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THE SCHREIER TECHNIQUE FOR SUBALGEBRAS OF A FREE LIE ALGEBRA

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ABSTRACT. In group theory Schreier's technique provides a basis for a subgroup of a free group. In this paper an analogue is developed for free Lie algebras. It hinges on the idea of cutting a Hall set into two parts. Using it, we show that proper subalgebras of finite codimension are not finitely generated and, following M. Hall, that a finitely generated subalgebra is a free factor of a subalgebra of finite codimension.

1. **Introduction.** The Schreier technique originated in Schreier's ([12]) proof of the Nielsen-Schreier theorem that a subgroup of a free group is free. In this paper we develop a similar method for free Lie algebras. Let us briefly describe the Schreier technique so that the similarity will be clear. Given a subgroup H of the free group $F(X)$, Schreier proves the existence of a collection T of representatives for the cosets Hg of H which is prefix-closed with respect to the given basis X. Then given $t \in T$ and $x \in X$ such that $tx \notin T$, there exists a $t' \in T$ with $Htx = Ht'$, so that $txt'^{-1} \in H$. The collection of all these elements gives a free set of generators for H. Hall and Radó ([7]) proved a converse that builds subgroups of $F(X)$ given the combinatorial structure described above (the collection T and the mapping of tx to t'). Hall ([4]) used this construction to prove that a finitely generated subgroup of a free group is a free factor of a subgroup of finite index.

Lewin ([9]) extended the method to one-sided ideals of free associative algebras and to group algebras of free groups (see also [11] in this connection). In these cases, the transversal T is replaced with a linear basis for a linear complement of the ideal, which consists of monomials and is prefix-closed and the element t' is replaced with a linear combination of the elements of T.

In Section 2 we recall the definition of Hall sets. Such a set constitutes a linear basis for the free Lie algebra $L(X)$ consisting of (parenthesized) monomials. The concept analogous to prefix-closed sets of monomials is that of a Hall cut. The concept first appeared in Širšov's celebrated paper ([15]), where he used it to obtain a result similar to Lazard's elimination theorem. An account of Širšov's work appears in [1]. Širšov's ideas were then used by Schützenberger in [13] for the construction of certain codes. He calls Hall cuts "partition de Širšov". An account of his work (which includes a proof of our Theorem 3.6) appears in [10, Theorem 5.16], where Hall cuts are called "upwards closed subset".

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Hall cuts are defined and combinatorially analyzed in Section 3. This analysis leads in Section 4 to a preliminary construction of subalgebras. As a special case, we get Lazard's elimination theorem ([8]). This construction is, however, not general enough to build all the subalgebras. For that, it is necessary to add the possibility of linear modifications. In Section 5, the concept of a Schreier system is defined. It is the packet of information needed for constructing subalgebras. In that section, a construction is described which associates to each Schreier system a free subalgebra of $L(X)$ with a designated base. This is an analogue of the theorem of Hall and Radó. Section 6 axiomatizes the concept of a Hall cut. In Section 7 we prove that every homogeneous subalgebra can be built by the construction of Section 5. The non-homogeneous case is treated in a standard manner and one gets the theorem of Širšov ([15]) and Witt ([16]) that a subalgebra of a free Lie algebra is free. Section 8 contains several applications of the technique. We prove that a non-zero proper subalgebra of $L(X)$ of finite codimension is not finitely generated. From this, we conclude Baumslag's theorem ([2]) that a non-zero proper ideal is not finitely generated as a Lie algebra. Finally, we prove (an analogue of Hall's theorem) that a finitely generated subalgebra of $L(X)$ is a free factor of a subalgebra of finite codimension.

2. **Hall Sets.** In all that follows, we let X be a set (the set of letters). We denote the free magma on X by $M(X)$. It is the set of binary trees whose leaves are marked by letters of X. Equivalently it is the set of fully parenthesized words on the alphabet X. Recall that every element of $M(X) \setminus X$ can be uniquely written as the product of two elements of $M(X)$. If u and v are in $M(X)$, we denote by uv^n the element

$$uv^n := \underbrace{\big(\cdots(uv)\cdots v\big)}_{n\ v\text{'s}}.$$

$L(X)$ will denote the free Lie algebra on X over a field k.

Let us first recall the definition of Hall sets. This is taken from [10, Chapter 4].

DEFINITION. Let H be a subset of $M(X)$. Then H is called a *Hall set* if the following conditions hold:

- H has a total order $\leq$.
- X is contained in H.
- For any tree $h = (h'h'')$ in $H \setminus X$, one has

(1) $$h'' \in H \text{ and } h < h''.$$

- For any tree $h = (h'h'')$ in $M(X) \setminus X$, one has $h \in H$ iff

(2) $$h', h'' \in H \text{ and } h' < h'',$$

and

(3) $$\text{either } h' \in X \text{ or } h' = (uv) \text{ and } v \geq h''.$$

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Hall sets first appeared in [5], where the condition $h < h''$ was replaced by the stronger condition that order respects degree. The present condition appears in [14], which also shows that it is in some sense optimal. For our purposes, use of that weaker condition is mandatory.

Every element in $M(X)$ gives a Lie polynomial, in $L(X)$, by interpreting the parentheses as Lie brackets. The polynomials corresponding to elements of a Hall set are called *Hall polynomials*. The basic property of these is the following:

THEOREM 2.1. *The Hall polynomials form a basis of the free Lie algebra (as a k-vector space).*

A proof of this appears in [10, Theorem 4.9].

3. **Combinatorics of Hall Cuts.**

DEFINITION. Let H be a Hall set in $M(X)$. Then a *Hall cut in* H is a subset $R \subset H$, which is greater than its complement, *i.e.* $u \in R$ and $v \in H \setminus R$ implies $v < u$.

DEFINITION. A subset R of $M(X)$ is said to be *right closed* if whenever $r = (r'r'') \in R \setminus X$, one has $r'' \in R$.

LEMMA 3.1. *Let R be a Hall cut in H. Then R is right closed.*

PROOF. Let $r = (r'r'') \in R \setminus X$. Then by (1), one has $r < r''$, so by the definition of a Hall cut one cannot have $r'' \in H \setminus R$. ∎

DEFINITION. Let H be a Hall set in $M(X)$ and R be a Hall cut in H. An element $z \in H \setminus R$ is called an *R-exit* if either $z \in X$ or $z = (z'z'') \notin X$ and $z'' \in R$.

When there is no danger of confusion, we may drop the R and call z an exit. We give an equivalent characterization of the exits.

LEMMA 3.2. *Let R be a Hall cut in H. Then an element $z \in H \setminus R$ is an R-exit iff it is not a product of two elements in $H \setminus R$.*

PROOF. Let $z \in H \setminus R$ be an element which is not a product of two elements in $H \setminus R$. If $z \in X$, then z is an exit. If not, write $z = (z'z'')$. We have to prove that $z'' \in R$. By (2), one has $z' < z''$, so by the definition of a Hall cut, if $z'' \notin R$, then $z' \notin R$, so that z is a product of two elements in $H \setminus R$.

Conversely, if $z \in X$ or if $z = (z'z'') \notin X$ and $z'' \in R$, then z clearly is not a product of two elements of $H \setminus R$. ∎

COROLLARY 3.3. *Let R be a Hall cut in H. Then every element in $H \setminus R$ is in the submagma of $M(X)$ generated by the set of R-exits.*

PROOF. Induction on the degree and use of the lemma. ∎

We want to show now that $H \setminus R$ is a Hall set on the set of R-exits. In order to consider a Hall set on a subset of $M(X)$, we have to show that this subset freely generates a free submagma of $M(X)$. We thus want to establish a criterion for that to happen.

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PROPOSITION 3.4. *Let $Z \subset M(X)$ and let $\langle Z \rangle$ be the submagma of $M(X)$ generated by Z. Then $\langle Z \rangle$ is a free submagma with basis Z iff no element in Z is a product of two elements in $\langle Z \rangle$.*

PROOF. Let Z' be a set in bijection with Z. The bijection $Z' \longrightarrow Z$ can be uniquely extended to a magma epimorphism $\phi: M(Z') \longrightarrow \langle Z \rangle$, where $M(Z')$ is the free magma on Z'. It is clear that $\langle Z \rangle$ is free with Z as its basis iff ϕ is injective.

If $z \in Z$ is a product of two elements in $\langle Z \rangle$, then clearly z is an image of two distinct elements of $M(Z')$, so ϕ is not injective. Therefore the condition is necessary.

On the other hand, let us assume that ϕ is not injective and let $u \in \langle Z \rangle$ be an element of least degree, such that there exist distinct $v, w \in M(Z')$ with $\phi(v) = \phi(w) = u$. If either v or w is in Z', then the other must not be in Z' (because the restriction of ϕ to Z' is a bijection of Z' with Z) and u (as an image of both v and w) is an element of Z which is a product of two elements of $\langle Z \rangle$. Otherwise, we have $v = (v'v'')$ and $w = (w'w'')$. Since $\phi(v) = \phi(w)$, we have $\phi(v') = \phi(w')$ and $\phi(v'') = \phi(w'')$. In addition, $v \neq w$, so either $v' \neq w'$ or $v'' \neq w''$, so that either $\phi(v')$ or $\phi(v'')$ is an element with two distinct pre-images and with a smaller degree than u, contradicting the choice of u. ∎

Let us apply this criterion to the set of R-exits of a Hall cut R.

PROPOSITION 3.5. *Let R be a Hall cut in H, and let Z be the set of R-exits. Then $\langle Z \rangle$ is a free magma with Z as its basis.*

PROOF. We check the condition of Proposition 3.4.

Let us first prove by induction on the degree of elements in $\langle Z \rangle$ that $\langle Z \rangle \cap R = \emptyset$. Z is disjoint from R by definition, so this gives the induction base. Suppose that no product of degree less than n of elements of Z is in R. Since every element $r = (r'r'')$ in $R \setminus X$ has $r'' \in R$ (Lemma 3.1), we see that no product of degree n of elements of Z is in R. Thus, we have by induction that $\langle Z \rangle \cap R = \emptyset$.

Now, every $z = (z'z'') \in Z \setminus X$ has $z'' \in R$, so we see that no element of Z is a product of two elements of $\langle Z \rangle$ and the condition holds. ∎

Corollary 3.3 and the last proposition enable us to identify $\langle Z \rangle$ with $M(Z)$ and to view $H \setminus R$ as a subset of the free magma $M(Z)$.

THEOREM 3.6. *Let H be a Hall set in $M(X)$ and R a Hall cut in H. Let Z be the set of R-exits. Then $H \setminus R$, viewed as a subset of the free magma $M(Z)$ and given order as a subset of the Hall set H, is a Hall set on Z.*

PROOF. We verify the demands on a Hall set.

Z is included in $H \setminus R$ by definition.

Let us prove (1). If $h = (h'h'') \in H \setminus R$ is not in Z, we have to prove that $h'' \in H \setminus R$ and $h < h''$. Since h is not in Z, we indeed cannot have $h'' \in R$. The fact that $h < h''$ follows since property (1) holds in H.

Let $h = (h'h'') \in M(Z) \setminus Z$ and assume that $h \in H \setminus R$. We have to prove that $h', h'' \in H \setminus R$, $h' < h''$ and either $h' \in Z$ or $h' = (uv)$ with $v \geq h''$. As $h \in H$, (2) tells us that h' and h'' are in H and $h' < h''$. As $h \in M(Z) \setminus Z$ and by the proof of Proposition 3.5

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(namely, by the fact that $\langle Z\rangle \cap R = \emptyset$), we have that h' and h'' are not in R. If $h' \notin Z$, then $h' = (uv)$ with $u, v \in H \setminus R$ and $v \geq h''$, by property (3) of the Hall set H.

Finally, let $h = (h'h'') \in M(Z) \setminus Z$ and assume that $h', h'' \in H \setminus R$, $h' < h''$ and either $h' \in Z$ or $h' = (uv)$ with $v \geq h''$. We have to prove that $h \in H \setminus R$. Since $h'' \in M(Z)$, we cannot have $h'' \in R$ and therefore $h \notin R$ (Lemma 3.1). It remains to show that $h \in H$. Since (2) holds, it remains to show (3). If $h' = (uv)$ with $v \geq h''$, then (3) holds. If $h' \in Z$, there are two possibilities. If $h' \in X$, then (3) clearly holds. If not, then $h' \in Z \setminus X$, and thus $h' = (uv)$ with $v \in R$. So v is in R and h'' is in $H \setminus R$, so $v > h''$. Therefore, (3) holds in that case too. ∎

It would be convenient to know the form of exits.

PROPOSITION 3.7. *Let R be a Hall cut in H and Z be the set of R-exits. Then $Z \cup R$ is the set of all elements of H of the form*

$$z = \Big(\big(\cdots (xr_n)\cdots r_2\big)r_1\Big), \tag{4}$$

where $n \geq 0$, $r_i \in R$ for $i = 1, \ldots, n$, $r_n \geq \cdots \geq r_2 \geq r_1$ and $x \in X$.

PROOF. Let z be an element of the given form. To begin with z is in H. Assume $z \notin R$. If $n = 0$, then $z \in X$. Otherwise, $z = (z'z'')$ where $z'' = r_1$ is in R. In both cases, $z \in Z$.

Conversely, let z be in $Z \cup R$. Then z is in H and we prove that z is of the form (4) by induction on the degree of z. If $z \in X$, then defining $x = z$ and $n = 0$, we see that z is of the required form. If $z \notin X$, write $z = (z'z'')$. Then $z'' \in R$. Define $r_1 = z''$. If $z' \in X$, then defining $x = z'$ we see that z is of the form (4) with $n = 1$. If $z' \notin X$, write $z' = (uv)$. (3) implies that $v \geq z''$. Since R is a Hall cut and $z'' \in R$, we see that $v \in R$. Therefore $z' \in Z \cup R$ and induction gives $z' = \big(\cdots (xr_n)\cdots r_2\big)$, where $n \geq 1$, $r_i \in R$ for $r = 2, \ldots, n$, $r_n \geq \cdots \geq r_2$ and $x \in X$. Recalling that $r_2 = v \geq z'' = r_1$, we see that z is indeed of the form (4). ∎

Finally, we wish to know how exits are changed when we add an element to the cut. It is here that Lazard's elimination theorem (as defined below) first appears.

PROPOSITION 3.8. *Let R be a Hall cut in H and assume that $H \setminus R$ has a maximal element r. Then*

$$Z_{R\cup\{r\}} = \big\{zr^n \mid n \geq 0 \text{ and } z \in Z_R \setminus \{r\}\big\}.$$

PROOF. By equation (4) of Proposition 3.7, an element of $Z_{R\cup\{r\}}$ is of the form

$$u = \Big(\big(\cdots (xr_m)\cdots r_2\big)r_1\Big),$$

with $m \geq 0$, $r_i \in R \cup \{r\}$, $r_m \geq \cdots \geq r_2 \geq r_1$ and $x \in X$. As r is the smallest element of $R \cup \{r\}$, there exists an $n \geq 0$, such that $r_1 = r_2 = \cdots = r_n = r$ and $r_{n+1}, \ldots, r_m \in R$. Define

$$z = \big(\cdots (xr_m)\cdots r_{n+1}\big).$$

(If $m = n$, let $z = x$). Then $u = zr^n$. Note that if $n = 0$, then $z = u < r$ and if $n > 0$, then (zr) is in H, so by (2), we again have $z < r$. Therefore $z \notin R \cup \{r\}$ and using Proposition 3.7 (or the definition of exits) we conclude that $z \in Z_R \setminus \{r\}$.

Conversely, let $u = zr^n$, with $n \geq 0$ and $z \in Z_R \setminus \{r\}$. Then, it is obvious that u is of the form (4), so it remains to show that $u \in H$. This is proved by induction on n. For $n = 0$, we have $u = z \in H$. For $n = 1$, we have that $z < r$ (because $z \neq r$ and r is the maximal element of $H \setminus R$) and either $z \in X$ or $z = (z'z'')$ with $z'' > r$ (because $z'' \in R$ and $r \in H \setminus R$). Therefore, (2) and (3) hold and $(zr) \in H$. Assuming that $zr^n \in H$ for some $n \geq 1$, we have by (1) that $zr^n < r$, so (2) is satisfied for zr^{n+1}. Condition (3) for that element amounts to the fact $r \geq r$, so that $zr^{n+1} \in H$. ∎

4. **Basic Construction and Examples.** In this section, we give the basic construction of subalgebras of $L(X)$ based on the combinatorics of Hall cuts. This construction is not general enough to supply all the subalgebras. For this, we shall have to add additional information, which will be treated in the next sections. However, the basic construction carries within it the basic idea. We denote by l the map that associates to each Hall element (in the free magma) the corresponding Hall polynomial (in the free Lie algebra).

THEOREM 4.1. *Let H be a Hall set in $M(X)$, R a Hall cut in H and Z the set of R-exits. Then the linear subspace spanned by $l(H \setminus R)$ is a free subalgebra A of $L(X)$ with basis $l(Z)$. $L(X)$ is the direct sum (as a k-vector space) of A and the linear subspace spanned by $l(R)$.*

PROOF. By Theorem 3.6, we know that $H \setminus R$ is a Hall set on Z. Combined with the basic result on linear bases for free Lie algebras given by Hall sets (Theorem 2.1), we get the theorem. ∎

We give some examples of subalgebras of $L(X)$ with the bases that Theorem 4.1 provides for them.

Let us start with the subalgebra generated by a subset Y of X. It would of course not be a great surprise to get Y as a basis for that subalgebra. Let H be a Hall set on X, such that every element in $\langle Y \rangle$ is smaller than every element that contains a letter in $X \setminus Y$. Such a Hall set clearly exists. Now let R be the set of elements of H which contain a letter in $X \setminus Y$. Then R is a Hall cut in H. There are no exits of degree more than 1, because if $z = (z'z'')$ with $z'' \in R$, then z'' (and therefore also z) contains a letter in $X \setminus Y$, so that $z \in R$. Therefore, the exits are the letters not in R, *i.e.* $Z = Y$. Thus $H \setminus R$ is a Hall set on Y and the theorem gives us that $l(Y)$ is a free basis of the subalgebra spanned by $l(H \setminus R)$.

A more interesting example, which was analyzed by Lazard (see [3]) is the ideal generated by a subset of X. We state the result as a corollary.

COROLLARY 4.2. *Let $Y \subset X$. Let H_Y be a Hall set on Y. Consider the set*

$$Z = \Big\{ \big(\cdots (xh_n) \cdots h_1 \big) \mid x \in X \setminus Y,\ n \geq 0,\ h_i \in H_Y \text{ and } h_n \geq \cdots \geq h_1 \Big\}.$$

Then the ideal of $L(X)$ generated by $X \setminus Y$ is a free subalgebra with basis $l(Z)$.

PROOF. Let H be a Hall set on X, which includes H_Y as the largest elements. Such a Hall set clearly exists. Then H_Y is a Hall cut relative to H. Proposition 3.7 tells us that the

H_Y-exits are of the form of the elements of Z. In view of Theorem 4.1, it remains to show that all the elements of Z are indeed H_Y-exits. Let $z \in Z$. Then $x \notin Y$, so $z \notin H_Y$. If $n = 0$ then $z \in X$, so $z \in H$ and is a H_Y-exit. If $n \geq 1$, then $z = (z'z'')$ with $z'' = h_1 \in H_Y$, so it only remains to show that $z \in H$. This follows by repeated applications of properties (2) and (3) of Hall sets. ∎

It should be noted that the basis Z of Corollary 4.2 is not the one that Bourbaki gives in [3, II.2.9, Proposition 10].

Taking Y to be a singleton $\{a\}$, we have $H_Y = \{a\}$ and the special case of the last corollary is the Lazard elimination theorem, which first appeared in [8]. Another proof appears in [10] or [14]. Since it lies at the core of the Schreier construction of the following sections, it deserves special mention.

COROLLARY 4.3. *(Lazard's elimination theorem). Let X be a set and $a \in X$. Then $L(X)$ is the direct sum (as a k-vector space) of ka and of a Lie subalgebra which is freely generated, as a Lie algebra, by the elements*

$$\big(-\operatorname{ad}(a)\big)^n(x), n \geq 0, x \in X \setminus \{a\}.$$

5. **Schreier Systems.** The combinatorial structure that holds the information that enables us to construct subalgebras is the Schreier system. Such a system contains a Hall cut and we will define a subalgebra by an inductive process that involves looking at subcuts that grow towards the given cut. In order to use (transfinite) induction, we shall assume that the Hall set H is (inversely) well-ordered, *i.e.* every non-empty subset of H has a maximum. Every cut R other than H itself is then determined by the maximal element m_R in $H \setminus R$ by way of the relation

$$R = \{h \in H \mid h > m_R\}.$$

The order relation on the elements of H can then be identified (after inversion) with the inclusion order on those cuts of H and when we take into account the trivial cut H, we see that the type of the order on the set of cuts of H is the inverse of that of H with an added element that is larger than all the elements. As H is well-ordered, the set of cuts is well-ordered too. We can thus use induction. If $R \neq H$, the cut $R \cup \{m_R\}$, which is the successor of R will be denoted by R^+. The set of R-exits will be denoted by Z_R, the R^+-exits by Z_R^+. For $r \in H$, we denote the cut defined by r by

$$R_r = \{h \in H \mid h > r\}.$$

The set of R_r-exits is denoted Z_r.

DEFINITION. A *Schreier system* (X, H, R, π) is composed of the following set of data:

- An alphabet X.
- A Hall set H in $M(X)$.

- An (inversely) well-ordered Hall cut $R \subset H \subset M(X)$. Let us denote by Z the set of R-exits.
- A map $\pi\colon Z \cup R \longrightarrow kR$ (the modification map), such that if $v \in Z \cup R$ and $r \in R$ appears with a non-zero coefficient in $\pi(v)$, then $v < r$ and in addition either $v \in X$ or $v = (v'v'')$ with $r < v''$ (*i.e.* v is an element of $Z_r \setminus \{r\}$). The coefficient of r in $\pi(v)$ will be denoted by $\pi(v)_r$.

We now define by transfinite induction on subcuts S of R, the following elements of $L(X)$:

- For each $v \in Z_S$, an element $b_S(v)$.
- If $S \neq R$, for each $v \in Z_S$, an element $l_S(v)$.
- If $S \neq R$, for each $v \in Z_S^+$, an element $e_S(v)$.

These elements should satisfy the following stability condition. If $T \subset S \subset R$ are subcuts of R and $v \in Z_S^+ \cap Z_T^+$, such that the linear combination $\pi(v)$ does not contain any element of the interval $S \setminus T$ with a non-zero coefficient, then $e_S(v) = e_T(v)$.

The basic idea is that the l-elements should be obtained from the b-elements by a linear change of basis (hence the letter l) and the e-elements should be obtained from the l-elements by Lazard elimination (hence the letter e).

Now that we know what we seek to define, let us get to the definition itself. First, we wish to define all these elements for the empty subcut $S = \emptyset$. Clearly $Z_\emptyset = X$ and for $x \in X$ we define $b_\emptyset(x) = \bar{x}$, where $\bar{x}$ is the basis element of $L(X)$ associated with the letter x. Assuming that $R \neq \emptyset$, let a be the largest element in H. Then a must be a letter. We define $l_\emptyset(a) = \bar{a}$ and if $x \in X$ other than a, let $l_\emptyset(x) = \bar{x} - \pi(x)_a \bar{a}$. We have

$$Z_\emptyset^+ = \left\{ xa^n \mid n \geq 0 \text{ and } x \in X \setminus \{a\} \right\}.$$

For $v = xa^n$, we define $e_\emptyset(v)$ by induction on n. If $n = 0$ (*i.e.* if $v \in X \setminus \{a\}$), let $e_\emptyset(v) = l_\emptyset(x)$. Suppose that we have defined $e_\emptyset(xa^n)$, let $e_\emptyset(xa^{n+1}) = [e_\emptyset(xa^n), \bar{a}]$. This completes the definition of the three types of elements for the empty subcut.

Suppose that we have defined the elements $e_T(v)$ (defined when $v \in Z_T^+$) for each subcut T of S and let us define the elements associated with the subcut S. If S is the successor subcut of T (*i.e.* if $S = T^+$), we define $b_S(v) = e_T(v)$ (note that it is well-defined). If S is a limit subcut, let $v \in Z_S$. Since $\pi(v)$ is a finite linear combination, there exists an $s \in S$, such that $\pi(v)$ does not contain any element t in the interval $S \setminus R_s$ with a non-zero coefficient. By the stability condition above, we have $e_{R_t}(v) = e_{R_s}(v)$ for each t in the interval $S \setminus R_s$, so that as t decreases, $e_{R_t}(v)$ becomes stable (after the point s). We define $b_S(v)$ to be that stable element.

Assume now that $S \neq R$ and let r be the maximal element of $H \setminus S$. We let $l_S(r) = b_S(r)$ and for $v \in Z_S$ different from r, we let $l_S(v) = b_S(v) - \pi(v)_r b_S(r)$. Finally, we reach the definition of the e's. Every v in Z_S^+ is (by Proposition 3.8) of the form $v = ur^n$ with unique $n \geq 0$ and $u \in Z_S \setminus \{r\}$. We define $e_S(v)$ by induction on n. For $n = 0$, that is when $v \in Z_S \setminus \{r\}$, we define $e_S(v) = l_S(v)$. Supposing that $e_S(ur^n)$ has already been defined, we define $e_S(ur^{n+1}) = [e_S(ur^n), l_S(r)]$.

THEOREM 5.1. *Let (X, H, R, π) be a Schreier system. Denote $B = \{b_R(v) \mid v \in Z\}$, where Z is the set of R-exits. Then B is a free generating set of the subalgebra A generated by it. The set $\{b_{R_r}(r) \mid r \in R\}$ is a linear basis for a k-linear complement of A in $L(X)$.*

PROOF. For each subcut $S \subset R$, we denote

$$\begin{aligned} B_S &:= \{b_S(v) \mid v \in Z_S\} \\ L_S &:= \{l_S(v) \mid v \in Z_S\} \\ E_S &:= \{e_S(v) \mid v \in Z_S^+\} \\ C_S &:= \{b_{R_s}(s) \mid s \in S\}. \end{aligned}$$

Let A_S be the subalgebra generated by B_S. We prove by transfinite induction on the subcuts S of R that A_S is a free Lie algebra with B_S as a basis and that as a k-vector space, we have $L(X) = A_S \oplus kC_S$ (with C_S k-linearly independent). The theorem is the case $S = R$.

Let us first verify this for the empty subcut $S = \emptyset$. By definition we have $B_\emptyset = \overline{X}$, so $A_\emptyset = L(X)$ and indeed $A_\emptyset$ is a free Lie algebra with $B_\emptyset$ as a basis. $C_\emptyset$ is empty, so that indeed $L(X) = A_\emptyset \oplus kC_\emptyset$.

Suppose that the inductive claim has been proven for every proper subcut $T \subset S$ and let us prove it for the subcut S. Let us first assume that S is the successor subcut to T (*i.e.* $S = T^+ = T \cup \{t\}$). By the induction hypothesis, A_T is a free Lie algebra on B_T and $L(X) = A_T \oplus kC_T$. The set L_T is obtained from B_T by changing some of the elements (not including $b_T(t)$) with a multiple of $b_T(t)$. Thus it is clear that L_T also is a basis of A_T. By the definition of E_T, E_T is obtained from L_T by applying Lazard elimination relative to the element $l_T(t)$ $\big(= b_T(t)\big)$. In addition, we have $E_T = B_S$. Therefore, by Lazard's elimination theorem (Theorem 4.3), A_S is a free Lie algebra with B_S as a basis and $A_T = A_S \oplus kb_T(t)$. Since $C_S = C_T \cup \{b_T(t)\}$, we get the required result.

In order to treat the case of a limit subcut, we need to further examine the decomposition $A_T = A_S \oplus kb_T(t)$ in the successor case. Let $l \in A_T$. Then l is a sum of monomials (with coefficients) in the generators B_T. Let us assume that l is a monomial of degree n of the form

$$l = b_1 \cdots b_n,$$

with $b_i \in B_T$ and with some parentheses which we can harmlessly omit. Let us first assume that none of the letters b_i is equal to $b_T(t)$. Then for each i there is a coefficient $\alpha_i \in k$, such that $b_i - \alpha_i b_T(t)$ is in A_S. Now we have

$$b_1 \cdots b_n = \Big(\big(b_1 - \alpha_1 b_T(t)\big) + \alpha_1 b_T(t)\Big) \cdots \Big(\big(b_n - \alpha_n b_T(t)\big) + \alpha_n b_T(t)\Big).$$

Expanding this expression yields a sum of 2^n monomials in the elements of L_T, the first of which is the original monomial (but with $b_i - \alpha_i b_T(t)$ substituted for b_i) and such that all the other monomials contain $b_T(t)$. Therefore, after the Lazard elimination, we get a monomial like the original monomial (but with the new basis elements) and additional monomials (in the elements of E_T) which are of a degree less than n (since Lazard elimination reduces the degree of every monomial that includes the eliminated letter). If

$n = 1$, we may also get a multiple of $b_T(t)$. Notice that if all the α_i's are zero, then no change occurs. If one of the letters b_i is equal to $b_T(t)$, then similar considerations give us a sum of monomials all of which are of a degree less than n in the elements of E_T (unless $n = 1$ in which case we get a multiple of $b_T(t)$).

Now let S be a limit subcut. B_S generates A_S by definition. We want to show that B_S is Lie-independent. Let $b_S(v_1), \ldots, b_S(v_n)$ with $v_1, \ldots, v_n \in Z_S$ be any n elements of B_S. Then there exist subcuts $T_1, \ldots, T_n$, such that $b_S(v_i) = b_T(v_i)$ for every cut T with $T_i \subset T \subset S$. Taking a proper subcut T of S that includes all the T_i, we get that the $b_S(v_i)$ are elements of a basis of A_T so they are Lie-independent. We thus get that A_S is free Lie algebra with B_S as a basis. It remains to be shown that $L(X) = A_S \oplus kC_S$. The linear independence of C_S modulo A_S follows again from the finite character of linear independence. Let us show that $L(X) = A_S + kC_S$. Let $l \in L(X)$. Then, by the induction hypothesis, for every proper subcut T of S, there is an expression

$$l = a_T + c_T \tag{5}$$

with $a_T \in A_T$ and $c_T \in kC_T$. We shall see that there exists a proper subcut T (depending on l), such that $a_T \in A_S$, so that the expression (5) stabilizes at this subcut (*i.e.* it remains constant for subcuts between T and S) and we have $l \in A_S + kC_T$. We may assume by induction that this is true for every limit proper subcut of S, so that every change in the expression (5) occurs at successor steps. From the discussion of the successor case it follows that the degree of the element a_T (as an expression in the elements of B_T) can not grow and therefore it stabilizes at a certain point. Moreover, the number of monomials (relative to some fixed Hall set) occurring in the leading term of a_T does not grow either so that it too stabilizes. Let $b_1 \cdots b_n$ be a monomial in the leading term of a_T after the point of stabilization of the degree and number of monomials of the leading term. Then each of the b_i's corresponds to an element of Z_S (otherwise it would disappear in the point of elimination of b_i). If v is the element of Z_S corresponding to some b_i, then $\pi(v)$ is a finite linear combination and we may take the cut T to contain all the elements of S which appear with a non-zero coefficient in $\pi(v)$. At this point $b_T(v) = b_S(v)$, and taking the subcut T to be so large that it contains all those elements for all the different b_i's in all the monomials of the leading term, we reach a point at which the leading term stabilizes and is in A_S. We may now perform this for the element of the next degree and then continue with decreasing degrees. After a finite number of steps, we get a proper subcut T of S at which point the expression (5) stabilizes, as required. ∎

The algebra A will be denoted by $A(X, H, R, \pi)$. Let us note that if π preserves degrees, that is if whenever r appears with a non-zero coefficient in $\pi(v)$, r and v have the same degree, then we get a base of homogeneous elements and the complement also consists of homogeneous elements. The Schreier system is called *homogeneous* in this case.

6. **Potential Hall Cuts.** We start with some motivation. Let R be a Hall cut in H. Proposition 3.7 implies that the elements of $Z \cup R$ are the elements of H of the form (4). However, let us assume that the set H is unknown and that we only know R with its order. This information is sufficient in order to define the R-exits, that is to determine when an element of the form (4) is in H.

LEMMA 6.1. *Let R be a Hall cut in H. Let*

$$z = \Big(\big(\cdots(xr_n)\cdots r_2\big)r_1\Big)$$

with $n \geq 0$, $r_k \in R$ for $k = 1,\ldots,n$, $r_n \geq \cdots \geq r_2 \geq r_1$ and $x \in X$. For $k = 1,\ldots,n$, denote

$$u_k := \big(\cdots(xr_n)\cdots r_{k+1}\big).$$

We agree that $u_n = x$. Then $z \in H$ iff for each $k = 1,\ldots,n$ the following condition holds:

$$(R_k)\text{: either } u_k \notin R \text{ or } u_k \in R \text{ and } u_k < r_k.$$

PROOF. Assume that $z \in H$. Then by (2), each u_k is in H and $u_k < r_k$ for every k whether u_k is in R or not.

Conversely, assume that (R_k) holds for every k. We prove by decreasing induction on k that u_k is in H. For $k = n$, $u_n = x$ which is in X (and thus in H). Assume that u_k is in H. If $u_k \notin R$, then $u_k < r_k$ because R is a Hall cut. If $u_k \in R$, then by (R_k), $u_k < r_k$. Therefore, in any case, $u_k < r_k$, so that (2) holds for $(u_k r_k) = u_{k-1}$. For $k = n$, $u_k \in X$ and for $k < n$, $r_{k+1} \geq r_k$, so that (3) holds too. Thus u_{k-1} is in H and the induction is complete. ∎

Thus we see that in order to identify the R-exits, we only need to know the alphabet X and the set R with its order. We axiomatize the structure that carries this information (in the free magma again).

DEFINITION. Let R be a subset of $M(X)$. Then R is called a *potential Hall cut* if the following conditions hold:

- R has a total order $\leq$.
- For any tree $r = (r'r'')$ in $R \setminus X$, one has

$$r'' \in R \text{ and } r < r''. \tag{6}$$

- Every $r \in R$ is of the form

$$r = \Big(\big(\cdots(xr_n)\cdots r_2\big)r_1\Big),$$

  with $n \geq 0$, $r_k \in R$ for $k = 1,\ldots,n$, $r_n \geq \cdots \geq r_2 \geq r_1$ and $x \in X$, such that for every $1 \leq k \leq n$, condition (R_k) holds.

As we saw, every Hall cut is a potential Hall cut. We will see that the converse is also true, in the sense that every potential Hall cut can be completed to a Hall set in which it is a Hall cut. This is done (by a converse to Theorem 3.6) by adding a Hall set on the set of exits. Let us first define these.

DEFINITION. Let R be a potential Hall cut. An element $z \in M(X)$ is called an *R-exit* if $z \notin R$, but is of the form (4) with $n \geq 0$, $r_k \in R$ for $k = 1,\ldots,n$, $r_n \geq \cdots \geq r_2 \geq r_1$, $x \in X$ and (R_k) holds for every $1 \leq k \leq n$.

As in Section 3, we first want to show that the exits are a basis for a free submagma of $M(X)$.

PROPOSITION 6.2. *Let R be a potential Hall cut and Z the set of R-exits. Then $\langle Z\rangle$ is a free magma with Z as its basis.*

PROOF. The proof follows the lines of the proof of Proposition 3.5. First, we prove by induction on the degree of elements in $\langle Z\rangle$ that $\langle Z\rangle \cap R = \emptyset$. The induction base follows from the fact that, by definition, elements of Z are not in R. For each element $r = (r'r'')$ in $R \setminus X$, one has $r'' \in R$, so if we know that products of degree less than n are not in R, it follows that products of degree n are also not in R. Therefore, no element of Z is a product of two elements in $\langle Z\rangle$, because for each $z = (z'z'')$ in $Z \setminus X$, one has $z'' \in R$. Now apply Proposition 3.4. ■

Proposition 6.2 allows us to consider a Hall set on Z as a subset of $M(X)$. We now wish to show that given a potential Hall cut, we can complete it to a Hall set by adding a Hall set on the exits.

THEOREM 6.3. *Let $R \subset M(X)$ be a potential Hall cut on X. Let Z be the set of R-exits. Let H_Z be an arbitrary Hall set on Z (considered as a subset of $M(X)$). On $H = H_Z \cup R$, we define a total order, so that it extends the given orders on R (as a potential Hall cut) and on H_Z (as a Hall set) and such that any element of R is larger than any element of H_Z. Then H is a Hall set on X.*

PROOF. We verify the demands on a Hall set.

If $x \in X$, then either x is in R (and thus in H), or is an R-exit and thus in $Z \subset H_Z \subset H$.

Let $h = (h'h'') \in H \setminus X$. We wish to prove that in this case, $h'' \in H$ and $h < h''$. If $h \in R$, then $h'' \in R$ and $h < h''$ by (6). If $h \in Z$, then $h'' \in R$ and $h < h''$ by the definition of Z and the order ($h \in H_Z$ and $h'' \in R$, so $h < h''$). Finally, if $h \in H_Z \setminus Z$, then $h'' \in H_Z$ and $h < h''$ by property (1) of H_Z as a Hall set on Z.

Let $h = (h'h'') \in M(X) \setminus X$ and suppose that $h \in H$. We have to prove (2) and (3). If $h \in Z \cup R$, then h is of the form (4). We thus see that $h' = \big(\cdots(xr_n)\cdots r_2\big)$ is also of the form (4) with $r_k \in R$, $r_n \geq \cdots \geq r_2$, $x \in X$ and (R_k) for $k = 2,\ldots,n$, so that $h' \in Z \cup R$. The condition $h' < h''$ holds in this case by condition (R_1). If $h' \notin X$ (*i.e.* if $n > 1$), then $h' = (u_2 r_2)$ with $r_2 \geq r_1$. Thus we have (2) and (3) if $h \in Z \cup R$. If $h \in H_Z \setminus Z$, then (2) and (3) hold by the corresponding properties of the hall set H_Z.

Let us now assume that $h = (h'h'') \in M(X) \setminus X$, such that $h', h'' \in H$, $h' < h''$ and either $h' \in X$ or $h' = (uv)$ with $v \geq h''$. We have to show that $h \in H$. If $h' \in H_Z \setminus Z$, then $h' = (uv)$ with $v \in H_Z$. Since $v \geq h''$ and by the definition of the order on H, we see that $h'' \in H_Z$. Therefore $h \in H_Z$, by the corresponding property of the hall set H_Z. The remaining case is when $h' \in Z \cup R$. If $h'' \in H_Z$, then because $h' < h''$, we have $h' \in Z$, so that (2) and (3) hold for h (relative to the alphabet Z) and $h \in H_Z$. Suppose now that $h'' \in R$. By the definitions of a potential Hall cut and of exits, $h' = \big(\cdots(xr_n)\cdots r_2\big)$ with $n \geq 1$, $r_k \in R$ for $k = 2,\ldots,n$, $r_n \geq \cdots \geq r_2$, $x \in X$ and (R_k) for $k = 2,\ldots,n$. Defining $r_1 = h''$, we see that if $n \geq 2$, then $r_2 = v \geq h'' = r_1$. Moreover, $u_1 = h' < h'' = r_1$, so that (R_1) holds and we see that $h \in Z \cup R$. Therefore, in that case too, $h \in H$ and since we considered all the possible cases, the proof is complete. ■

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COROLLARY 6.4. *Let R be a potential Hall cut on X and Z be the set of R-exits. Let $Y \subset Z$ and let $Y \cup R$ have an order that extends the order on R and so that every element of Y is smaller than every element of R. Then $Y \cup R$ is a potential Hall cut on X.*

PROOF. Choose a Hall set H_Z on Z, such that Y is a Hall cut in H_Z and on Y the order is that restricted from $Y \cup R$. Since Y is a subset of Z, this clearly is possible. Then Theorem 6.3 implies that $H_Z \cup R$ is a Hall set on X and $Y \cup R$ is an ordinary Hall cut in it and therefore a potential Hall cut. ∎

Examining the definition of Schreier systems in Section 5, we see that in the definition and in the constructions, we use only the cut with its order and the exits. Now that we know that any potential Hall cut is indeed a Hall cut of some Hall set, we can dispense with H and for the cut, we may be content with a potential Hall cut.

The following property will be useful for applications.

THEOREM 6.5. *Let H be a Hall set and let R be a subset of H, such that whenever $\Big(\big(\cdots(xr_n)\cdots r_2\big)r_1\Big) \in R$ with $x \in X$, one has $r_1, \ldots, r_n \in R$. Then R (with the induced order) is a potential Hall cut.*

PROOF. Examining the definition of a potential Hall cut, we see that all the demands on the order are satisfied because H is a Hall set. ∎

7. **Analyzing Subalgebras.** The target of this section is to show that the subalgebra construction method described in Section 5 is general enough to give information on subalgebras of $L(X)$. We shall see that the method builds all the homogeneous subalgebras. An easy and standard method reduces the general case to the homogeneous one.

THEOREM 7.1. *Let X be an alphabet, $L(X)$ be the free Lie algebra on X and $A \subset L(X)$ be a homogeneous subalgebra. Then there exists a homogeneous Schreier system (X, R, π) (where R is a potential Hall cut on X) such that $A = A(X, R, \pi)$.*

PROOF. Let $A = \oplus_{i=1}^{\infty} A_i$ and $L = \oplus_{i=1}^{\infty} L_i$ be the decompositions of A and of L to homogeneous components. We shall inductively define Schreier systems (X, R_n, π_n) (the set of R_n-exits will be denoted Z_n), such that:

1. $R_{n-1} \subset R_n \subset R_{n-1} \cup Z_{n-1}$ is a subcut (*i.e.* every element of $R_n \setminus R_{n-1}$ is smaller than every element of R_{n-1}).
2. All the elements of $R_n \setminus R_{n-1}$ are of degree n.
3. π_n extends π_{n-1} on elements of degree less than n.
4. π_n preserves degree, that is whenever $u \in R_n \cup Z_n$ and v appears in $\pi_n(u)$ with a non-zero coefficient, the degrees of u and v are equal.
5. $A(X, R_n, \pi_n) = A + L^{n+1}$, where $L^{n+1} = \oplus_{i=n+1}^{\infty} L_i$ is the $(n+1)$-th ideal in the lower central series.

Note that every R_n-exit of degree n or less is an R_m-exit for every $m \geq n$ and that $\pi_n(v)$ can be non-zero only for v of degree n or less. Thus the meaning of condition 3 is that the value of π on elements of degree n is determined at the n-th stage in the induction process.

Suppose that we have defined these. We claim that the Schreier system (X, R, π) defined as the union of the systems (X, R_n, π_n) is the desired Schreier system. The algebra $A(X, R, \pi)$ is homogeneous. In order to show that it is equal to A, one must show that the n-th components are identical for every n. By the conditions above, the set of generators of degree less than or equal to n becomes stable after the n-th stage, so the n-th component of $A(X, R, \pi)$ is equal to the n-th component of $A(X, R_n, \pi_n)$ which by condition 5 is the n-th component of A.

So all that remains is to perform the inductive construction. The induction base is achieved by taking $R_0 = \emptyset$. It is easy to verify the conditions. Suppose that we have already constructed R_{n-1} and π_{n-1}. Then we know that $A+L^n = A(X, R_{n-1}, \pi_{n-1})$. Theorem 5.1 supplies us with a set B of free generators for $A+L^n$ in one-to-one correspondence with Z_{n-1}. The form of the Schreier system implies that they are homogeneous elements, so we can write $B = \cup B_i$, where B_i is the set of generators of degree i. Let A' be the subalgebra generated by $B_1 \cup \cdots \cup B_{n-1}$. It is clear that $A' \subset A$. Let A'_n be the n-th homogeneous component of A'. Since B is a basis of $A + L^n$, whose n-th component is L_n, it follows that $L_n = A'_n \oplus kB_n$. Choose (by Zorn's lemma) a subset $Y \subset B_n$ that is linearly independent modulo A_n and maximal with respect to that property. Let U be the elements of Z_{n-1} which correspond to the elements of Y and let V be the elements of Z_{n-1} of degree n which are not in U. Order U with an arbitrary (inverse) well-ordering and define $R_n = U \cup R_{n-1}$, with an order that extends those of U and R_{n-1} and such that the elements of U are smaller than those of R_{n-1}. R_n is a potential Hall cut by Corollary 6.4.

Let us describe the set of R_n-exits. The exits of degree less than n are the R_{n-1}-exits of degree less than n, because on those degrees nothing changed. The exits of degree n are the elements of Z_{n-1} of degree n, which are not in U (*i.e.* the set V). The exits of degree more than n will not concern us. We define π_n to extend π_{n-1} on the exits of degree less than n and to be zero on the exits of degree more than n. If z is an R_n-exit of degree n, then the corresponding element is not in Y and therefore (by the maximality of Y) it is linearly dependent modulo A_n on Y. Therefore, there exists a linear combination of elements in Y which is equal to z modulo A. We take $\pi_n(z)$ to be the corresponding linear combination of the elements of U. The difference between the elements of Y and the corresponding linear combinations are elements of A_n. Denoting them by F, we see that $kB_n = kF \oplus kY$. Therefore $L_n = A'_n \oplus kF \oplus kY$, with $A'_n \oplus kF \subset A_n$ and Y linearly independent modulo A_n. Therefore $A'_n \oplus kF = A_n$.

Let $A'' = A(X, R_n, \pi_n)$. Then A'' is a homogeneous subalgebra. Theorem 5.1 describes a linear complement of A''. The structure of the Schreier system (X, R_n, π_n) implies that all the elements of this complement are of degree n or less. Therefore, A'' includes the ideal L^{n+1}. A'' coincides with $A(X, R_{n-1}, \pi_{n-1})$ on degrees less than n, therefore the component at those degrees coincide with those of A. The construction of R_n and π_n ensure that the n-th component is $A'_n \oplus kF$ which is equal to A_n. Collecting all of this information, we have $A'' = A + L^{n+1}$. ∎

The general (not necessarily homogeneous) case can be reduced to the homogeneous case.

PROPOSITION 7.2. *Let A be a subalgebra of $L(X)$. Let A' be the subalgebra generated by the leading terms of elements of A. Suppose that Y' is a homogeneous basis of A' and C a homogeneous linear complement. Suppose that $Y \subset A$, such that the mapping that associates with each element its leading term is a one-to-one correspondence between Y and Y'. Then A is a free Lie algebra with Y as a basis and C as a complement.*

PROOF. First, it is clear that every homogeneous element of A' is a leading term of an element of A. Therefore, the set Y always exists. Let H be a Hall set on an alphabet in bijection with Y (and Y'). Then for each $h \in H$, the leading term of the substitution in h of the elements of Y is the substitution in h of the elements of Y'. Therefore, they are linearly independent, so that Y generates a free Lie subalgebra B of A. We claim that $B = A$. If not, choose an element of $A \setminus B$ of least degree. Then its leading term is an element of A', which can thus be expressed as a combination of Hall expressions in the basis Y'. Subtracting the corresponding expression in the basis Y yields an element of $A \setminus B$ of smaller degree and we get a contradiction.

If $A + C \neq L(X)$, then choosing an element in $L(X) \setminus (A + C)$ of least degree and using a similar argument to the above gives a contradiction. In a similar fashion, assuming that $A \cap C \neq 0$ and choosing a non-zero element of $A \cap C$ of least degree gives a contradiction. ■

Combining the last two theorems with the fact (proved in Section 5) that a subalgebra associated with a Schreier system is free, we get the Širšov-Witt theorem.

THEOREM 7.3. *Every subalgebra of a free Lie algebra is free.*

Motivated by the last result, we wish to slightly modify the Schreier construction so that it would build all the subalgebras (and not just the homogeneous ones).

DEFINITION. A *modified homogeneous Schreier system* consists of a homogeneous Schreier system together with a mapping $m: Z \longrightarrow L(X)$, such that if $z \in Z$ of degree n, then $m(z)$ is an element of $L(X)$ of degree less than n.

Given a modified homogeneous Schreier system, we define the subalgebra associated to it in the usual manner, except that to each element of the basis (which corresponds to an exit z) we add the element $m(z)$. In this way, by the previous discussion, we can construct all the subalgebras of $L(X)$.

8. **Applications.**

In a finitely generated free group, subgroups of finite index are finitely generated. Lazard's result [8] already indicates that in free Lie algebras the situation is quite the opposite and this is the contents of the next theorem. In the following, notice that though the situation here is the opposite of that in free group theory, the method of proof is similar.

THEOREM 8.1. *Let $L(X)$ be the free Lie algebra on a set X with $|X| \geq 2$ and let A be a proper subalgebra of finite codimension. Then A is (free) of infinite rank.*

PROOF. By Proposition 7.2, we may assume that A is homogeneous. Then by Theorem 7.1 there exists a Schreier system (X, R, π), such that $A = A(X, R, \pi)$. Since R stands in one-to-one correspondence with a linear basis of a complement of A, we see that R is a non-empty finite set. Let a be the largest element in R. Then $a \in X$. Let b be any letter in $X \setminus \{a\}$. Then all the elements ba^n are in $Z \cup R$. Since only finitely many of them can be in R, we conclude that Z is infinite. As Z stands in one-to-one correspondence with a basis of A, the theorem follows. ∎

We can now apply this in order to get a simple proof of a result of B. Baumslag [2].

THEOREM 8.2. *Let $L(X)$ be the free Lie algebra on a set X and let I be a proper non-zero ideal of $L(X)$. Then I is not finitely generated as a Lie algebra.*

PROOF. Choose an element $a \notin I$. Consider $I + ka$. Since I is an ideal, $I + ka$ is a subalgebra of $L(X)$ and I is a subalgebra of it of codimension one. Using Theorem 8.1, we get the result. ∎

We recall that the idealizer of a subalgebra is the largest subalgebra in which it is an ideal. The last theorem is easily seen to be equivalent to the following.

COROLLARY 8.3. *Let A be a non-zero finitely generated subalgebra of $L(X)$. Then A is its own idealizer.*

Another result which can be deduced from the Schreier technique is the existence of a Hall complement.

THEOREM 8.4. *Let A be a finitely generated subalgebra of $L(X)$ and let Y be a finite subset of $L(X)$ which is linearly independent modulo A. Then there exists a subalgebra B of finite codimension in $L(X)$, which contains A as a free factor (i.e. B has a basis which contains a basis of A), such that Y is linearly independent modulo B.*

PROOF. The proof goes along the lines of the proof of the analogous result for free groups. For the sake of simplicity, we first assume that Y is empty. By Theorem 7.1 and Proposition 7.2, there exists a modified homogeneous Schreier system (X, R, π, m), such that A is the subalgebra associated to that system. Since A is finitely generated, the set Z of exits is finite. Every exit in Z defines (via the mapping π) a basis element of A. This definition is achieved by a certain induction and involves a finite number of elements of R. Let us take all the elements of R which are involved in the definition of all the basis elements. This is a finite set. Let us now enlarge it, so that if $\Big(\big(\cdots(xr_n)\cdots r_2\big)r_1\Big)$ is in this set, then $r_1, r_2, \ldots, r_n \in R$. This can be done by adding to the set some of the subtrees of elements of the set which are in R. Thus, we have a finite set of elements of R, which includes all the elements involved in the definition of the basis elements of A and which is closed as described above. By Theorem 6.5, it is a potential Hall cut R'. All the elements of Z are exits with respect to R' and we may construct a modified homogeneous Schreier system (based on the information in the original system) with R' as the cut, such that all the basis elements of A are basis elements of the subalgebra B associated to the new system. Since R' is finite and since R' corresponds to a linear basis of a linear complement of B, we conclude that B is a subalgebra of $L(X)$ of finite codimension and that B has a basis which contains a basis of A, as required.

Suppose now that we wish to ensure that Y is linearly independent modulo B. Each of the elements of Y can be (by Theorem 5.1) written as a sum of an element of A and a linear combination of elements which correspond to elements of R ($b_{R_r}(r)$ in the notation of Theorem 5.1). These linear combinations are linearly independent by assumption. In order to make the elements of Y linearly independent modulo B, we need only make sure that all these elements would correspond to elements of R' and this may be ensured by enlarging R' by some finite subset of R. ∎

Following M. Hall ([6]), on the free Lie algebra, we consider the topology defined by the basis consisting of the subalgebras of finite codimension. Then the last theorem implies that every finitely generated subalgebra is closed in that topology.

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