and as shown in my former letter,

$$v\frac{da}{dv} = a_x + p_x v a_{x+1} + {}_2p_x v^2 a_{x+2} + \dots.$$

or

$$\frac{da}{dv} = \frac{a_x}{v} + \frac{p_x v a_{x+1}}{v} + \frac{{}_2p_x v^2 a_{x+2}}{v} + \dots.$$

If, now, a_{x+1}, a_{x+2}, ... are none of them greater than a_x,

then

$$\frac{da}{dv} < \frac{a_x}{v} + \frac{p_x v a_x}{v} + \frac{{}_2p_x v^2 a_x}{v} + \dots.$$

Hence, under the same condition, we shall certainly have

$$(1+a)^2 > \frac{da}{dv},$$

if

$$(1+a_x) + p_x v(1+a_x) + {}_2p_x v^2(1+a_x) + \dots. > \frac{a_x}{v} + \frac{p_x v a_x}{v} + \frac{{}_2p_x v^2 a_x}{v} + \dots.$$

that is, if

$$1+a_x > \frac{a_x}{v},$$

or

$$i \cdot \frac{1}{i} + a_x > a_x + i a_x,$$

or

$$\frac{1}{i} > a_x;$$

that is, if the value of a perpetuity of 1 is greater than the value of a life annuity of 1, the rate of interest being the same in both cases.

In other words, since the value of the perpetuity is necessarily the greater, $\frac{dP}{dv}$ is positive; therefore P_x, the net premium, increases as the rate of interest decreases, provided that a_x is not less than a_{x+1}, a_{x+2}, ...

I am, Sir,
Your obedient servant,
W. SUTTON.

18 *Lincoln's Inn Fields*,
1 *March* 1873.

ON THE FORMULA FOR THE MARKET VALUE OF A COMPLETE ANNUITY.

To the Editor of the Journal of the Institute of Actuaries.

SIR,—The usefulness of the expression for the value of a life annuity in terms of d and p, first proposed by the late Griffith Davies, is obvious, whether from a theoretical or practical point of view. From the theoretical, in that it shows the elements of which the value consists; and from the practical, in that it is of universal application, equally valid whether p and d be based on the same rate of interest or not, or when p is a purely arbitrary quantity.

In his work on Annuities, David Jones has introduced a modification of this expression, as a formula for determining the value of an annuity payable to the moment of death. I am not aware that the correctness of his formula has ever been called in question, and I recollect that it has, in at least one instance, been cited as correct in the pages of the *Journal* of the Institute. Notwithstanding this authority, however, and the long immunity of the formula from criticism, I think I shall be able to demonstrate that it is erroneous, and to show the correct expression which should be substituted for it.

The formula in question, which is given at p. 190 of Jones's work, and repeated at p. 217, is $\frac{1-p}{i+p}$. What may be called the rational basis on which this formula is constructed is this:—If the purchaser deducts from every 1 of outlay the annual premium for assuring that 1, and receives at the end of every completed year of the annuitant's life thereafter $i+p$, and the due proportion of $i+p$ for such part of the last year of life as shall have been lived through at the end of that year, these payments will have exactly yielded him interest on the 1 laid out and secured the repayment of the same. It is clear that up to the end of the last completed year of life the required conditions are fulfilled; interest and sinking fund are duly provided for—and it is *assumed* that the proportionate payment to be made in respect of the last uncompleted year will exactly meet the requirements of the case, neither exceeding nor falling short of them. But it is precisely this assumption that renders the formula erroneous. For what *are* the requirements? It must be borne in mind that, p being paid each year in advance, nothing more is wanted to complete the sinking fund after the payment at end of the last completed year; the assurance of 1 at end of the following year is secured, and all that the purchaser is then entitled to receive is i, the interest on his outlay for the year. But by the hypothesis on which the formula is based, what the purchaser will actually receive is the proportionate part of the payment he has annually received during the annuitant's life, viz., $\frac{i+p}{2}$. Therefore, when p is greater than i, he will receive more than he is entitled to; when p is less than i, he will receive less than he is entitled to; and only when p and i happen to be equal will he receive exactly his due.

Proceeding to construct a true formula, let I be the annual payment to be made in respect of 1 outlay. Then, since $\frac{\mathrm{I}}{2}$ will be received at the end of the year in which death occurs, in addition to *such assured sum as together with it will make up the outlay*, it follows that this assured sum must be, not 1, but $1-\frac{\mathrm{I}}{2}$. From every 1 of outlay, then, the purchaser will deduct $d+p\left(1-\frac{\mathrm{I}}{2}\right)$; that is, a year's interest in advance and the premium required for assuring what is not otherwise secured to him of his outlay. In consideration of the price paid, $1-d-p\left(1-\frac{\mathrm{I}}{2}\right)$, he will receive an annual payment of I during life, so that the value of an annuity of 1 during life with proportionate payment after death is $\dfrac{1-d-p\left(1-\frac{I}{2}\right)}{I}$. But from the equation $I=d+p\left(1-\frac{I}{2}\right)$ we can assign a value to I in terms of d and p, and we find that value to be $\dfrac{d+p}{1+\frac{p}{2}}$. Substituting this value in the foregoing formula, we have

$$\frac{1-d-p\left(1-\dfrac{\frac{d+p}{2}}{1+\frac{p}{2}}\right)}{\dfrac{d+p}{1+\frac{p}{2}}};$$

and this expression is finally reducible to the simple form $\dfrac{1-\left(d+\frac{p}{2}\right)}{d+p}$ which accordingly is the correct formula we have been seeking.

It is true that this formula might have been deduced with much more apparent simplicity from the old expression of Francis Baily $a_x+\dfrac{A_x}{2}$; for not only a_x but A_x also is capable of being expressed in terms of d and p, so that this expression becomes $\dfrac{1-(d+p)}{d+p}+\dfrac{\frac{p}{2}}{d+p}$, or $\dfrac{1-\left(d+\frac{p}{2}\right)}{d+p}$; but I preferred constructing it on what I call a rational as distinguished from a purely formal basis, because it is not at once apparent whether the d and p of the second member of Baily's transformed expression are functions identical with those of the first member. From my construction it is evident that they are, and we thus arrive at the certitude which the expression requires to make it of universal application.

It will be found that the difference between the results obtained from Jones's formula and that which I propose to substitute for it, is too small to be of practical importance. I hope, however, it will not be thought useless to correct a theoretical error in a work which is so extensively used as a text book by the students of our profession.

I am, Sir,

Your obedient servant,

1 *Old Broad Street, London.* ANDREW BADEN.

23 *June* 1873.