectly correlated. Otherwise, they are perfectly anticorrelated.

More commonly, we might take many such joint readings. Suppose the probability of finding perfect correlation is p. Therefore, the probability of anticorrelation is $1 - p$. Assigning a value $+1$ for every case of perfect correlation and a value of -1 for every case of perfect anticorrelation, we deduce that the average correlation C will be given by $C = 2p - 1$.

Single Qubit Temporal Correlation

Quantum registers in practice may be vast, and the description of change and persistence of labstates will be accordingly complicated. It is reasonable to focus attention on the simplest SUO first, in order to appreciate what we are faced with.

Consider an experiment that is dealing with an SUO represented by a single qubit. Suppose that, by stage Σ_0, the observer has prepared a labstate $\mathbf{\Psi}_0$ in the rank-one quantum register $\mathcal{Q}_0$ given by the CBR expression

$$\mathbf{\Psi}_0 \equiv \alpha \mathbf{0}_0 + \beta \mathbf{1}_0, \tag{18.11}$$

where $|\alpha|^2 + |\beta|^2 = 1$.

Now suppose the labstate $\mathbf{\Psi}_0$ is allowed to evolve undisturbed to stage Σ_1 under semi-unitary evolution, into labstate $\mathbf{\Psi}_1$ in a rank-one quantum register

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$\mathcal{Q}_1$. Then we write $\mathbf{\Psi}_1 \equiv \mathbb{U}_{1,0}\mathbf{\Psi}_0$, where the evolution operator $\mathbb{U}_{1,0}$ is semi-unitary. Taking the matrix representation of this operator in standard form, shown in Eq. (11.26), with the phase E set to zero, we have the rules

$$\begin{aligned} \mathbb{U}_{1,0}\mathbf{0}_0 &= a\mathbf{0}_1 - b^*\mathbf{1}_1, \\ \mathbb{U}_{1,0}\mathbf{1}_0 &= b\mathbf{0}_1 + a^*\mathbf{1}_1, \end{aligned} \tag{18.12}$$

where $|a|^2 + |b|^2 = 1$.

We now come to a critical point, one that impinges on the quantum Hamming measures of dissimilarity discussed above. Consider a persistent array of detectors from stage Σ_m to stage Σ_n, where $n > m$, such that detector i_m at stage Σ_m is regarded as the same as i_n at stage Σ_n. Given the labstates $\mathbf{\Psi}_m$ and $\mathbf{\Psi}_n$, there are two very different quantities that could be measured:

Correlation of Probabilities

Suppose the observer measured P^i_m, the probability of a positive signal in i_m and, separately, P^i_n, the probability of a signal in i_n. These two quantities give important information about what goes on at detector i at two separate stages. But there is no direct correlation between those two pieces of information. Every observation of i_m at stage Σ_m cannot possibly be influenced by what could happen at i_n at stage Σ_n, and vice-versa, simply because each observation in any given run necessarily precludes the other observation, in that run.

The argument goes as follows. If the observer looks at stage Σ_m to see what the signal status of detector i_m is, then that stops the run and so i_n cannot be investigated *during that run*. Likewise, if the observer decides to see what the signal status is of detector i_n at stage Σ_n, that means they cannot have looked at i_m at stage Σ_m, because that would have stopped that run immediately. In brief, a comparison of the signal status i_m and i_n in any given run is ruled out. The only information the observer has are the two probabilities P^i_m and P^i_n, and these have been measured separately over many different runs.

If $\widehat{P}^i_m \equiv 1 - P^i_m$, $\widehat{P}^i_n \equiv 1 - P^i_n$, we define the *correlation of probabilities* $P^i_{n,m}$ as follows:

$$\begin{aligned} P^i_{n,m} &\equiv P^i_n P^i_m + \widehat{P}^i_n \widehat{P}^i_m - P^i_n \widehat{P}^i_m - \widehat{P}^i_n P^i_m \\ &= (2P^i_n - 1)(2P^i_m - 1) = C^i_n C^i_m, \end{aligned} \tag{18.13}$$

where C^i_n and C^i_m are detector i correlations at stage Σ_n and Σ_m, respectively.

Probability of Correlation

What the observer is really after is the probability of finding a signal at i_n knowing for sure that there was a signal in i_m. This is then a genuine temporal correlation, denoted $C^i_{n,m}$. In the following, we simplify the discussion by taking the register to consist of just one detector.

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Definition 18.6 Given initial state $\mathbf{\Psi}_m \equiv \alpha \mathbf{0}_m + \beta \mathbf{1}_m$, where $|\alpha|^2 + |\beta|^2 = 1$, and the evolution operator is $\mathbb{U}_{n,m}$, the *temporal correlation* $C_{n,m}$ is given by

$$\begin{aligned} C_{n,m} &\equiv \Pr(\mathbf{0}_m|\mathbf{\Psi}_m)\Pr(\mathbf{0}_n|\mathbf{0}_m) + \Pr(\mathbf{1}_m|\mathbf{\Psi}_m)\Pr(\mathbf{1}_n|\mathbf{1}_m) \\ &\quad - \Pr(\mathbf{1}_m|\mathbf{\Psi}_m)\Pr(\mathbf{0}_n|\mathbf{1}_m) - \Pr(\mathbf{0}_m|\mathbf{\Psi}_m)\Pr(\mathbf{1}_n|\mathbf{0}_m) \\ &= 2\Pr(\mathbf{0}_m|\mathbf{\Psi}_m)\Pr(\mathbf{0}_n|\mathbf{0}_m) + 2\Pr(\mathbf{1}_m|\mathbf{\Psi}_m)\Pr(\mathbf{1}_n|\mathbf{1}_0)_n, \\ &= 2|\alpha|^2|\overline{\mathbf{0}}_n\mathbb{U}_{n,m}\mathbf{0}_m|^2 + 2|\beta|^2|\overline{\mathbf{1}}_n\mathbb{U}_{n,m}\mathbf{1}_m|^2 - 1. \end{aligned} \tag{18.14}$$

We point out that the probabilities $\Pr(\mathbf{0}_m|\mathbf{\Psi}_m) \equiv |\alpha|^2$ and $\Pr(\mathbf{1}_m|\mathbf{\Psi}_m) \equiv |\beta|^2$ are found during the calibration process, prior to the experiment proper starting.

From (18.12), we readily find that

$$C_{n,m} = 2|a|^2 - 1, \tag{18.15}$$

which, significantly, is independent of the components α, β, of the initial labstate. This result makes sense: if the dynamics is trivial, in that the only effect is to multiply the initial labstate by some complex phase factor, then $|a| = 1$ and the correlation is perfect, that is, $C_{n,m} = 1$. If, conversely, the dynamics flips the initial labstate to an orthogonal labstate, then the correlation is -1.

We will use the above result (18.15) in the next chapter on the Leggett–Garg inequalities.

The generalization of these calculations to higher rank registers will undoubtedly be more complicated but, in principle, amenable to the same logic as above: in all cases, the observer should decide what it is that they can actually measure, and then the formalism will give unambiguous predictions.

A final point we need to make is that we expect our apparatus to be large rank and calibrated. Then the observables will be (say) the *up* and *down* components of spin, and interest will be in spin correlations. The QDN analysis in such a case assigns *two* qubits, one for spin *up* and one for spin *down*. This effectively doubles the size of the quantum register. If we labeled spin up by index $2i$, then spin down would be indexed by $2i + 1$, and our temporal correlations would involve signals in detectors $2i$ and $2i + 1$.

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