Canad. Math. Bull. Vol. **41** (4), 1998 pp. 404–412

L^p-BOUNDEDNESS OF A SINGULAR INTEGRAL OPERATOR

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ABSTRACT. Let $b(t)$ be an L^∞ function on $\mathbf{R}$, $\Omega(y')$ be an H^1 function on the unit sphere satisfying the mean zero property (1) and $Q_m(t)$ be a real polynomial on $\mathbf{R}$ of degree m satisfying $Q_m(0) = 0$. We prove that the singular integral operator

$$T_{Q_m,b}(f)(x) = p.v.\int_{\mathbf{R}}^{n} b(|y|)\Omega(y)|y|^{-n} f\left(x - Q_m(|y|)y'\right)\,dy$$

is bounded in $L^p(\mathbf{R}^n)$ for $1 < p < \infty$, and the bound is independent of the coefficients of $Q_m(t)$.

Let S^{n-1} be the unit sphere in R^n, $n \geq 2$, with normalized Lebesgue measure $d\sigma = d\sigma(x')$. Let $\Omega(x)$ be a homogeneous function of degree zero, with $\Omega \in L^1(S^{n-1})$ and

$$\int_{\mathbf{S}^{n-1}} \Omega(x')\,d\sigma(x') = 0, \tag{1}$$

where $x' = x/|x|$ for any $x \neq 0$.

Suppose $b(t)$ is an $L^\infty(\mathbf{R})$ function. In this paper we investigate the L^p boundedness of the singular integral operator

$$T_{Q_m,b}(f)(x) = p.v.\int_{\mathbf{R}^n} K(y) f\big(x - Q_m(|y|)y'\big)\,dy \tag{2}$$

where $y' = y/|y| \in \mathbf{S}^{n-1}$, $K(y) = b(|y|)\Omega(y)|y|^{-n}$ and $Q_m(t) = \sum_{k=1}^{m} \beta_k t^k$ is real.

The study of such kind of operators has a long history. Readers can see [5] for more detailed background. In particular, the following theorem can be found in [5].

THEOREM A. *Let $T_{Q_m,b}$ be the singular integral operator defined in (2). If $\Omega \in H^1(\mathbf{S}^{n-1})$ satisfies (1), then this operator is bounded in $L^2(\mathbf{R}^n)$.*

The proof of Theorem A is based on the Plancherel theorem, so that the authors in [5] were only able to obtain the L^2 boundedness. The main purpose of this note is to extend Theorem A to the following L^p boundedness theorem.

THEOREM 1. *Let $T_{Q_m,b}$ be the singular integral operator defined by (2) and $1 < p < \infty$. If $\Omega \in H^1(\mathbf{S}^{n-1})$ satisfies (1) then this operator is bounded in $L^p(\mathbf{R}^n)$.*

More precisely, we have

$$\|T_{Q_m,b}(f)\|_p \leq C\|\Omega\|_{H^1(\mathbf{S}^{n-1})}\,\|f\|_p \tag{3}$$

Received by the editors June 4, 1997; revised May 6, 1998.
AMS subject classification: 42B20.
Key words and phrases: singular integral, rough kernel, Hardy space.

https://doi.org/10.4153/CMB-1998-054-5 Published online by Cambridge University Press

where C is a constant independent of Ω, f and the coefficients of Q_m.

The proof of Theorem 1 is based on a lemma in [4] by Fan and Pan (see Lemma 3 below).

Throughout this paper, the letter C will denote a positive constant that may vary at each occurrence but is independent of the essential variables.

To prove Theorem 1, we recall the definition of Hardy spaces on the unit sphere, as well as some lemmas.

The Poisson kernel on $\mathbf{S}^{n-1}$ is defined by

$$P_{ry'}(x') = (1 - r^2)/|ry' - x'|^n$$

where $0 \le r < 1$ and $x', y' \in \mathbf{S}^{n-1}$.

For any $\Omega \in L^1(\mathbf{S}^{n-1})$, we define the radial maximal function $P^+\Omega(x')$ by

$$P^+\Omega(x') = \sup_{0 \le r < 1} \left| \int_{\mathbf{S}^{n-1}} \Omega(y') P_{rx'}(y')\, d\sigma(y') \right|.$$

If $\|P^+\Omega\|_{L^1(\mathbf{S}^{n-1})} < \infty$, we say that Ω belongs to the Hardy space $H^1(\mathbf{S}^{n-1})$ with H^1-norm $\|\Omega\|_{H^1(\mathbf{S}^{n-1})} = \|P^+\Omega\|_{L^1(\mathbf{S}^{n-1})}$.

The space $H^1(\mathbf{S}^{n-1})$ was studied in [1] (see also [2]). In particular, it is known that

$$L^1(\mathbf{S}^{n-1}) \supseteq H^1(\mathbf{S}^{n-1}) \supseteq L \operatorname{Log}^+ L(\mathbf{S}^{n-1}) \supseteq L^q(\mathbf{S}^{n-1})$$

for any $q > 1$.

Another important property of $H^1(\mathbf{S}^{n-1})$ is the atomic decomposition, which will be reviewed below.

An *exceptional* atom is an L^∞ function $E(x)$ satisfying $\|E\|_\infty \le 1$.

An ∞-*atom* is an L^∞ function $a(.)$ that satisfies

(4) $$\operatorname{supp}(a) \subset \{x' \in S^{n-1}, |x' - x_0'| < \rho \text{ for some } x_0' \in S^{n-1} \text{ and } \rho > 0\};$$

(5) $$\int_{\mathbf{S}^{n-1}} a(\xi')\, d\sigma(\xi') = 0,$$

(6) $$\|a\|_\infty \le \rho^{-(n-1)}.$$

From [1] or [2], we find that any $\Omega \in H^1(\mathbf{S}^{n-1})$ has an atomic decomposition $\Omega = \sum \lambda_j a_j$, where the a_j's are either exceptional atoms or ∞-atoms and $\sum |\lambda_j| \le C\|\Omega\|_{H^1(\mathbf{S}^{n-1})}$. In particular, if $\Omega \in H^1(\mathbf{S}^{n-1})$ has the mean zero property (1) then all the atoms a_j in the atomic decomposition can be chosen to be ∞-atoms.

LEMMA 1 [VAN DER CORPUT [8]]. *Suppose ϕ and h are smooth functions on $[a, b]$ and ϕ is real-valued. If $|\phi^{(k)}(x)| \ge 1$ for $x \in [a, b]$ then*

$$\left| \int_a^b e^{i\lambda\phi(t)} h(t)\, dt \right| \le C_k |\lambda|^{\frac{-1}{k}} \left[\|h\|_\infty + \|h'\|_1 \right]$$

holds when

(i) $k \ge 2$

(ii) or $k = 1$, if in addition it is assumed that $\phi'(t)$ is monotonic.

https://doi.org/10.4153/CMB-1998-054-5 Published online by Cambridge University Press

LEMMA 2 [SEE [7]]. *Suppose* $P(y) = \sum_{|\alpha|\le m} a_\alpha y^\alpha$ *is a polynomial of degree m in* $\mathbf{R}^n$ *and* $\varepsilon < \frac{1}{m}$. *Then*

$$\int_{|y|\le 1} |P(y)|^{-\varepsilon}\, dy \le A_\varepsilon \Big(\sum_{|\alpha|\le m} |a_\alpha|\Big)^{-\varepsilon}.$$

The bound A_ϵ *may depend on* ε, *m, and the dimension n, but it is independent of the coefficients* $\{a_\alpha\}$.

We shall need the following results from [4] (see also [3]).

LEMMA 3. *Let* $m \in \mathbf{N}$, $s = 0, 1, \ldots, m$, $k \in \mathbf{Z}$ *and* $\{\sigma_{s,k}\}$ *be a family of measures on* $\mathbf{R}^n$ *with* $\hat\sigma_{0,k} = 0$ *for every* $k \in \mathbf{Z}$. *Let* $\{\alpha_{sj} : s = 1, 2, \ldots, m \text{ and } j = 1, 2\} \subset \mathbf{R}^+$, $\{\eta_s : s = 1, 2, \ldots, m\} \subset \mathbf{R}^+ \setminus \{1\}$, $\{N_s : s = 1, 2, \ldots, m\} \subset \mathbf{N}$, *and* $L_s \colon \mathbf{R}^n \longrightarrow \mathbf{R}^n$ *be linear transformations for* $s = 1, 2, \ldots, m$. *Suppose for* $s = 1, 2, \ldots, m$

(a) $\|\hat\sigma_{s,k}\| \le 1$ *for* $k \in \mathbf{Z}$*;*

(b) $|\hat\sigma_{s,k}(\xi)| \le C(\eta_s^k |L_s\xi|)^{-\alpha_{s2}}$ *for* $\xi \in \mathbf{R}^n$ *and* $k \in \mathbf{Z}$*;*

(c) $|\hat\sigma_{s,k}(\xi) - \hat\sigma_{s-1,k}(\xi)| \le C(\eta_s^k |L_s\xi|)^{\alpha_{s1}}$ *for* $\xi \in \mathbf{R}^n$ *and* $k \in \mathbf{Z}$*;*

(d) For some $q > 1$ *there exists* $A_q > 0$ *such that*

$$\Big\| \sup_{k\in\mathbf{Z}} \big| |\sigma_{s,k}| * f \big| \Big\|_{L^q(\mathbf{R}^n)} \le A_q \|f\|_{L^q(\mathbf{R}^n)}$$

for all $f \in L^q(\mathbf{R}^n)$.

Then for every $p \in \Big(\frac{2q}{q+1}, \frac{2q}{q-1}\Big)$, *there exists a positive constant* C_p *such that*

$$\Big\| \sum_{k\in\mathbf{Z}} \sigma_{m,k} * f \Big\|_{L^p(\mathbf{R}^n)} \le C_p \|f\|_{L^p(\mathbf{R}^n)} \tag{7}$$

and

$$\Big\| \Big(\sum_{k\in\mathbf{Z}} |\sigma_{m,k} * f|^2\Big)^{\frac{1}{2}} \Big\|_{L^p(\mathbf{R}^n)} \le C_p \|f\|_{L^p(\mathbf{R}^n)} \tag{8}$$

holds for all $f \in L^p(\mathbf{R}^n)$. *The constant* C_p *is independent of the linear transformations* $\{L_s\}_{s=1}^m$.

Now we are in the position to prove Theorem 1.

Note that $T_{Q_m,b}(f)$ is equal to

$$\int_{\mathbf{R}^n} |y|^{-n} b(|y|)\Omega(y') f\big(x - Q_m(|y|)y'\big)\, dy \tag{9}$$

where $\Omega \in H^1(S^{n-1})$ satisfies the mean zero property (1). We can write $\Omega = \sum \lambda_j a_j$, where $\sum |\lambda_j| \le C\|\Omega\|_{H^1(S^{n-1})}$ and each a_j is an ∞-atom.

So

$$\|T_{Q_m,b}(f)\|_p \le C \sum |\lambda_j| \|B_j(f)\|_p$$

where

$$B_j(f)(x) = \int_{\mathbf{R}^n} b(|y|)|y|^{-n} a_j(y') f\big(x - Q_m(|y|)y'\big)\, dy$$

with a_j being an ∞-atom.

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Therefore, it suffices to show

$$\|B_j f\|_p \le C\|f\|_p \tag{10}$$

where C is independent of the coefficients of the polynomial $Q_m(t)$ and the atoms $a_j(.)$.

For simplicity, we denote $a_j(.)$ by $a(.)$ and $B_j(f)$ by $B(f)$. Noting that $\mathrm{supp}(a)$ is in the ball $B(x'_0, \rho) \cap \mathbf{S}^{n-1}$ for some $x'_0 \in \mathbf{S}^{n-1}$, without loss of generality we may assume $x'_0 = \mathbf{1} = (1, 0, \dots, 0)$.

Let $I_k = (2^k, 2^{k+1})$, then $B(f)(x)$ is equal to

$$\int_{\mathbf{R}^n} b(|y|)|y|^{-n} a(y') \sum_{k=-\infty}^{\infty} \chi_{I_k}(|y|) f\big(x - Q_m(|y|)y'\big)\, dy = \sum_{k=-\infty}^{\infty} \sigma_{m,k} * f(x)$$

where

$$\hat{\sigma}_{m,k}(\xi) = \int_{2^k \le |y| \le 2^{k+1}} b(|y|)|y|^{-n} a(y') e^{-iQ_m(|y|)\langle y', \xi\rangle}\, dy$$

with m being the degree of the polynomial Q_m.

Define

$$\hat{\sigma}_{m-s,k}(\xi) = \int_{2^k}^{2^{k+1}} b(t)t^{-1} e^{-i\sum_{j=m-s+1}^{m} \beta_j t^j \langle x'_0, \xi\rangle} \int_{\mathbf{S}^{n-1}} a(y') e^{-iQ_{m-s}(t)\langle y', \xi\rangle}\, d\sigma(y')\, dt,$$

$s = 1, 2, \dots, m-1$. Noting that $Q_m(t) = \sum_{k=1}^m \beta_k t^k$, we have $Q_0(t) = 0$.

So we define

$$\hat{\sigma}_{0,k}(\xi) = \int_{2^k}^{2^{k+1}} b(t)t^{-1} \int_{\mathbf{S}^{n-1}} a(y') e^{-iQ_0(t)\langle y', \xi\rangle}\, d\sigma(y')\, dt$$

then

$$\hat{\sigma}_{0,k}(\xi) = \int_{2^k}^{2^{k+1}} b(t)t^{-1} \int_{\mathbf{S}^{n-1}} a(y')\, d\sigma(y')\, dt = 0, \quad \text{by (5).}$$

Also, we easily see

$$\|\hat{\sigma}_{m-s,k}\| \le \int_{2^k}^{2^{k+1}} |b(t)|t^{-1} \int_{\mathbf{S}^{n-1}} |a(y')|\, d\sigma(y')\, dt \le C$$

for all $k \in \mathbf{Z}$ and $s = 0, 1, \dots, m-1$.

In the rest of this paper, for any non-zero $\xi = (\xi_1, \dots, \xi_n) \in \mathbf{R}^n$, we write $\xi/|\xi| = \xi' = (\xi'_1, \dots, \xi'_n) \in \mathbf{S}^{n-1}$.

Suppose $n \ge 3$ and $a(.)$ is an ∞-atom on $\mathbf{S}^{n-1}$ with $\mathrm{supp}(a) \subseteq \mathbf{S}^{n-1} \cap B(\xi', \rho)$, where $B(\xi', \rho)$ is the ball in $\mathbf{R}^n$ centered at ξ'. Let

$$F_a(\tau, \xi') = (1-\tau^2)^{(n-3)/2} \chi_{(-1,1)}(\tau) \int_{\mathbf{S}^{n-2}} a\big(\tau, (1-\tau^2)^{1/2} y'\big)\, d\sigma(y').$$

Then, we have the following estimates for F_a when $n \ge 3$.

LEMMA 4. *Up to a constant multiplier independent of* $a(.)$, $F_a(\tau,\xi')$ *is an* ∞*-atom on* $\mathbf{R}$. *More precisely, there is a constant* C *which is independent of* $a(.)$ *such that*

$$\operatorname{supp}(F_a) \subseteq \big(\xi_1' - 2r(\xi'), \xi_1' + 2r(\xi')\big);$$
$$\|F_a\|_\infty \le C/r(\xi');$$
$$\int_{\mathbf{R}} F_a(\tau)\,d\tau = 0,$$

where $r(\xi') = |\xi|^{-1}|A_\rho\xi|$ *and* $A_\rho\xi = (\rho^2\xi_1, \rho\xi_2, \dots, \rho\xi_n)$.

The function F_a can be similarly defined in the case $n = 2$. Suppose $n = 2$ and $a(.)$ is an ∞-atom on $\mathbf{S}^1$ satisfying (4)–(6). The center of the support of $a(.)$ is $\xi' = (\xi_1', \xi_2') \in \mathbf{S}^1$. Let

$$f_a(\tau,\xi') = (1-\tau^2)^{-1/2}\chi_{(-1,1)}(\tau)\Big(a\big(\tau,(1-\tau^2)^{1/2}\big) + a\big(\tau,-(1-\tau^2)^{1/2}\big)\Big).$$

Similar to Lemma 4, we have

LEMMA 5. *Up to a constant multiplier independent of* $a(.)$, $f_a(\tau,\xi')$ *is a* q*-atom on* $\mathbf{R}$, *where* q *is any fixed number in the interval* $(1,2)$*. The radius of its support is* $r(\xi') = |\xi|^{-1}\{\rho^4\xi_1^2 + \rho^2\xi_2^2\}^{1/2}$, *and the center of its support is* ξ_1'.

The proofs of the above two Lemmas can be found in [6].

By inspecting the proof of Theorem 1 using Lemma 3, we only need to check that the family $\{\sigma_{m-s,k}\}$ satisfies the following conditions: for $s = 0, 1, 2, \dots, m-1$

(i) $|\hat\sigma_{m-s,k}| \le C|2^{(m-s)k}|A_\rho\xi||\beta_{m-s}||^{-\frac{1}{4(m-s)}}$ and $\beta_{m-s} \ne 0$;
(ii) $|\hat\sigma_{m-s,k}(\xi) - \hat\sigma_{m-s-1,k}(\xi)| \le C|2^{(m-s)k}|A_\rho\xi||\beta_{m-s}||$;
(iii) $\big\|\sup_{k\in Z}|\sigma_{m-s,k} * f|\big\|_{L^p(R^n)} \le C\|f\|_{L^p(R^n)}$

where $C > 0$ is independent of $k \in \mathbf{Z}$, $\xi \in \mathbf{R}^n$, and the coefficients of the polynomial Q_{m-s}.

We will only prove the case $n > 2$, since the proof for $n = 2$ is essentially the same (using Lemma 5 instead of Lemma 4) with a slight modification.

In fact for $s = 1, 2, \dots, m-1$ we have

$$\begin{aligned}
&\big|\hat\sigma_{m-s,k}(\xi) - \hat\sigma_{m-s-1,k}(\xi)\big| \\
&\quad= \Big|\int_{2^k}^{2^{k+1}} b(t)t^{-1}e^{-i\sum_{j=m-s+1}^m \beta_j t^j\langle x_0',\xi\rangle}\int_{\mathbf{S}^{n-1}} a(y')e^{-iQ_{m-s}(t)\langle y',\xi\rangle}\,d\sigma(y')\,dt \\
&\qquad - \int_{2^k}^{2^{k+1}} b(t)t^{-1}e^{-i\sum_{j=m-s}^m \beta_j t^j\langle x_0',\xi\rangle}\int_{\mathbf{S}^{n-1}} a(y')e^{-iQ_{m-s-1}(t)\langle y',\xi\rangle}\,d\sigma(y')\,dt\Big|.
\end{aligned}$$

Let O be the rotation such that $O(\xi) = |\xi|\mathbf{1}$ and O^{-1} be its inverse. Noting that $a(y')$ has support in $B(\mathbf{1},\rho)$, so by an argument of rotation

$$\begin{aligned}
\int_{\mathbf{S}^{n-1}} a(y')e^{-iQ_{m-s}(t)\langle y',\xi\rangle}\,d\sigma(y') &= \int_{\mathbf{S}^{n-1}} a(O^{-1}y')e^{-iQ_{m-s}(t)|\xi|\langle y',\mathbf{1}\rangle}\,d\sigma(y') \\
&= \int_{\mathbf{S}^{n-1}} A(y')e^{-iQ_{m-s}(t)|\xi|\langle y',\mathbf{1}\rangle}\,d\sigma(y')
\end{aligned}$$

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where $A(y')$ is an ∞-atom supported in $B(\xi',\rho)$. Thus, let $y'=(\tau,y'_2,\dots,y'_n)$, then by the definition of $F_a(\tau,\xi')$ and Lemma 4, we have

$$\begin{aligned}
&|\hat\sigma_{m-s,k}(\xi)-\hat\sigma_{m-s-1,k}(\xi)|\\
&=\Big|\int_{2^k}^{2^{k+1}} b(t)t^{-1}e^{-i\sum_{j=m-s+1}^m\beta_jt^j\langle x'_0,\xi\rangle}\int_{\mathbf R}F_a(\tau,\xi')e^{-iQ_{m-s}(t)|\xi|\tau}\,d\tau\,dt\\
&\qquad-\int_{2^k}^{2^{k+1}} b(t)t^{-1}e^{-i\sum_{j=m-s}^m\beta_jt^j\langle x'_0,\xi\rangle}\int_{\mathbf R}F_a(\tau,\xi')e^{-iQ_{m-s-1}(t)|\xi|\tau}\,d\tau\,dt\Big|\\
&=\Big|\int_{2^k}^{2^{k+1}} b(t)t^{-1}e^{-i\sum_{j=m-s}^m\beta_jt^j\langle x'_0,\xi\rangle}e^{-iQ_{m-s}(t)|\xi|\tau}\\
&\qquad\times\int_{\mathbf R}F_a(\tau,\xi')\{e^{i\beta_{m-s}t^{m-s}(|\xi|\tau-\langle x'_0,\xi\rangle)}-1\}\,d\tau\,dt\Big|\\
&\le\Big|\int_{2^k}^{2^{k+1}}|b(t)|t^{-1}\int_{\mathbf R}|F_a(\tau,\xi')||e^{i\beta_{m-s}t^{m-s}(|\xi|\tau-\langle x'_0,\xi\rangle)}-1|\,d\tau\,dt\Big|\\
&\le\Big|\int_{2^k}^{2^{k+1}}|b(t)|t^{-1}\int_{\mathbf R}|F_a(\tau,\xi')|\,\Big|\beta_{m-s}t^{m-s}\big(|\xi|\tau-\langle x'_0,\xi\rangle\big)\Big|\,d\tau\,dt\Big|.
\end{aligned}$$

Since $x'_0=\mathbf 1=(1,0,\dots,0)$, then

$$\begin{aligned}
\big||\xi|\tau-\langle x'_0,\xi\rangle\big| &= \big||\xi|\tau-\langle\mathbf 1,\xi\rangle\big|=\big||\xi|\tau-|\xi|\xi'_1\big|\\
&=\big||\xi|(\tau-\xi'_1)\big|\le C\big||\xi|r(\xi')\big|,\qquad\text{for all }\tau\in\operatorname{supp}(F_a).
\end{aligned}$$

Thus

$$\begin{aligned}
|\hat\sigma_{m-s,k}(\xi)-\hat\sigma_{m-s-1,k}(\xi)| &\le C\Big|\int_{2^k}^{2^{k+1}}|b(t)|t^{-1}\int_{\mathbf R}|F_a(\tau,\xi')|\,\Big|\beta_{m-s}t^{m-s}\big(\big||\xi|r(\xi')\big|\big)\Big|\,d\tau\,dt\Big|\\
&\le C\|b\|_\infty|\beta_{m-s}|\,|\xi|\,|r(\xi')|\int_{2^k}^{2^{k+1}}t^{m-s-1}\,dt\\
&\le C2^{(m-s)k}|\beta_{m-s}|\,|A_\rho\xi|.
\end{aligned}$$

Similarly, we can prove that

$$|\hat\sigma_{m,k}(\xi)-\hat\sigma_{m-1,k}(\xi)|\le C2^{mk}|\beta_m||A_\rho\xi|.$$

This proves (ii).

Now

$$\hat\sigma_{m-s,k}(\xi)=\int_{2^k}^{2^{k+1}} b(t)t^{-1}e^{-i\sum_{j=m-s+1}^m\beta_jt^j\langle x'_0,\xi\rangle}\int_{\mathbf R}F_a(\tau,\xi')e^{-iQ_{m-s}(t)|\xi|\tau}\,d\tau\,dt.$$

By Lemma 4, without loss of generality we may assume that F_a is a 2-atom with support in $\big(-2r(\xi'),2r(\xi')\big)$, where $r(\xi')=|\xi|^{-1}|A_\rho\xi|$, and $A_\rho\xi=(\rho^2\xi_1,\rho\xi_2,\dots,\rho\xi_n)$.

Thus $A(\tau)=r(\xi')F_a\big(r(\xi')\tau,\xi'\big)$ is a 2-atom with support in the interval $(-1,1)$. After change of variables we have

$$\hat\sigma_{m-s,k}(\xi)=\int_{2^k}^{2^{k+1}} b(t)t^{-1}e^{-i\sum_{j=m-s+1}^m\beta_jt^j\langle x'_0,\xi\rangle}\int_{\mathbf R}A(\tau)e^{-iQ_{m-s}(t)r(\xi')|\xi|\tau}\,d\tau\,dt.$$

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By Hölder's inequality,

$$\begin{aligned}|\hat{\sigma}_{m-s,k}(\xi)| &\leq C\|b\|_\infty 2^{-k/2}\Big\{\int_{2^k}^{2^{k+1}}\Big|\int_{\mathbf{R}} A(\tau)e^{-iQ_{m-s}(t)r(\xi')|\xi|\tau}\,d\tau\Big|^2\,dt\Big\}^{1/2}\\ &= C\|b\|_\infty 2^{-k/2}2^{k/2}\Big\{\int_1^2\Big|\int_{\mathbf{R}} A(\tau)e^{-iQ_{m-s}(2^kt)r(\xi')|\xi|\tau}\,d\tau\Big|^2\,dt\Big\}^{1/2}\\ &= C\|b\|_\infty\Theta_k.\end{aligned}$$

To estimate Θ_k, we choose a function $\psi \in C^\infty(\mathbf{R})$ satisfying

$$\psi(t) \equiv 1 \text{ for } |t| \leq 1, \quad \psi(t) \equiv 0 \text{ for } |t| \geq 2.$$

Define T_k by

$$(T_k f)(t) = \chi_{(1,2)}(t)\int_{\mathbf{R}} e^{-iQ_{m-s}(2^kt)r(\xi')|\xi|\tau}\psi(\tau)f(\tau)\,d\tau.$$

Then

$$T_kT_k^* f(t) = \int_{\mathbf{R}} L(t,\tau)f(\tau)\,d\tau,$$

where

$$L(t,\tau) = \int_{\mathbf{R}} e^{iv\left(Q_{m-s}(2^kt)-Q_{m-s}(2^k\tau)\right)r(\xi')|\xi|}\psi^2(v)\,dv\,\chi_{(1,2)}(t)\chi_{(1,2)}(\tau).$$

We easily see that

$$|L(t,\tau)| \leq C\chi_{(1,2)}(t)\chi_{(1,2)}(\tau).$$

On the other hand, by Lemma 1, we have

$$|L(t,\tau)| \leq C\Big\{|Q_{m-s}(2^kt)-Q_{m-s}(2^k\tau)|\,|r(\xi')|\,|\xi|\Big\}^{-1}\chi_{(1,2)}(t)\chi_{(1,2)}(\tau).$$

Thus

$$|L(t,\tau)| \leq C\Big\{|Q_{m-s}(2^kt)-Q_{m-s}(2^k\tau)|\,|r(\xi')|\,|\xi|\Big\}^{\frac{-1}{2(m-s)}}\chi_{(1,2)}(t)\chi_{(1,2)}(\tau).$$

By invoking Lemma 2 we have

$$\begin{aligned}\sup_{\tau>0}\int_{\mathbf{R}}|L(t,\tau)|\,dt &\cong \sup_{t>0}\int_{\mathbf{R}}|L(\tau,t)|\,d\tau\\ &\leq C\Big\{|r(\xi')|\,|\xi|\Big\}^{\frac{-1}{2(m-s)}}\sup_{t>0}\int_{\mathbf{R}}\Big\{|Q_{m-s}(2^k\tau)-Q_{m-s}(2^kt)|\Big\}^{\frac{-1}{2(m-s)}}\,d\tau\\ &\leq C\Big\{|r(\xi')|\,|\xi|\Big\}^{\frac{-1}{2(m-s)}}|2^k\beta_{m-s}|^{\frac{-1}{2(m-s)}}.\end{aligned}$$

This shows

$$\|T_k f\|_2 \leq C\Big|r(\xi')|\xi|2^k|\beta_{m-s}|\Big|^{\frac{-1}{4(m-s)}}$$

which leads to

$$|\hat{\sigma}_{m-s,k}(\xi)| \leq C\Big|r(\xi')|\xi|2^k|\beta_{m-s}|\Big|^{\frac{-1}{4(m-s)}} = C\Big||A_\rho\xi|2^k|\beta_{m-s}|\Big|^{\frac{-1}{4(m-s)}}.$$

This shows, for $s = 0, 1, 2, \ldots, m-2$,

$$|\hat{\sigma}_{m-s,k}(\xi)| \leq \Big|2^k|\beta_{m-s}|\,|A_\rho\xi|\Big|^{-\frac{1}{4(m-s)}}.$$

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For $s = m - 1$, following the same argument above, we easily obtain

$$|\hat{\sigma}_{1,k}(\xi)| \leq \left|2^k |\beta_1|\, |A_\rho \xi|\right|^{-\frac{1}{4}}.$$

This proves (i).

Finally, we need to check (iii).

For $s = 1, 2, \ldots, m - 1$, we can write

$$\sigma_{m-s,k} * f(x) = \int_{2^k \leq |y| \leq 2^{k+1}} b(|y|)|y|^{-n} a(y') f\Big(x - \Big(Q_{m-s}(|y|)y' - \sum_{j=m-s+1}^{m} \beta_j |y|^j x_0'\Big)\Big)\, dy.$$

Write $Q_{m-s}(|y|)y' + \sum_{j=m-s+1}^{m} \beta_j |y|^j x_0' = \big(P_1(|y|), P_2(|y|), \ldots, P_n(|y|)\big)$, where each P_j is a polynomial of $|y|$ whose coefficients depends on y', x_0' and $\beta_j' s$.

We denote $\tilde{P}(|y|) = \big(P_1(|y|), P_2(|y|), \ldots, P_n(|y|)\big)$, then

$$\sigma_{m-s,k} * f(x) = \int_{2^k \leq |y| \leq 2^{k+1}} b(|y|)|y|^{-n} a(y') f\big(x - \tilde{P}(|y|)\big)\, dy.$$

Thus

$$\begin{aligned}
\sup_{k \in Z} |\sigma_{m-s,k} * f(x)| &\leq \sup_{k \in Z} \|b\|_\infty 2^{-nk} \int_{2^k \leq |y| \leq 2^{k+1}} \left|a(y) f\big(x - \tilde{P}(|y|)\big)\right| dy \\
&\approxeq \sup_{k \in Z} 2^{-k} \int_{|t| \leq 2^{k+1}} \int_{\mathbf{S}^{n-1}} |a(y')| \left| f\big(x - \tilde{P}(t)\big)\right| d\sigma(y')\, dt.
\end{aligned}$$

So we have

$$\Big(\sup_{k \in Z} |\sigma_{m-s,k} * f(x)|\Big)^p \leq \int_{\mathbf{S}^{n-1}} \Big(\sup_{r>0} \frac{1}{r} \int_0^r \left| f\big(x - \tilde{P}(t)\big)\right| dt\Big)^p |a(y')|\, d\sigma(y').$$

Therefore, we have

$$\Big\|\sup_{k \in Z} |\sigma_{m-s,k} * f|\Big\|_{L^p(R^n)}^p \leq \int_{\mathbf{S}^{n-1}} |a(y')| \left\|\sup_{r>0} \frac{1}{r} \int_0^r \left| f\big(. - \tilde{P}(t)\big)\right| dt\right\|_{L^p(R^n)}^p d\sigma(y').$$

It was shown that

$$\left\|\sup_{r>0} \frac{1}{r} \int_0^r \left| f\big(. - \tilde{P}(t)\big)\right| dt\right\|_{L^p(R^n)}^p \leq C \|f\|_{L^p(R^n)}^p \quad \text{for } 1 < p < \infty$$

with C independent of the coefficients of $\tilde{P}$ (thus independent of y', x_0' and $\beta_j' s$) (see [8, pp. 476–478]). So we have for $s = 1, 2, \ldots, m - 1$

$$\Big\|\sup_{k \in Z} |\sigma_{m-s,k} * f|\Big\|_{L^p(R^n)} \leq C \|f\|_{L^p(R^n)} \quad \text{for } 1 < p < \infty.$$

Using the exact same argument, we can check that

$$\Big\|\sup_{k \in Z} |\sigma_{m,k} * f|\Big\|_{L^p(R^n)} \leq C \|f\|_{L^p(R^n)} \quad \text{for } 1 < p < \infty.$$

The theorem is proved.

https://doi.org/10.4153/CMB-1998-054-5 Published online by Cambridge University Press

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https://doi.org/10.4153/CMB-1998-054-5 Published online by Cambridge University Press