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GLOBAL HYPOELLIPTICITY OF SUMS OF SQUARES ON COMPACT MANIFOLDS

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Abstract We present necessary and sufficient conditions for an operator of the type sum of squares to be globally hypoelliptic on $T \times G$, where T is a compact Riemannian manifold and G is a compact Lie group. These conditions involve the global hypoellipticity of a system of vector fields on G and are weaker than Hörmander's condition, while generalizing the well known Diophantine conditions on the torus. Examples of operators satisfying these conditions in the general setting are provided.

Introduction

The problem of characterizing hypoellipticity, in its several flavors, of partial differential operators is a central one in PDE theory. This includes sums of squares of vector fields, which have a prominent role due to their wide applicability and prevalence across diverse fields, ranging from geometry to probability theory. In this regard, as far as the *local* theory is concerned, one of the best known results deals with Hörmander's bracket condition [14], which we briefly recall. Given $\mathrm{X}_0, \mathrm{X}_1, \ldots, \mathrm{X}_r$ real, smooth vector fields, and c a smooth real-valued function, say on an open set $\Omega \subset \mathbb{R}^N$, where we define $P \doteq \sum_{j=1}^{r} \mathrm{X}_j^2 + \mathrm{X}_0 + c$, if the Lie algebra generated by the vector fields X_j spans the tangent space $T_x\Omega$ for every $x \in \Omega$, then P is hypoelliptic. The converse holds true when the coefficients of P are real-analytic [9]. In particular, the bracket condition implies *global* hypoellipticity, but it is far from necessary even for real-analytic operators. One of the

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reasons, which also makes the global problem difficult to deal with, is the appearance of small divisors phenomena, encoded in what are generally called Diophantine conditions, which in their simplest form are illustrated by the famous result of Greenfield and Wallach [10]: the operator $\partial_t + \alpha\partial_x$ is globally hypoelliptic on the torus $\mathbb{T}^2$ if and only if α is an irrational non-Liouville number.

These Diophantine conditions are specially common when dealing with an operator of *tube type*, that is, an operator defined on a product $\mathbb{T}^n_t \times \mathbb{T}^m_x$ whose coefficients depend only on the t variable. The natural approach is then to attack the problem using partial Fourier series on the x variable, a powerful tool that justifies why most of the results in the literature deal exclusively with this environment. This poses the question as to what extent can one find analogous characterizations for global hypoellipticity of operators defined on more general compact manifolds, and, in that case, how one should formulate or otherwise replace Diophantine conditions entirely; they must, therefore, be understood from a more abstract viewpoint.

Concerning sums of squares of tube type on tori, one of the most general class of operators considered in the literature so far is the subject of [3]. There, the authors introduce operators with real ultradifferentiable coefficients of the form

$$-\Delta_t - \sum_{\ell=1}^{N}\Big(\sum_{j=1}^{m} a_{\ell j}(t)\partial_{x_j} + \sum_{k=1}^{n} b_{\ell k}(t)\partial_{t_k}\Big)^2, \tag{0.1}$$

where $N \in \mathbb{N}$ is arbitrary, $-\Delta_t = -\partial^2_{t_1} - \ldots - \partial^2_{t_n}$ is the usual Laplace-Beltrami operator on $\mathbb{T}^n$, and the vector fields $\sum_{k=1}^{n} b_k(t)\partial_{t_k}$ are skew-symmetric on $\mathbb{T}^n$. Since the Diophantine condition presented there is very technical, we will not reproduce it here, but we mention that it involves only the coefficients $a_{\ell j}$. This state-of-the-art result is our starting point: In the present work, we introduce a class of operators that naturally encompasses (0.1) but which are allowed to live on a much more general ambient, and whose global hypoellipticity we address.

More precisely, let T be a compact, connected, and orientable smooth manifold and G be a compact and connected Lie group. Our main result concerns the global hypoellipticity of operators on $T \times G$ of the following kind:

$$P \doteq \Delta_T - \sum_{\ell=1}^{N}\Big(\sum_{j=1}^{m} a_{\ell j}(t)\mathrm{X}_j + \mathrm{W}_\ell\Big)^2, \tag{0.2}$$

where Δ_T is the Laplace-Beltrami operator on T associated to a given Riemannian metric, $\mathrm{W}_1, \ldots, \mathrm{W}_N$ are skew-symmetric, real, smooth vector fields on T, while $a_{\ell j} \in \mathscr{C}^\infty(T;\mathbb{R})$ for every $\ell \in \{1, \ldots, N\}$ and $j \in \{1, \ldots, m\}$, and $\mathrm{X}_1, \ldots, \mathrm{X}_m$ is a basis of real left-invariant vector fields on G. The inspiration to consider tube operators of the type sum of squares comes from [4, 6, 7, 8], where global *analytic*-hypoellipticity of such operators is investigated under Hörmander's condition. As mentioned before, the characterization of (smooth) global hypoellipticity on the N-dimensional torus has traditionally been done in terms of some kind of Diophantine condition — which, unlike Hörmander's, is not local (see

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e.g. [11, 12, 13] and references therein). It is worth mentioning that for each class of regularity — for instance, smooth, Gevrey, real-analytic — there is a corresponding type of Diophantine condition, and as the operator under study takes a more general form, the corresponding condition becomes increasingly difficult to treat. Our framework is more general, and so we propose a new and invariant condition about certain system of vector fields on G that characterize the global hypoellipticity on $T\times G$ of operators of the type sum of squares like in (0.2) that avoid any reference to Diophantine conditions.

In order to not overload this section with notations, we will briefly describe our results after introducing a convention: Throughout this work, $\mathfrak{g}$ denotes the Lie algebra of G. Note that when $G=\mathbb{T}^m$, $\mathfrak{g}$ is the space of $\mathbb{R}$-linear combinations of $\partial_{x_1},\dots,\partial_{x_m}$. Let us consider the following set of left-invariant vector fields on G:

$$\mathcal{L}\doteq\Big\{\mathrm{L}\in\mathfrak{g}\ :\ \mathrm{L}=\sum_{j=1}^{m}a_{\ell j}(t)\mathrm{X}_j \text{ for some } \ell\in\{1,\dots,N\} \text{ and some } t\in T\Big\}.$$

Recall that the system $\mathcal{L}$ is globally hypoelliptic in G if any distribution u in G satisfying $\mathrm{L}u\in\mathscr{C}^\infty(G)$ for every $\mathrm{L}\in\mathcal{L}$ is already smooth. Our main results tell that the global hypoellipticity of the system $\mathcal{L}$ on G is necessary for global hypoellipticity of P (see Theorem 3.3 and Proposition 7.2). Under an additional hypothesis, Theorem 3.5 says that this condition is also sufficient.

This additional hypothesis comes from the fact that we are also allowing G to be a noncommutative Lie group and, as we stress, when $G=\mathbb{T}^m$, this hypothesis is always fulfilled by our operator P. It was carefully chosen in order to allow us to provide interesting examples and applications when G is not the m-dimensional torus (see Example 8.1 and the discussion around it), and, moreover, we show in Section 8.3 that a slightly stronger assumption would force G to be Abelian. Nevertheless, we do not know at the moment how restrictive it is, and even suspect that such a condition is not necessary for the validity of Theorem 3.5. A soft evidence for this suspicion is that, according to that theorem's statement, perturbations of P not satisfying the additional hypothesis do not destroy its global hypoellipticity.

As an immediate consequence of our main results, we can give a new characterization of the global hypoellipticity of the operator (0.2) when $G=\mathbb{T}^m$ and T is an arbitrary compact Riemannian manifold. This result is already new in two aspects: there is no mention of Diophantine conditions, and the first factor of $T\times\mathbb{T}^m$ can be much more general than an n-dimensional torus. This corollary goes as follows:

Theorem 1. *Let T be a compact manifold as above and consider the LPDO on $T\times\mathbb{T}^m$ defined by*

$$P\doteq\Delta_T-\sum_{\ell=1}^{N}\Big(\sum_{j=1}^{m}a_{\ell j}(t)\partial_{x_j}+\mathrm{W}_\ell\Big)^2.$$

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Then P is globally hypoelliptic in $T \times \mathbb{T}^m$ if and only if the system of vector fields with constant coefficients

$$\mathcal{L} = \Big\{ \mathrm{L} \in \mathfrak{g} \ : \ \mathrm{L} = \sum_{j=1}^{m} a_{\ell j}(t)\partial_{x_j} \quad \textit{for some } \ell \in \{1, \ldots, N\} \textit{ and some } t \in T \Big\}$$

is globally hypoelliptic in $\mathbb{T}^m$.

It turns out that this condition about $\mathcal{L}$ is equivalent to the Diophantine condition presented in [3] (see Section 8.1 for more details), thus our result above generalizes [3, Theorem 1.5]. However, stated as such, our new condition is much easier to check than the number-theoretic one in many practical situations: For instance, if one is able to find certain finitely many $\mathrm{L}_1, \ldots, \mathrm{L}_r \in \mathcal{L}$ spanning the whole $\mathfrak{g}$, then automatically $\mathcal{L}$ is globally hypoelliptic. This is interesting even when G is a torus; see Example 8.4, which gives the general idea for constructing other examples using results already known in the literature.

Interesting applications are also yielded by Lemma 5.1, which allows us to replace, in our results, the system $\mathcal{L}$ by $\mathrm{Lie}\,\mathcal{L}$, the Lie algebra generated by $\mathcal{L}$. In Hörmander's condition, the tangent space $T_x\Omega$ must be generated by the vector fields $\mathrm{X}_0, \ldots, \mathrm{X}_r$, together with their higher order brackets, at each point x; in the construction of $\mathrm{Lie}\,\mathcal{L}$, however, we are allowed to take brackets of our vector fields evaluated at *different points* of the manifold. This shows a new and surprising sufficient condition for the global hypoellipticity that is much weaker than the Hörmander's. Incidentally, a big difference between the commutative and the noncommutative cases is revealed, since in the first one, there is no gain in considering $\mathrm{Lie}\,\mathcal{L}$. In Example 8.1 (a generalization of [1, Theorem 3]), we construct a class of operators that illustrates a phenomenon that can not occur in $G = \mathbb{T}^m$. Following these lines, Hörmander's condition will be more explored in Section 8.2. On the one hand, a finite type condition at a single point implies that the system $\mathcal{L}$ is globally hypoelliptic (Corollary 8.6). On the other hand, Example 8.1, with convenient choices of coefficients, yields operators that are globally hypoelliptic while the finite type condition fails to be true everywhere — actually, operators that are clearly not locally hypoelliptic.

Section 1 is intended to recollect a few basic results on this business — especially those aspects peculiar to Lie groups — and also settle the notation. In Section 2, we develop the basic machinery — a suitable substitute to partial Fourier series — that was used throughout the other sections. It is based on the spectral theory of the (partial) Laplace-Beltrami operator on G; although most of the results here are known, we decided to keep some of their proofs (or sketches) in the text, as we did not find some of them in the literature in the exact form employed.

The notation established in the first two sections allows us to state in Section 3 our main results in a concise way. Theorem 3.5 is our main result regarding sufficient conditions for global hypoellipticity, and we present its proof at the end of Section 6. Theorem 3.3 is the keystone to obtain a necessary condition for global hypoellipticity, and Section 7 is devoted to prove it. Section 8 has a series of examples and remarks aiming to put our work in perspective, especially when we consider known results in the torus, Hörmander's condition, and also a necessary condition based on Sussmann's orbits. We end applying the techniques here developed to prove in Section 9 broader versions of [3, Theorem 1.9] (Theorem 9.1) and of [1, Theorem 1] (Theorem 9.3).

1. Preliminaries

Let M be a compact, connected, smooth manifold, which, for simplicity, we further require to be orientable and, in fact, oriented. We endow it with a Riemannian metric, and we denote by $\mathrm{d}V$ its underlying volume form. The L^2 norms below are always taken with respect to this measure, which we assume without loss of generality (w.l.o.g.) to be normalized. Let $\mathrm{d} : \mathscr{C}^\infty(M;\mathbb{R}) \to \mathscr{C}^\infty(M;T^*M)$ be the exterior derivative and $\mathrm{d}^* : \mathscr{C}^\infty(M;T^*M) \to \mathscr{C}^\infty(M;\mathbb{R})$ its formal adjoint: The Laplace-Beltrami operator is then defined as the second-order differential operator

$$\Delta \doteq \mathrm{d}^*\mathrm{d} : \mathscr{C}^\infty(M;\mathbb{R}) \longrightarrow \mathscr{C}^\infty(M;\mathbb{R}).$$

Their action can be complexified by allowing all the objects involved to take values in $\mathbb{C}$.

We recall the main properties of Δ which will be of fundamental importance to us. It is an elliptic operator, and positive semidefinite, that is, $\langle \Delta f, f\rangle_{L^2(M)} \geq 0$ for all $f \in \mathscr{C}^\infty(M)$. We denote by $\sigma(\Delta) \subset \mathbb{R}_+$ its spectrum, that is, the set of all eigenvalues of Δ: This set is countably infinite, and for each $\lambda \in \sigma(\Delta)$, we denote by E_λ the eigenspace associated with λ, which is a finite dimensional vector space containing smooth functions only. These eigenspaces are pairwise orthogonal in $L^2(M)$, and E_0 is precisely the space of constant functions since M is connected. The Spectral Theorem tells us that if we endow each E_λ with the L^2 inner product then, as Hilbert spaces,

$$L^2(M) \cong \widehat{\bigoplus_{\lambda\in\sigma(\Delta)}} E_\lambda.$$

Moreover, the following consequence of Weyl's asymptotic formula [5, p. 155] holds:

$$\sum_{\lambda\in\sigma(\Delta)\setminus 0} (\dim E_\lambda)\lambda^{-2m} < \infty, \quad \text{where } m \doteq \dim\ M. \tag{1.1}$$

If for each $\lambda \in \sigma(\Delta)$, we denote by $\mathcal{F}_\lambda : L^2(M) \to E_\lambda$ the corresponding orthogonal projection, then every $f \in L^2(M)$ can be written as

$$f = \sum_{\lambda\in\sigma(\Delta)} \mathcal{F}_\lambda(f),$$

where convergence takes place in $L^2(M)$. In the same spirit, we may extend the projection maps $\mathcal{F}_\lambda$ to act on distributions and identify many spaces of (generalized) functions on M by analyzing the growth of their corresponding sequences of projections, in a Paley-Wiener-like fashion.

The space $\mathscr{C}^\infty(M)$ of all complex-valued smooth functions on M is naturally endowed with a locally convex topology (uniform convergence of all derivatives on compact coordinate sets). As our volume form $\mathrm{d}V$ allows us to identify the space of all smooth densities on M with $\mathscr{C}^\infty(M)$, by the same token, we may identify the topological dual of the latter with $\mathscr{D}'(M)$, the space of Schwartz distributions on M. The measure

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dV further allows us to embed all the classical spaces of functions in $\mathscr{D}'(M)$: We interpret each $f \in L^1(M)$ as a distribution on M by letting it act on a test function $\phi \in \mathscr{C}^\infty(M)$ as

$$\langle f,\phi\rangle \doteq \int_M f\phi \; \mathrm{d}V.$$

If $f \in \mathscr{D}'(M)$ for each $\lambda \in \sigma(\Delta)$, we have $f|_{E_\lambda} \in E_\lambda^*$, and we denote by $\mathcal{F}_\lambda(f)$ the unique element in E_λ that satisfies

$$\langle \mathcal{F}_\lambda(f),\phi\rangle_{L^2(M)} = \langle f,\overline{\phi}\rangle, \quad \forall \phi \in E_\lambda.$$

Concretely, if $\{\phi_i^\lambda \;:\; 1 \le i \le \dim E_\lambda\}$ is an orthonormal basis for E_λ, then

$$\mathcal{F}_\lambda(f) = \sum_{i=1}^{d_\lambda} \langle \mathcal{F}_\lambda(f),\phi_i^\lambda\rangle_{L^2(M)} \; \phi_i^\lambda = \sum_{i=1}^{d_\lambda} \langle f,\overline{\phi_i^\lambda}\rangle \; \phi_i^\lambda, \quad \forall f \in \mathscr{D}'(M),$$

where $d_\lambda \doteq \dim E_\lambda$; it coincides with the original definition of $\mathcal{F}_\lambda(f)$ when $f \in L^2(M)$. We denote by $\mathcal{F}(f)$ the sequence $(\mathcal{F}_\lambda(f))_{\lambda\in\sigma(\Delta)}$. The following result can be found, for example, in [2].

Proposition 1.1. *For a sequence $a = (a(\lambda))_{\lambda\in\sigma(\Delta)}$, where $a(\lambda) \in E_\lambda$ for all $\lambda \in \sigma(\Delta)$, the following characterizations hold:*

1. *$a = \mathcal{F}(f)$ for some $f \in \mathscr{C}^\infty(M)$ if and only if for every $s > 0$, there exists $C > 0$, such that*
$$\|a(\lambda)\|_{L^2(M)} \le C(1+\lambda)^{-s}, \quad \forall \lambda \in \sigma(\Delta).$$
2. *$a = \mathcal{F}(f)$ for some $f \in \mathscr{D}'(M)$ if and only if there exist $C,s > 0$, such that*
$$\|a(\lambda)\|_{L^2(M)} \le C(1+\lambda)^{s}, \quad \forall \lambda \in \sigma(\Delta).$$

1.1. Riemannian metrics on compact Lie groups

Let G be a compact and connected Lie group, whose dimension as a manifold we denote by m. We denote by $\mathfrak{g}$ the Lie algebra of all *real* vector fields on G that are left-invariant: This is a finite dimensional vector space, canonically isomorphic to T_eG — where $e \in G$ stands for the identity element.

Any basis $\mathrm{X}_1,\dots,\mathrm{X}_m \in \mathfrak{g}$ forms a global frame for TG, and if $\chi_1,\dots,\chi_m \in \mathfrak{g}^*$ is the corresponding dual basis — which we regard as left-invariant 1-forms on G — they form a global frame for T^*G. In particular, $\chi \doteq \chi_1 \wedge \cdots \wedge \chi_m$ is a nonvanishing left-invariant top-degree form on G, and it is easy to check that any other such form must be a multiple of χ: One often calls

$$\mathrm{d}V_G \doteq \Big(\int_G \chi\Big)^{-1} \chi$$

the *Haar volume form* associated with the orientation given by the frame $\mathrm{X}_1,\dots,\mathrm{X}_m$. Left-invariant Riemannian metrics on G are in one-to-one correspondence with inner

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products on $\mathfrak{g}$: Any such inner product, which we regard as an inner product on T_eG, can be pushed forward by the left-translation $L_x : G \to G$ (which is a diffeomorphism of G onto itself) to an inner product on T_xG for every $x \in G$, thus producing the desired left-invariant Riemannian metric. Now, if we fix an inner product $\langle \cdot,\cdot \rangle$ on $\mathfrak{g}$ and, as above, select $\mathrm{X}_1,\dots,\mathrm{X}_m$ an orthonormal basis for $\mathfrak{g}$, then χ is precisely the Riemannian volume form with respect to (w.r.t.) the left-invariant Riemannian metric $\langle \cdot,\cdot \rangle$ and compatible with the orientation of G given by $\mathrm{X}_1,\dots,\mathrm{X}_m$. In particular, the Riemannian volume form w.r.t. a left-invariant Riemannian metric is always left-invariant, hence a constant multiple of the Haar volume form. As such, with respect to such a metric, any left-invariant vector field $\mathrm{X} \in \mathfrak{g}$ is *(formally) skew-symmetric*, that is

$$\langle \mathrm{X}f,g \rangle_{L^2(G)} = -\langle f,\mathrm{X}g \rangle_{L^2(G)}, \quad \forall f,g \in \mathscr{C}^\infty(G).$$

Particularly relevant to what comes next are the so-called ad-*invariant metrics*: These are left-invariant Riemannian metrics $\langle \cdot,\cdot \rangle$ on G with the additional property that

$$\langle [\mathrm{X},\mathrm{Y}],\mathrm{Z} \rangle = -\langle \mathrm{Y},[\mathrm{X},\mathrm{Z}] \rangle, \quad \forall \mathrm{X},\mathrm{Y},\mathrm{Z} \in \mathfrak{g}. \tag{1.2}$$

Such metrics always exist since we are assuming G to be compact [16, Proposition 4.24]. The key point is that, in that case, if $\mathrm{X}_1,\dots,\mathrm{X}_m \in \mathfrak{g}$ is an orthonormal basis, then the Laplace-Beltrami operator Δ_G associated to $\langle \cdot,\cdot \rangle$ can be written as

$$\Delta_G = -\sum_{j=1}^{m} \mathrm{X}_j^2, \tag{1.3}$$

and, moreover, every left-invariant vector field on G commutes with Δ_G.

2. Partial Fourier projection maps on product manifolds

Let T,G be two compact, connected, smooth manifolds, orientable and oriented, and also carrying Riemannian metrics, just like M did in Section 1, and whose dimensions will be denoted by $n \doteq \dim T$ and $m \doteq \dim G$. Then their product enjoys the very same properties. Moreover, $T \times G$ carries the product metric. If we denote by $\mathrm{d}V$ (respectively, $\mathrm{d}V_T,\mathrm{d}V_G$) the Riemannian volume form of $T \times G$ (respectively, T,G) with respect to the metric introduced above, then one can prove the following version of Fubini's Theorem:

Proposition 2.1. *For every $f \in \mathscr{C}^\infty(T \times G)$, we have*

$$\int_{T\times G} f(t,x)\ \mathrm{d}V(t,x) = \int_T \Big(\int_G f(t,x)\ \mathrm{d}V_G(x) \Big) \mathrm{d}V_T(t).$$

Let Δ (respectively, Δ_T,Δ_G) be the Laplace-Beltrami operator on $T \times G$ (respectively, T,G) associated to the underlying metric(s) above: In the next statement, we summarize their most significant relationships. Notice that given any differential operator P on T (or on G), we will also denote its lift to $T \times G$ by P.

Proposition 2.2.

1. *$\Delta = \Delta_T + \Delta_G$ as differential operators on $T \times G$.*
2. *If for each $\mu \in \sigma(\Delta_T)$ (respectively, $\lambda \in \sigma(\Delta_G)$), we denote by $E^T_\mu \subset \mathscr{C}^\infty(T)$ (respectively, $E^G_\lambda \subset \mathscr{C}^\infty(G)$) the eigenspace of Δ_T (respectively, Δ_G) associated to μ (respectively, λ), and choose bases for them*

$$\{\psi_i^\mu \ : \ 1 \leq i \leq d_\mu^T\}, \quad \text{where } d_\mu^T \doteq \dim E_\mu^T,$$
$$\{\phi_j^\lambda \ : \ 1 \leq j \leq d_\lambda^G\}, \quad \text{where } d_\lambda^G \doteq \dim E_\lambda^G,$$

which are orthonormal w.r.t. the inner products inherited from $L^2(T)$,$L^2(G)$, respectively, then the set

$$\mathcal{S} \doteq \{\psi_i^\mu \otimes \phi_j^\lambda \ : \ 1 \leq i \leq d_\mu^T, \ 1 \leq j \leq d_\lambda^G, \ \mu \in \sigma(\Delta_T), \ \lambda \in \sigma(\Delta_G)\}$$

is a Hilbert basis for $L^2(T \times G)$.
3. *Every $\alpha \in \sigma(\Delta)$ is of the form $\alpha = \mu + \lambda$ for some $\mu \in \sigma(\Delta_T)$ and $\lambda \in \sigma(\Delta_G)$.*
4. *If for each $\alpha \in \mathbb{R}_+$, we define*

$$\mathcal{P}(\alpha) \doteq \{(\mu,\lambda) \in \sigma(\Delta_T) \times \sigma(\Delta_G) \ : \ \mu + \lambda = \alpha\},$$

then the eigenspace of Δ associated to $\alpha \in \sigma(\Delta)$ is precisely

$$E_\alpha = \bigoplus_{(\mu,\lambda) \in \mathcal{P}(\alpha)} E_\mu^T \otimes E_\lambda^G$$

and an orthonormal basis for this space w.r.t. the $L^2(T \times G)$ inner product is

$$\{\psi_i^\mu \otimes \phi_j^\lambda \ : \ 1 \leq i \leq d_\mu^T, \ 1 \leq j \leq d_\lambda^G, \ (\mu,\lambda) \in \mathcal{P}(\alpha)\}.$$

Remark 2.3. For $\alpha \in \mathbb{R}_+$, the set $\mathcal{P}(\alpha)$ may contain more than one pair, that is, there may exist distinct $(\mu,\lambda),(\mu',\lambda') \in \sigma(\Delta_T) \times \sigma(\Delta_G)$ for which $\mu + \lambda = \mu' + \lambda'$. However, such a set is necessarily finite, since both $\sigma(\Delta_T)$ and $\sigma(\Delta_G)$ are discrete and unbounded.

Now, let $f \in \mathscr{C}^\infty(T \times G)$ and, given $t \in T$, we once more regard $f(t,\cdot)$ as a smooth function on G, for which we consider its orthogonal expansion

$$f(t,\cdot) = \sum_{\lambda \in \sigma(\Delta_G)} \mathcal{F}_\lambda^G(f(t,\cdot)),$$

where $\mathcal{F}_\lambda^G(f(t,\cdot)) \in E_\lambda^G$ can be written, in terms of our previously chosen basis, as

$$\mathcal{F}_\lambda^G(f(t,\cdot)) = \sum_{j=1}^{d_\lambda^G} \langle f(t,\cdot), \phi_j^\lambda \rangle_{L^2(G)} \phi_j^\lambda = \sum_{j=1}^{d_\lambda^G} \Big(\int_G f(t,x) \overline{\phi_j^\lambda(x)} \mathrm{d}V_G(x) \Big) \phi_j^\lambda. \tag{2.1}$$

Allowing now t to vary in T, we see at once that for each given $\lambda \in \sigma(\Delta_G)$, the map

$$t \in T \longmapsto \mathcal{F}_\lambda^G(f(t,\cdot)) \in E_\lambda^G$$

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is smooth, hence an element of $\mathscr{C}^\infty(T;E^G_\lambda) \cong \mathscr{C}^\infty(T) \otimes E^G_\lambda$, which we denote by $\mathcal{F}^G_\lambda(f)$. We can then consider the E^G_λ-valued orthogonal expansion w.r.t. Δ_T of $\mathcal{F}^G_\lambda(f) \in \mathscr{C}^\infty(T;E^G_\lambda)$: given $\mu \in \sigma(\Delta_T)$, we have

$$\mathcal{F}^T_\mu \mathcal{F}^G_\lambda(f) = \sum_{j=1}^{d^G_\lambda} \mathcal{F}^T_\mu \Big(\int_G f(\cdot,x)\overline{\phi^\lambda_j(x)}\mathrm{d}V_G(x)\Big) \otimes \phi^\lambda_j = \sum_{i=1}^{d^T_\mu}\sum_{j=1}^{d^G_\lambda} \Big\langle f,\psi^\mu_i \otimes \phi^\lambda_j \Big\rangle_{L^2(T\times G)} \psi^\mu_i \otimes \phi^\lambda_j$$

which is an element of $E^T_\mu \otimes E^G_\lambda$. By Proposition 2.2(4), this is nothing but a portion of $\mathcal{F}_\alpha(f)$, and we actually conclude that

$$\mathcal{F}_\alpha(f) = \sum_{(\mu,\lambda)\in\mathcal{P}(\alpha)} \mathcal{F}^T_\mu \mathcal{F}^G_\lambda(f), \quad \forall \alpha \in \sigma(\Delta). \tag{2.2}$$

On time, we notice that for every $\lambda \in \sigma(\Delta_G)$, we have

$$\mathscr{C}^\infty(T;E^G_\lambda) = \{f \in \mathscr{C}^\infty(T\times G) \ : \ \Delta_G f = \lambda f\} \tag{2.3}$$

— as one easily sees by analyzing the orthogonal expansion of any $f \in \mathscr{C}^\infty(T\times G)$ w.r.t. our Hilbert basis $\mathcal{S}$ — and that $\mathcal{F}^G_\lambda : \mathscr{C}^\infty(T\times G) \to \mathscr{C}^\infty(T;E^G_\lambda)$ is a projection. Indeed, given $f \in \mathscr{C}^\infty(T\times G)$, it follows from (2.1) that $\mathcal{F}^G_\lambda(f)$ is characterized as the unique element in $\mathscr{C}^\infty(T;E^G_\lambda)$ with the property that

$$\langle \mathcal{F}^G_\lambda(f),\psi\rangle_{L^2(T\times G)} = \langle f,\psi\rangle_{L^2(T\times G)}, \quad \forall \psi \in \mathscr{C}^\infty(T;E^G_\lambda) \tag{2.4}$$

which can be easily checked by expanding any such ψ in terms of an orthonormal basis. It follows at once that $\mathcal{F}^G_\lambda : \mathscr{C}^\infty(T\times G) \to \mathscr{C}^\infty(T;E^G_\lambda)$ acts as the identity on $\mathscr{C}^\infty(T;E^G_\lambda)$. In order to extend the definitions above to distributions $f \in \mathscr{D}'(T\times G)$, given $\lambda \in \sigma(\Delta_G)$, we expect to construct an object $\mathcal{F}^G_\lambda(f) \in \mathscr{D}'(T;E^G_\lambda)$. First of all, notice that we may identify $(E^G_\lambda)^*$ with E^G_λ itself by means of the anti-Riesz isomorphism $\phi \in E^G_\lambda \mapsto \langle\cdot,\bar{\phi}\rangle_{L^2(G)} \in (E^G_\lambda)^*$, for which $\{\overline{\phi^\lambda_j} \ : \ 1 \le j \le d^G_\lambda\}$ is the corresponding dual basis. Thus, an element $g \in \mathscr{C}^\infty(T;(E^G_\lambda)^*)$ can be written uniquely as

$$g = \sum_{j=1}^{d^G_\lambda} g_j \otimes \overline{\phi^\lambda_j}, \quad g_j \in \mathscr{C}^\infty(T).$$

Note that when $f \in \mathscr{C}^\infty(T\times G)$, we have seen (2.4) that we can apply $\mathcal{F}^G_\lambda(f)$, as an element of $\mathscr{D}'(T;E^G_\lambda)$, to $g \in \mathscr{C}^\infty(T;(E^G_\lambda)^*)$ and obtain

$$\langle \mathcal{F}^G_\lambda(f),g\rangle = \sum_{j=1}^{d^G_\lambda} \Big(\int_T \int_G f(t,x)g_j(t)\overline{\phi^\lambda_j(x)}\mathrm{d}V_G(x)\mathrm{d}V_T(t)\Big) \langle \phi^\lambda_j,\phi^\lambda_j\rangle_{L^2(G)} = \langle f,g\rangle.$$

Now, for $f \in \mathscr{D}'(T\times G)$, its projection $\mathcal{F}^G_\lambda(f) \in \mathscr{D}'(T;E^G_\lambda)$ is also written uniquely as

$$\mathcal{F}^G_\lambda(f) = \sum_{j=1}^{d^G_\lambda} F_j \otimes \phi^\lambda_j, \quad F_j \in \mathscr{D}'(T),$$

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where, as one can now easily guess,

$$\langle F_j, \psi \rangle \doteq \langle f, \psi \otimes \overline{\phi_j^\lambda} \rangle, \quad \forall \psi \in \mathscr{C}^\infty(T).$$

We have thus defined a linear map $\mathcal{F}_\lambda^G : \mathscr{D}'(T \times G) \to \mathscr{D}'(T; E_\lambda^G)$ which is essentially the transpose of the inclusion map $\mathscr{C}^\infty(T; E_\lambda^G) \hookrightarrow \mathscr{C}^\infty(T \times G)$. We can now characterize smoothness in terms of the double partial Fourier maps.

Proposition 2.4. *A distribution $f \in \mathscr{D}'(T \times G)$ is smooth if and only if for every $s > 0$, there exists $C > 0$, such that*

$$\|\mathcal{F}_\mu^T \mathcal{F}_\lambda^G(f)\|_{L^2(T \times G)} \leq C(1 + \mu + \lambda)^{-s}, \quad \forall (\mu, \lambda) \in \sigma(\Delta_T) \times \sigma(\Delta_G).$$

The next two corollaries of Proposition 2.4 are fundamental to our approach later on. Before we state (and prove) them, we will need the following remark.

Remark 2.5. For $f, g \in \mathscr{C}^\infty(T; E_\lambda^G)$ given by

$$f = \sum_{i=1}^{d_\lambda^G} f_i \otimes \phi_i^\lambda, \ g = \sum_{i'=1}^{d_\lambda^G} g_{i'} \otimes \phi_{i'}^\lambda, \quad f_i, g_{i'} \in \mathscr{C}^\infty(T),$$

we have by Proposition 2.1

$$\langle f, g \rangle_{L^2(T \times G)} = \int_{T \times G} \sum_{i,i'=1}^{d_\lambda^G} f_i(t) \phi_i^\lambda(x) \overline{g_{i'}(t) \phi_{i'}^\lambda(x)} \mathrm{d}V(t,x) = \sum_{i=1}^{d_\lambda^G} \langle f_i, g_i \rangle_{L^2(T)}.$$

Moreover, we have

$$\mathcal{F}_\mu^T(f) = \sum_{i=1}^{d_\lambda^G} \mathcal{F}_\mu^T(f_i) \otimes \phi_i^\lambda, \quad \forall \mu \in \sigma(\Delta_T),$$

hence

$$\|f\|_{L^2(T \times G)}^2 = \sum_{i=1}^{d_\lambda^G} \|f_i\|_{L^2(T)}^2 = \sum_{i=1}^{d_\lambda^G} \sum_{\mu \in \sigma(\Delta_T)} \|\mathcal{F}_\mu^T(f_i)\|_{L^2(T)}^2 = \sum_{\mu \in \sigma(\Delta_T)} \|\mathcal{F}_\mu^T(f)\|_{L^2(T \times G)}^2.$$

Corollary 2.6. *If $f \in \mathscr{C}^\infty(T \times G)$, then for every $s > 0$, there exists $C > 0$, such that*

$$\|\mathcal{F}_\lambda^G(f)\|_{L^2(T \times G)} \leq C(1 + \lambda)^{-s}, \quad \forall \lambda \in \sigma(\Delta_G). \tag{2.5}$$

Proof. By the computations done in Remark 2.5, we have

$$\|\mathcal{F}_\lambda^G(f)\|_{L^2(T \times G)}^2 = \sum_{\mu \in \sigma(\Delta_T)} \|\mathcal{F}_\mu^T \mathcal{F}_\lambda^G(f)\|_{L^2(T \times G)}^2.$$

By Proposition 2.4, for each $s > 0$, there exists $C > 0$, such that

$$\|\mathcal{F}_\lambda^G(f)\|_{L^2(T \times G)}^2 \leq \sum_{\mu \in \sigma(\Delta_T)} C^2(1 + \mu + \lambda)^{-2s-2n} \leq C^2(1 + \lambda)^{-2s} \sum_{\mu \in \sigma(\Delta_T)} (1 + \mu)^{-2n},$$

where $n = \dim T$ and the last series converges thanks to Weyl's formula (1.1) for Δ_T. $\square$

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Corollary 2.7. *If $f \in \mathscr{D}'(T \times G)$ is such that*

1. *for every $s > 0$, there exists $C > 0$, such that (2.5) holds and*
2. *for every $s' > 0$, there exist $C' > 0$ and $\theta \in (0,1)$, such that*

$$\|\mathcal{F}_\mu^T \mathcal{F}_\lambda^G(f)\|_{L^2(T\times G)} \leq C'(1+\mu+\lambda)^{-s'}, \quad \forall(\mu,\lambda) \in \Lambda_\theta, \tag{2.6}$$

where

$$\Lambda_\theta \doteq \{(\mu,\lambda) \in \sigma(\Delta_T) \times \sigma(\Delta_G) \ : \ (1+\lambda) \leq (1+\mu)^\theta\}. \tag{2.7}$$

Then $f \in \mathscr{C}^\infty(T \times G)$.

Proof. Let $\Lambda_\theta^c \subset \sigma(\Delta_T) \times \sigma(\Delta_G)$ denote the complement of Λ_θ. For $(\mu,\lambda) \in \Lambda_\theta^c$, we have

$$1+\mu+\lambda < (1+\lambda)^{\frac{1}{\theta}} + \lambda \leq (1+\lambda)^{1+\frac{1}{\theta}} \leq (1+\lambda)^{\frac{2}{\theta}}$$

since $1/\theta > 1$. Therefore, given $s' > 0$, we define $s \doteq 2\theta^{-1}s'$, hence for $(\mu,\lambda) \in \Lambda_\theta^c$, we have

$$(1+\lambda)^{-s} \leq (1+\mu+\lambda)^{-\frac{\theta s}{2}} = (1+\mu+\lambda)^{-s'}.$$

Let then $C, C' > 0$ be such that (2.5) and (2.6) hold, hence

$$\|\mathcal{F}_\mu^T \mathcal{F}_\lambda^G(f)\|_{L^2(T\times G)} \leq \begin{cases} C(1+\mu+\lambda)^{-s'}, & \text{in } \Lambda_\theta^c, \\ C'(1+\mu+\lambda)^{-s'}, & \text{in } \Lambda_\theta. \end{cases}$$

Combining both estimates, it follows from Proposition 2.4 that $f \in \mathscr{C}^\infty(T \times G)$. □

Before we end this section, we will prove a result about LPDOs which commute with one of the partial Laplace-Beltrami operators on $T \times G$: such LPDOs will also commute with the partial Fourier projection map associated to the corresponding factor. This is a key property that all of our operators of interest in the forthcoming sections will enjoy.

Proposition 2.8. *Let P be an LPDO in $T \times G$ which commutes with Δ_G. If $u \in \mathscr{D}'(T \times G)$, then $\mathcal{F}_\lambda^G(Pu) = P\mathcal{F}_\lambda^G(u)$ for every $\lambda \in \sigma(\Delta_G)$.*

Proof. We will be content to prove the assertion when u is smooth. First, notice that P maps $\mathscr{C}^\infty(T; E_\lambda^G)$ to itself: indeed, if $f \in \mathscr{C}^\infty(T; E_\lambda^G)$, then by (2.3)

$$\Delta_G f = \lambda f \Longrightarrow \Delta_G(Pf) = P(\Delta_G f) = \lambda(Pf)$$

from which we conclude that $Pf \in \mathscr{C}^\infty(T; E_\lambda^G)$. We claim that P^* — the formal adjoint of P — also commutes with Δ_G: For $f, g \in \mathscr{C}^\infty(T \times G)$, we have $\langle \Delta_G f, g\rangle_{L^2(T\times G)} = \langle f, \Delta_G g\rangle_{L^2(T\times G)}$, hence

$$\langle P^*\Delta_G f, g\rangle_{L^2(T\times G)} = \langle f, \Delta_G P g\rangle_{L^2(T\times G)} = \langle f, P\Delta_G g\rangle_{L^2(T\times G)} = \langle \Delta_G P^* f, g\rangle_{L^2(T\times G)},$$

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and since this holds for all $f,g \in \mathscr{C}^\infty(T \times G)$, our claim follows. In particular, P^* also preserves $\mathscr{C}^\infty(T;E_\lambda^G)$ for each $\lambda \in \sigma(\Delta_G)$. Now, for $u \in \mathscr{C}^\infty(T \times G)$, we have, for all $\psi \in \mathscr{C}^\infty(T;E_\lambda^G)$,

$$\langle \mathcal{F}_\lambda^G(Pu),\psi\rangle_{L^2(T\times G)} = \langle Pu,\psi\rangle_{L^2(T\times G)} = \langle u,P^*\psi\rangle_{L^2(T\times G)} = \langle \mathcal{F}_\lambda^G(u),P^*\psi\rangle_{L^2(T\times G)}$$

thanks to (2.4): notice that in the last equality, we used that $P^*\psi \in \mathscr{C}^\infty(T;E_\lambda^G)$. After a final transposition, we conclude that

$$\langle \mathcal{F}_\lambda^G(Pu),\psi\rangle_{L^2(T\times G)} = \langle P\mathcal{F}_\lambda^G(u),\psi\rangle_{L^2(T\times G)}, \quad \forall \psi \in \mathscr{C}^\infty(T;E_\lambda^G),$$

which yields our conclusion, since both $\mathcal{F}_\lambda^G(Pu)$ and $P\mathcal{F}_\lambda^G(u)$ belong to $\mathscr{C}^\infty(T;E_\lambda^G)$. □

3. A class of sublaplacians on product manifolds

From now on, we will assume some extra structure in the environment postulated in the previous sections: namely, G will be a Lie group (with $\dim G = m$), while T will remain a smooth manifold (with $\dim T = n$), both of them compact, connected, and oriented. We impose no conditions on the Riemannian metric on T but will require the one on G to be ad-invariant (1.2). We denote by $\mathfrak{g}$ the Lie algebra of G.

Let $\mathfrak{a} : T \to \mathfrak{g}$ be a smooth map. If $\mathrm{X}_1,\ldots,\mathrm{X}_m$ is a basis of $\mathfrak{g}$, then

$$\mathfrak{a}(t) = \sum_{j=1}^m a_j(t)\mathrm{X}_j, \quad t \in T,$$

where $a_1,\ldots,a_m \in \mathscr{C}^\infty(T;\mathbb{R})$ are uniquely determined. We thus regard $\mathfrak{a}$ as a first-order LPDO on $T \times G$, which we may sometimes write $\mathfrak{a}(t,\mathrm{X})$ when we want to stress this point of view. Notice that

$$\mathfrak{a}(t,\mathrm{X})(\psi \otimes \phi) = \sum_{j=1}^m (a_j\psi) \otimes (\mathrm{X}_j\phi), \quad \forall \psi \in \mathscr{D}'(T),\ \phi \in \mathscr{D}'(G),$$

hence, in particular, $\mathfrak{a}(t,\mathrm{X})(\psi \otimes 1_G) = 0$ for every $\psi \in \mathscr{D}'(T)$.

We introduce the class of LPDOs on $T \times G$ which is the main theme of this work. Define

$$P \doteq Q - \sum_{\ell=1}^N \Big(\mathfrak{a}_\ell(t,\mathrm{X}) + \mathrm{W}_\ell\Big)^2, \tag{3.1}$$

where $\mathfrak{a}_1,\ldots,\mathfrak{a}_N : T \to \mathfrak{g}$ are smooth maps, $\mathrm{W}_1,\ldots,\mathrm{W}_N$ are real, smooth vector fields on T, and Q is a real, positive semidefinite LPDO on T — meaning that $\langle Q\psi,\psi\rangle_{L^2(T)} \geq 0$ for every $\psi \in \mathscr{C}^\infty(T)$ — which is a wildcard in our model: We will slowly add hypotheses to it, but for now, we will assume that

$$\tilde{P} \doteq Q - \sum_{\ell=1}^N \mathrm{W}_\ell^2 \tag{3.2}$$

is a second-order LPDO on T that kills constants (i.e., has no zero order term). The main examples we will explore afterward are $Q = \Delta_T$ and $Q = 0$. Our aim in this work

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is to study necessary and sufficient conditions for an operator P as above to be *globally hypoelliptic*, or (GH) for short, in $T \times G$:

$$\forall u \in \mathscr{D}'(T \times G),\ Pu \in \mathscr{C}^\infty(T \times G) \Longrightarrow u \in \mathscr{C}^\infty(T \times G).$$

Since $\mathfrak{a}_1, \dots, \mathfrak{a}_N : T \to \mathfrak{g}$ are smooth, for each $\ell \in \{1, \dots, N\}$, we may write

$$\mathfrak{a}_\ell(t) = \sum_{j=1}^{m} a_{\ell j}(t) \mathrm{X}_j, \quad t \in T, \tag{3.3}$$

with $a_{\ell 1}, \dots, a_{\ell m} \in \mathscr{C}^\infty(T; \mathbb{R})$. Then, given $\psi \in \mathscr{D}'(T)$ and $\phi \in \mathscr{D}'(G)$, we have, unwinding the square in the definition of P,

$$P(\psi \otimes \phi) = (\tilde{P}\psi) \otimes \phi - \sum_{\ell=1}^{N} \Big(\sum_{j,j'=1}^{m} (a_{\ell j'} a_{\ell j} \psi) \otimes (\mathrm{X}_{j'} \mathrm{X}_j \phi) + \sum_{j=1}^{m} \Big((2 a_{\ell j} \mathrm{W}_\ell + \mathrm{W}_\ell a_{\ell j}) \psi \Big) \otimes (\mathrm{X}_j \phi) \Big). \tag{3.4}$$

Roughly speaking, P has "separated variables" with "constant coefficients" on G and hence behaves nicely under partial the Fourier projection maps on that factor. Rigorously, operators such as Q and W_ℓ commute with Δ_G, as they act on independent variables, but so does $\mathfrak{a}_\ell(t, \mathrm{X})$ since each X_j commutes with Δ_G (as pointed out at the end of Section 1.1). Thus, P also commutes with Δ_G; to all of them, Proposition 2.8 applies.

On time, we point out the following energy identity, which will be fundamental later on. Its proof is purely computational, and we leave it to the reader. Recall that a real vector field W on T is *skew-symmetric* (also often called *skew-adjoint*, or *divergence free*) if

$$\langle \mathrm{W} f, g \rangle_{L^2(T)} = -\langle f, \mathrm{W} g \rangle_{L^2(T)}, \quad \forall f, g \in \mathscr{C}^\infty(T).$$

Lemma 3.1. *Let P be as in (3.1). If we further assume that $\mathrm{W}_1, \dots, \mathrm{W}_N$ are skew-symmetric on T, then for each $\lambda \in \sigma(\Delta_G)$, we have*

$$\langle P\psi, \psi \rangle_{L^2(T \times G)} = \langle Q\psi, \psi \rangle_{L^2(T \times G)} + \sum_{\ell=1}^{N} \|\mathrm{Y}_\ell \psi\|^2_{L^2(T \times G)}, \quad \forall \psi \in \mathscr{C}^\infty(T; E^G_\lambda),$$

where $\mathrm{Y}_\ell \doteq \mathfrak{a}_\ell(t, \mathrm{X}) + \mathrm{W}_\ell$ for $\ell \in \{1, \dots, N\}$.

3.1. Main results

We start by discussing necessary conditions for global hypoellipticity of P in (3.1).

Proposition 3.2. *If P is* (GH) *in $T \times G$, then $\tilde{P}$ is* (GH) *in T.*

Proof. Let $u \in \mathscr{D}'(T)$ be such that $\tilde{P}u \in \mathscr{C}^\infty(T)$. Then, by (3.4), we have that $P(u \otimes 1_G) = (\tilde{P}u) \otimes 1_G$ is smooth on $T \times G$, hence, by hypothesis, $u \otimes 1_G \in \mathscr{C}^\infty(T \times G)$ — which can only happen if $u \in \mathscr{C}^\infty(T)$. ☐

Motivated by this remark, we shall be mostly concerned with the case when $\tilde{P}$ is an *elliptic* operator in T, an assumption that will allow us to make use of microlocal methods. Now, we come to our second necessary condition for global hypoellipticity of P.

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Theorem 3.3. *If P is* (GH) *in $T \times G$, then the following regularity condition holds:*

$$\forall u \in \mathscr{D}'(G),\ \mathfrak{a}_\ell(t,\mathrm{X})(1_T \otimes u) \in \mathscr{C}^\infty(T \times G)\ \forall \ell \in \{1,\ldots,N\} \Longrightarrow u \in \mathscr{C}^\infty(G). \tag{3.5}$$

Its proof is not as simple: We postpone it to Section 7. Under additional conditions, we will see that the necessary conditions in Proposition 3.2 and Theorem 3.3 are also sufficient. But first, let us restate (3.5) in terms of a system of left-invariant vector fields on G. To do so, we must recall the notion of global hypoellipticity for such systems:

Definition 3.4. Let M be a smooth, compact manifold as in Section 1. A family $\mathcal{L}$ of smooth vector fields on M is said to be *globally hypoelliptic* in M if for every $u \in \mathscr{D}'(M)$, we have

$$\mathrm{L}u \in \mathscr{C}^\infty(M),\ \forall \mathrm{L} \in \mathcal{L} \Longrightarrow u \in \mathscr{C}^\infty(M).$$

From now on, we denote by $\mathcal{L}$ the system of vector fields on G defined as follows:

$$\mathcal{L} \doteq \bigcup_{\ell=1}^{N} \operatorname{ran} \mathfrak{a}_\ell \subset \mathfrak{g}. \tag{3.6}$$

Thus, a left-invariant vector field L belongs to $\mathcal{L}$ if and only if there exist $\ell \in \{1,\ldots,N\}$ and $t \in T$, such that $\mathrm{L} = \mathfrak{a}_\ell(t)$. Moreover, for each $\ell \in \{1,\ldots,N\}$, we let

$$\mathcal{L}_\ell \doteq \operatorname{span}_{\mathbb{R}} \operatorname{ran} \mathfrak{a}_\ell \subset \mathfrak{g}. \tag{3.7}$$

We will prove in Proposition 7.2 that condition (3.5) is equivalent to ask that $\mathcal{L}$ is (GH) in G. In Section 8.1, we explore in detail such condition when G is a torus and equate it with the notion of nonsimultaneous approximability of a collection of vectors, a Diophantine condition already known to be connected with global hypoellipticity of operators like (3.1) when both T and G are tori [3].

When $Q = \Delta_T$, we can state our sufficiency result as follows:

Theorem 3.5. *Let*

$$P = \Delta_T - \sum_{\ell=1}^{N} \Big(\mathfrak{a}_\ell(t,\mathrm{X}) + \mathrm{W}_\ell\Big)^2, \tag{3.8}$$

and suppose that $\mathrm{W}_1,\ldots,\mathrm{W}_N$ are skew-symmetric real vector fields in T. Assume, moreover, that:

1. *For each given $\ell \in \{1,\ldots,N\}$, we have that $\mathfrak{a}_\ell(t_1),\mathfrak{a}_\ell(t_2)$ commute as vector fields in G, for any $t_1,t_2 \in T$. In other words, each $\mathcal{L}_\ell \subset \mathfrak{g}$ as defined in (3.7) spans a commutative Lie subalgebra.*
2. *The system $\mathcal{L} \subset \mathfrak{g}$ in* (3.6) *is* (GH) *in G.*

Then P is (GH) *in $T \times G$. Furthermore, if R is an LPDO in $T \times G$ of the form*

$$R \doteq -\sum_{\kappa=1}^{M} \Big(\mathfrak{b}_\kappa(t,\mathrm{X}) + \mathrm{V}_\kappa\Big)^2, \tag{3.9}$$

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where $\mathrm{V}_1,\dots,\mathrm{V}_M$ *are skew-symmetric real vector fields in* T *and* $\mathfrak{b}_1,\dots,\mathfrak{b}_M \in \mathscr{C}^\infty(T;\mathfrak{g})$ *do not necessarily satisfy the commutativity condition above, then* $P_0 \doteq P + R$ *is also* (GH) *in* $T \times G$.

Note that, for any $\ell \in \{1,\dots,N\}$, we can assume that $t \in T \mapsto \mathfrak{a}_\ell(t) \in \mathfrak{g}$ is not identically zero. For operators P as in (3.8), we have that

$$\tilde{P} = \Delta_T - \sum_{\ell=1}^{N} \mathrm{W}_\ell^2$$

is elliptic in T. Additionally, note that if $G = \mathbb{T}^m$, then $\mathcal{L}_\ell$ is always commutative, so Proposition 3.2 and Theorems 3.3 and 3.5 together yield Theorem 1, hence our result generalizes [3, Theorem 1.5]. Let us also point out that we were able to prove global hypoellipticity of P (3.1) in Theorem 9.1 and in Theorem 9.3 when Q is any positive semidefinite operator in T, where, on the other hand, we impose more restrictive assumptions on the vector fields $\mathfrak{a}_\ell(t,\mathrm{X})$, for $\ell \in \{1,\dots,N\}$.

4. Consequences of the ellipticity of $\tilde{P}$ on the Fourier projections

Let us evaluate the principal symbol of $\tilde{P}$ (3.2) at $(t_0,\tau_0) \in T^*T \setminus 0$ by taking any $\psi \in \mathscr{C}^\infty(T;\mathbb{R})$, such that $\mathrm{d}_T\psi(t_0) = \tau_0$: We have

$$\tilde{P}_2(t_0,\tau_0) = \lim_{\rho\to\infty} \rho^{-2} e^{-i\rho\psi}\Big(Q(e^{i\rho\psi}) - \sum_{\ell=1}^{N} \mathrm{W}_\ell^2(e^{i\rho\psi})\Big)\Big|_{t_0} = Q_2(t_0,\tau_0) + \sum_{\ell=1}^{N} (\mathrm{W}_\ell\psi)(t_0)^2.$$

In particular, if Q_2 is a nonnegative function and the system of vector fields $\mathrm{W}_1,\dots,\mathrm{W}_N$ is elliptic in T, then certainly $\tilde{P}$ is elliptic. If, on the other hand, $Q = \Delta_T$, then Q_2 may be evaluated by means of the local expression of the Laplace-Beltrami operator: In that case, $\tilde{P}$ is automatically elliptic — no assumptions needed on $\mathrm{W}_1,\dots,\mathrm{W}_N$.

Lemma 4.1. *Suppose that* $\tilde{P}$ *is elliptic and that* $u \in \mathscr{D}'(T \times G)$ *is such that* $Pu \in \mathscr{C}^\infty(T \times G)$. *Then for every* $\phi \in \mathscr{C}^\infty(G)$, *we have that* $\tilde{u}(\phi) \doteq \langle u, \cdot \otimes \phi\rangle \in \mathscr{C}^\infty(T)$.

Proof. First, we will show that

$$\{(t,\tau) \in T^*T \setminus 0 \ : \ (t,\tau,x,0) \in \mathrm{Char}(P) \text{ for some } x \in G\} = \emptyset \tag{4.1}$$

which is a direct consequence of the ellipticity of $\tilde{P}$. Indeed, we compute the principal symbol of P at $(t,\tau,x,0) \in T_t^*T \times T_x^*G \cong T^*_{(t,x)}(T \times G)$ by taking $\psi \in \mathscr{C}^\infty(T;\mathbb{R})$, such that $\mathrm{d}_T\psi(t) = \tau$, hence $f \doteq \psi \otimes 1_G$ satisfies $\mathrm{d}f(t,x) = (\tau,0)$ and

$$P_2(t,\tau,x,0) = \lim_{\rho\to\infty} \rho^{-2} e^{-i\rho f} P(e^{i\rho f})|_{(t,x)} = \lim_{\rho\to\infty} \rho^{-2} e^{-i\rho\psi} \tilde{P}(e^{i\rho\psi})|_t = \tilde{P}_2(t,\tau)$$

so (4.1) follows since $\tilde{P}$ is elliptic. Now, let $\phi \in \mathscr{C}^\infty(G)$: at first, we only know that $\tilde{u}(\phi) \in \mathscr{D}'(T)$. Using partitions of unity, we may assume w.l.o.g. that $\operatorname{supp}\phi$ is contained

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in a coordinate open set $U \subset G$, where we apply [15, Theorem 2.5.12] to conclude that

$$(t,\tau) \in \mathrm{WF}(\tilde{u}(\phi)) \Longrightarrow (t,\tau,x,0) \in \mathrm{WF}(u) \text{ for some } x \in U,$$

which is further contained in $\mathrm{Char}(P)$ since Pu is everywhere smooth. □

The following is an easy consequence of Lemma 4.1.

Corollary 4.2. *Suppose that $\tilde{P}$ is elliptic and that $u \in \mathscr{D}'(T \times G)$ is such that $Pu \in \mathscr{C}^\infty(T \times G)$. Then, $\mathcal{F}^G_\lambda(u) \in \mathscr{C}^\infty(T; E^G_\lambda)$ for every $\lambda \in \sigma(\Delta_G)$.*

For the next lemma, recall that for M, a compact manifold as in Section 1, the topology of $\mathscr{C}^\infty(M)$ can be given by the system of (semi)norms, defined, for $f \in \mathscr{C}^\infty(M)$, by

$$\|f\|_{\mathscr{H}^s(M)} \doteq \|(I+\Delta)^s f\|_{L^2(M)}, \quad s \in \mathbb{Z}_+.$$

We use this fact below with $M = T, G$ and $\Delta = \Delta_T, \Delta_G$, respectively.

Lemma 4.3. *Suppose that $u \in \mathscr{D}'(T \times G)$ is such that $\tilde{u}(\phi) = \langle u, \cdot \otimes \phi \rangle \in \mathscr{C}^\infty(T)$ for every $\phi \in \mathscr{C}^\infty(G)$. Then for each $s > 0$ there exist $C > 0$ and $\theta \in (0,1)$ such that*

$$\|\mathcal{F}^T_\mu \mathcal{F}^G_\lambda(u)\|_{L^2(T\times G)} \leq C(1+\mu+\lambda)^{-s}, \quad \forall (\mu,\lambda) \in \Lambda_\theta,$$

where Λ_θ is defined in (2.7).

Proof. The hypothesis means that the range of the continuous linear map $\tilde{u} : \mathscr{C}^\infty(G) \to \mathscr{D}'(T)$ actually lies in $\mathscr{C}^\infty(T)$. This yields a new linear map $\tilde{u} : \mathscr{C}^\infty(G) \to \mathscr{C}^\infty(T)$ which is continuous by the Closed Graph Theorem: it follows that for each $s \in \mathbb{Z}_+$, there exist $C > 0$ and $s' \in \mathbb{Z}_+$, such that

$$\|\tilde{u}(\phi)\|_{\mathscr{H}^s(T)} \leq C\|\phi\|_{\mathscr{H}^{s'}(G)}, \quad \forall \phi \in \mathscr{C}^\infty(G).$$

Taking $\phi = \overline{\phi^\lambda_j}$ — one of our orthonormal basis elements of E^G_λ — we obtain

$$\|\tilde{u}(\overline{\phi^\lambda_j})\|_{\mathscr{H}^s(T)} \leq C\|\overline{\phi^\lambda_j}\|_{\mathscr{H}^{s'}(G)} = C(1+\lambda)^{s'},$$

while on the other hand

$$\|\tilde{u}(\overline{\phi^\lambda_j})\|^2_{\mathscr{H}^s(T)} = \sum_{\mu\in\sigma(\Delta_T)} (1+\mu)^{2s} \|\mathcal{F}^T_\mu[\tilde{u}(\overline{\phi^\lambda_j})]\|^2_{L^2(T)} = \sum_{\mu\in\sigma(\Delta_T)} (1+\mu)^{2s} \sum_{i=1}^{d^T_\mu} |\langle u, \overline{\psi^\mu_i \otimes \phi^\lambda_j}\rangle|^2,$$

hence

$$\sum_{j=1}^{d^G_\lambda} \|\tilde{u}(\overline{\phi^\lambda_j})\|^2_{\mathscr{H}^s(T)} = \sum_{\mu\in\sigma(\Delta_T)} (1+\mu)^{2s} \sum_{i=1}^{d^T_\mu}\sum_{j=1}^{d^G_\lambda} |\langle u, \overline{\psi^\mu_i \otimes \phi^\lambda_j}\rangle|^2 = \sum_{\mu\in\sigma(\Delta_T)} (1+\mu)^{2s} \|\mathcal{F}^T_\mu \mathcal{F}^G_\lambda(u)\|^2_{L^2(T\times G)}$$

from which we conclude that

$$(1+\mu)^{2s}\|\mathcal{F}^T_\mu \mathcal{F}^G_\lambda(u)\|^2_{L^2(T\times G)} \leq \sum_{j=1}^{d^G_\lambda} \|\tilde{u}(\overline{\phi^\lambda_j})\|^2_{\mathscr{H}^s(T)} \leq d^G_\lambda C^2 (1+\lambda)^{2s'}, \quad \forall (\mu,\lambda) \in \sigma(\Delta_T)\times\sigma(\Delta_G),$$

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and thus, since $d_\lambda^G = \mathrm{O}(\lambda^{2m})$ thanks to (1.1),

$$(1+\mu)^s \|\mathcal{F}_\mu^T \mathcal{F}_\lambda^G(u)\|_{L^2(T\times G)} \leq C(1+\lambda)^{s'+m}, \quad \forall (\mu,\lambda)\in\sigma(\Delta_T)\times\sigma(\Delta_G).$$

Let $\theta\in(0,1)$ be so small that $\theta(s'+m)\leq s/2$: for $(\mu,\lambda)\in\Lambda_\theta$, we then have

$$\|\mathcal{F}_\mu^T \mathcal{F}_\lambda^G(u)\|_{L^2(T\times G)} \leq C(1+\lambda)^{s'+m}(1+\mu)^{-s} \leq C(1+\mu)^{\theta(s'+m)-s} \leq C(1+\mu)^{-s/2}.$$

Moreover, on Λ_θ, we have $1+\mu+\lambda\leq(1+\mu)^2$ from which our conclusion follows. □

Combining Lemmas 4.1 and 4.3, we conclude:

Corollary 4.4. *Suppose that $\tilde{P}$ is elliptic. If $u\in\mathscr{D}'(T\times G)$ is such that $Pu\in\mathscr{C}^\infty(T\times G)$, then for every $s>0$, there exist $C>0$ and $\theta\in(0,1)$, such that*

$$\|\mathcal{F}_\mu^T \mathcal{F}_\lambda^G(u)\|_{L^2(T\times G)} \leq C(1+\mu+\lambda)^{-s}, \quad \forall (\mu,\lambda)\in\Lambda_\theta.$$

5. Interlude: global hypoellipticity of certain systems of vector fields

In this section, we derive some general results regarding global hypoellipticity of systems of vector fields (Definition 3.4) which are needed to pave the way for the proofs of Theorem 3.5 and related results later on. We consider M a compact Riemannian manifold enjoying all the properties described in Section 1, from where we also borrow the notation. We denote its Laplace-Beltrami operator simply by Δ, and $\mathcal{L}$ will stand for any system of smooth vector fields in M.

Lemma 5.1. *The following are equivalent:*

1. $\mathcal{L}$ *is* (GH) *in* M.
2. $\mathrm{span}_{\mathbb{R}}\mathcal{L}$ *is* (GH) *in* M.
3. $\mathrm{Lie}\,\mathcal{L}$, *the Lie algebra generated by* $\mathcal{L}$, *is* (GH) *in* M.

Proof. It is clear that if $\mathcal{L}\subset\mathcal{L}'$ are two families of vector fields and $\mathcal{L}$ is (GH) in M, then so is $\mathcal{L}'$. This observation takes care of the implications (1) ⇒ (2) ⇒ (3) since $\mathcal{L}\subset\mathrm{span}_{\mathbb{R}}\mathcal{L}\subset\mathrm{Lie}\,\mathcal{L}$. Since, moreover

$$\mathrm{Lie}\,\mathcal{L} = \mathrm{span}_{\mathbb{R}} \bigcup_{\nu\in\mathbb{N}} \{[\mathrm{X}_1,[\cdots[\mathrm{X}_{\nu-1},\mathrm{X}_\nu]\cdots]] \ : \ \mathrm{X}_j\in\mathcal{L},\ 1\leq j\leq\nu\}$$

implication (3) ⇒ (1) follows immediately. □

The main advantage of the previous lemma is that it enables us to transition between different sets of generators of a given system. The next technical proposition characterizes global hypoellipticity of certain finitely generated systems in terms of manageable inequalities. Not only this will be required in the proof of our main result but also later on in Section 8.1. There, we discuss the case when G is a torus and show a direct application of Proposition 5.2 to an important system of vector fields with constant coefficients on $\mathbb{T}^m$. This allows us to connect our abstract conditions with the notion of simultaneous

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approximability — an inequality that can be read from the symbols of the vector fields à la Greenfield-Wallach.

Proposition 5.2. *Suppose that* $\mathrm{L}_1,\dots,\mathrm{L}_r$ *are smooth vector fields on* M *which commute with* Δ*. Then, the system* $\{\mathrm{L}_1,\dots,\mathrm{L}_r\}$ *is* (GH) *in* M *if and only if there exist* $C,\rho>0$ *and* $\lambda_0\in\sigma(\Delta)$*, such that*

$$\Big(\sum_{j=1}^{r}\|\mathrm{L}_j\phi\|^2_{L^2(M)}\Big)^{\frac{1}{2}}\geq C(1+\lambda)^{-\rho}\|\phi\|_{L^2(M)},\quad \forall\phi\in E_\lambda,\ \forall\lambda\geq\lambda_0. \tag{5.1}$$

Proof. Suppose that $C,\rho>0$ and $\lambda_0\in\sigma(\Delta)$ are such that (5.1) holds, and let $u\in\mathscr{D}'(M)$ be such that $\mathrm{L}_1u,\dots,\mathrm{L}_ru\in\mathscr{C}^\infty(M)$. Given $s>0$, for each $j\in\{1,\dots,r\}$, there exists $C_j>0$, such that

$$\|\mathcal{F}_\lambda(\mathrm{L}_ju)\|_{L^2(M)}\leq C_j(1+\lambda)^{-s-\rho},\quad \forall\lambda\in\sigma(\Delta),$$

by Proposition 1.1. Since $\mathcal{F}_\lambda(\mathrm{L}_ju)=\mathrm{L}_j\mathcal{F}_\lambda(u)$ (for L_j commutes with Δ: use Proposition 2.8 with $T\doteq\{\mathrm{pt}\}$, or see [2, Proposition 2.2]), we have for $\lambda\geq\lambda_0$ that

$$\|\mathcal{F}_\lambda(u)\|_{L^2(M)}\leq C^{-1}(1+\lambda)^{\rho}\Big(\sum_{j=1}^{r}\|\mathrm{L}_j\mathcal{F}_\lambda(u)\|^2_{L^2(M)}\Big)^{\frac{1}{2}}\leq C^{-1}\Big(\sum_{j=1}^{r}C_j^2\Big)^{\frac{1}{2}}(1+\lambda)^{-s}.$$

As the set $\{\lambda\in\sigma(\Delta)\,:\,\lambda<\lambda_0\}$ is finite, we conclude by Proposition 1.1 that $u\in\mathscr{C}^\infty(M)$.

For the converse, suppose that for each $\nu\in\mathbb{N}$, there exist $\lambda_\nu\in\sigma(\Delta)$ with $\lambda_\nu\geq\nu$ and $\phi_\nu\in E_{\lambda_\nu}$, such that

$$\Big(\sum_{j=1}^{r}\|\mathrm{L}_j\phi_\nu\|^2_{L^2(M)}\Big)^{\frac{1}{2}}<2^{-\nu}(1+\lambda_\nu)^{-\nu}\|\phi_\nu\|_{L^2(M)}.$$

W.l.o.g., we assume $\|\phi_\nu\|_{L^2(M)}=1$ and that $\{\lambda_\nu\}_{\nu\in\mathbb{N}}$ is strictly increasing. Then

$$u\doteq\sum_{\nu\in\mathbb{N}}\phi_\nu\in\mathscr{D}'(M)\setminus\mathscr{C}^\infty(M)$$

by Proposition 1.1. On the other hand, for $j\in\{1,\dots,r\}$, we have, given $s>0$:

- if $\lambda=\lambda_\nu$ for some $\nu\geq s$:

$$\|\mathcal{F}_\lambda(\mathrm{L}_ju)\|_{L^2(M)}=\|\mathrm{L}_j\phi_\nu\|_{L^2(M)}\leq 2^{-\nu}(1+\lambda_\nu)^{-\nu}\leq(1+\lambda)^{-s};$$

- if $\lambda\neq\lambda_\nu$ for every $\nu\in\mathbb{N}$:

$$\|\mathcal{F}_\lambda(\mathrm{L}_ju)\|_{L^2(M)}=0\leq(1+\lambda)^{-s}.$$

Thus, $\mathrm{L}_ju\in\mathscr{C}^\infty(M)$ since $\{\nu\in\mathbb{N}\,:\,\nu<s\}$ is finite; hence, $\{\mathrm{L}_1,\dots,\mathrm{L}_r\}$ is not (GH). □

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6. Sufficiency for operators subject to commutativity assumptions

Our aim in this section is to prove Theorem 3.5, which still requires some preparation. For each $\ell \in \{1,\dots,N\}$, we write

$$\mathfrak{a}_\ell(t) = \sum_{j=1}^{m} a_{\ell j}(t)\mathrm{X}_j, \quad t \in T,$$

which we assume to be not identically zero, hence, among $a_{\ell 1},\dots,a_{\ell m}$, there are exactly $m^\ell \geq 1$ functions that are $\mathbb{R}$-linearly independent. We denote them by $\alpha_{\ell 1},\dots,\alpha_{\ell m^\ell}$: writing the remaining coefficients as linear combinations of these allows us to write $\mathfrak{a}_\ell$ as

$$\mathfrak{a}_\ell(t) = \sum_{p=1}^{m^\ell} \alpha_{\ell p}(t)\mathrm{L}_p^\ell,$$

where $\mathrm{L}_1^\ell,\dots,\mathrm{L}_{m^\ell}^\ell$ are linear combinations of $\mathrm{X}_1,\dots,\mathrm{X}_m$, hence also elements of $\mathfrak{g}$. One can prove that $\mathrm{L}_1^\ell,\dots,\mathrm{L}_{m^\ell}^\ell$ are linearly independent, and actually a basis for $\mathcal{L}_\ell$ as defined in (3.7) (see Section 8.1, where we derive explicit expressions for these vector fields w.r.t. the choice $\alpha_{\ell p} \doteq a_{\ell j_p^\ell}$ for $p \in \{1,\dots,m^\ell\}$).

Linear independence of $\alpha_{\ell 1},\dots,\alpha_{\ell m^\ell}$ means that if we define $D_\ell : T \times \mathbb{R}^{m^\ell} \to \mathbb{R}$ by

$$D_\ell(t,\gamma) \doteq \left(\sum_{p=1}^{m^\ell} \alpha_{\ell p}(t)\gamma_p\right)^2, \quad t \in T,\ \gamma \in \mathbb{R}^{m^\ell},$$

then for each $\gamma \neq 0$, the function $t \in T \mapsto D_\ell(t,\gamma) \in \mathbb{R}$ cannot be identically zero. We then have, arguing as in the proof of [3, Lemma 3.1]:

Lemma 6.1. *There are constants $\alpha,\delta > 0$, such that for every $\gamma \in \mathbb{R}^{m^\ell}$, there exists a nonempty open set $A_\gamma \subset T$ with $\mathrm{vol}(A_\gamma) \geq \delta$, such that*

$$D_\ell(t,\gamma) \geq \alpha|\gamma|^2, \quad \forall t \in A_\gamma.$$

Next, we state without proof a fundamental estimate, which generalizes [12, Equation (2.10)].

Proposition 6.2. *Given $\delta > 0$, there exists a constant $C > 0$, such that for every open set $A \subset T$ with $\mathrm{vol}(A) \geq \delta$, one has*

$$\|\psi\|_{L^2(T)}^2 \leq C\Big(\|\psi\|_{L^2(A)}^2 + \|\mathrm{d}_T\psi\|_{L^2(T)}^2\Big), \quad \forall \psi \in \mathscr{C}^\infty(T).$$

We also mention the following easy result that will be useful in some arguments below.

Lemma 6.3. *Let W be any vector field on T. Then, there exists $C > 0$, such that*

$$\|\mathrm{W}\psi\|_{L^2(T)} \leq C\|\mathrm{d}_T\psi\|_{L^2(T)}, \quad \forall \psi \in \mathscr{C}^\infty(T).$$

All of this allows us to prove the following:

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Proposition 6.4. *Under the hypotheses of Theorem 3.5, there exist $C,\rho > 0$ and $\lambda_0 \in \sigma(\Delta_G)$, such that*

$$\langle P\psi,\psi\rangle_{L^2(T\times G)} \geq C(1+\lambda)^{-\rho}\|\psi\|^2_{L^2(T\times G)}, \quad \forall \psi \in \mathscr{C}^\infty(T;E^G_\lambda),\ \lambda \geq \lambda_0. \tag{6.1}$$

Proof. By hypothesis (1), the set of left-invariant vector fields $\mathcal{L}_\ell$ acts as a family of commuting, skew-symmetric — hence normal — linear endomorphisms of E^G_λ for each $\lambda \in \sigma(\Delta_G)$, which then admits an orthonormal basis

$$\phi_1^{\lambda,\ell},\ldots,\phi_{d^G_\lambda}^{\lambda,\ell} \in E^G_\lambda$$

which are common eigenvectors to all operators in $\mathcal{L}_\ell$; their associated eigenvalues are purely imaginary

$$\mathrm{L}^\ell_p\phi_i^{\lambda,\ell} = \sqrt{-1}\gamma_{i,p}^{\lambda,\ell}\phi_i^{\lambda,\ell}, \quad \gamma_{i,p}^{\lambda,\ell} \in \mathbb{R},$$

and we may bound their absolute values thanks to the following easy remark.

Lemma 6.5. *For every $\mathrm{X} \in \mathfrak{g}$, we have*

$$\|\mathrm{X}\phi\|_{L^2(G)} \leq \|\mathrm{X}\|_{\mathfrak{g}}\lambda^{1/2}\|\phi\|_{L^2(G)}, \quad \forall \phi \in E^G_\lambda,\ \forall \lambda \in \sigma(\Delta_G),$$

where $\|\cdot\|_{\mathfrak{g}}$ is the norm on $\mathfrak{g}$ induced by the underlying ad-*invariant inner product.*

Proof of Lemma 6.5. Assume w.l.o.g. $\mathrm{X} \neq 0$. Let then $\mathrm{X}_1,\ldots,\mathrm{X}_m$ be an orthonormal basis for $\mathfrak{g}$, such that $\mathrm{X}_1 = \mathrm{X}/\|\mathrm{X}\|_{\mathfrak{g}}$. As the sum of their squares equals $-\Delta_G$ (1.3), we have, for $\phi \in E^G_\lambda$,

$$\|\mathrm{X}_1\phi\|^2_{L^2(G)} \leq \sum_{j=1}^m \|\mathrm{X}_j\phi\|^2_{L^2(G)} = -\sum_{j=1}^m \langle \mathrm{X}_j^2\phi,\phi\rangle_{L^2(G)} = \langle \Delta_G\phi,\phi\rangle_{L^2(G)} = \lambda\|\phi\|^2_{L^2(G)}.$$

□

It follows immediately that

$$|\gamma_{i,p}^{\lambda,\ell}|^2 \leq \|\mathrm{L}^\ell_p\|^2_{\mathfrak{g}}\lambda.$$

For each $i,i' \in \{1,\ldots,d^G_\lambda\}$ and $p,p' \in \{1,\ldots,m^\ell\}$:

$$\langle \mathrm{L}^\ell_p\phi_i^{\lambda,\ell},\mathrm{L}^\ell_{p'}\phi_{i'}^{\lambda,\ell}\rangle_{L^2(G)} = \gamma_{i,p}^{\lambda,\ell}\gamma_{i',p'}^{\lambda,\ell}\langle \phi_i^{\lambda,\ell},\phi_{i'}^{\lambda,\ell}\rangle_{L^2(G)} = \delta_{ii'}\gamma_{i,p}^{\lambda,\ell}\gamma_{i',p'}^{\lambda,\ell},$$

so, in particular, for each given $t \in T$, we have

$$\begin{aligned}\langle \mathfrak{a}_\ell(t)\phi_i^{\lambda,\ell},\mathfrak{a}_\ell(t)\phi_{i'}^{\lambda,\ell}\rangle_{L^2(G)} &= \sum_{p,p'=1}^{m^\ell} \alpha_{\ell p}(t)\alpha_{\ell p'}(t)\langle \mathrm{L}^\ell_p\phi_i^{\lambda,\ell},\mathrm{L}^\ell_{p'}\phi_{i'}^{\lambda,\ell}\rangle_{L^2(G)} \\ &= \sum_{p,p'=1}^{m^\ell} \alpha_{\ell p}(t)\alpha_{\ell p'}(t)\delta_{ii'}\gamma_{i,p}^{\lambda,\ell}\gamma_{i',p'}^{\lambda,\ell} = \delta_{ii'}D_\ell(t,\gamma_i^{\lambda,\ell}),\end{aligned}$$

where $\gamma_i^{\lambda,\ell} \in \mathbb{R}^{m^\ell}$ is defined in the obvious manner.

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A general $\psi \in \mathscr{C}^\infty(T;E_\lambda^G)$ is written, given $\ell \in \{1,\dots,N\}$, as

$$\psi = \sum_{i=1}^{d_\lambda^G} \psi_i^\ell \otimes \phi_i^{\lambda,\ell},$$

so, for each $t \in T$ given, we have that

$$\begin{aligned} \|\mathfrak{a}_\ell(t)\psi(t)\|_{L^2(G)}^2 &= \int_G \Big| \sum_{i=1}^{d_\lambda^G} \psi_i^\ell(t)(\mathfrak{a}_\ell(t)\phi_i^{\lambda,\ell})(x)\Big|^2 \mathrm{d}V_G(x) \\ &= \sum_{i,i'=1}^{d_\lambda^G} \psi_i^\ell(t)\overline{\psi_{i'}^\ell(t)}\langle \mathfrak{a}_\ell(t)\phi_i^{\lambda,\ell}, \mathfrak{a}_\ell(t)\phi_{i'}^{\lambda,\ell}\rangle_{L^2(G)} = \sum_{i=1}^{d_\lambda^G} |\psi_i^\ell(t)|^2 D_\ell(t,\gamma_i^{\lambda,\ell}) \end{aligned} \tag{6.2}$$

but also

$$\|\mathrm{L}_p^\ell\psi(t)\|_{L^2(G)}^2 = \int_G \Big|\sum_{i=1}^{d_\lambda^G} \psi_i^\ell(t)(\mathrm{L}_p^\ell\phi_i^{\lambda,\ell})(x)\Big|^2 \mathrm{d}V_G(x) = \sum_{i=1}^{d_\lambda^G} |\psi_i^\ell(t)|^2|\gamma_{i,p}^{\lambda,\ell}|^2.$$

Recall that $\mathrm{L}_1^\ell,\dots,\mathrm{L}_{m^\ell}^\ell$ form a basis for $\mathcal{L}_\ell$ (3.7) for each $\ell \in \{1,\dots,N\}$, hence the set

$$\{\mathrm{L}_p^\ell \ : \ p \in \{1,\dots,m^\ell\},\ \ell \in \{1,\dots,N\}\} \tag{6.3}$$

generates $\mathrm{span}_{\mathbb{R}}\mathcal{L}$. By Lemma 5.1, our hypothesis (2) of global hypoellipticity of $\mathcal{L}$ in G entails the same property for $\mathrm{span}_{\mathbb{R}}\mathcal{L}$ and hence for (6.3). As these commute with Δ_G, Proposition 5.2 then provides us constants $C,\rho > 0$ and $\lambda_0 \in \sigma(\Delta_G)$, such that

$$\Big(\sum_{\ell=1}^N\sum_{p=1}^{m^\ell} \|\mathrm{L}_p^\ell\phi\|_{L^2(G)}^2\Big)^{\frac{1}{2}} \geq C(1+\lambda)^{-\rho}\|\phi\|_{L^2(G)}, \quad \forall \phi \in E_\lambda^G,\ \lambda \geq \lambda_0. \tag{6.4}$$

Fix $\lambda \geq \lambda_0$. We apply (6.4) to $\phi = \psi(t)$, for some $t \in T$ given

$$\|\psi(t)\|_{L^2(G)}^2 \leq C^{-2}(1+\lambda)^{2\rho}\sum_{\ell=1}^N\sum_{p=1}^{m^\ell}\|\mathrm{L}_p^\ell\psi(t)\|_{L^2(G)}^2 = C^{-2}(1+\lambda)^{2\rho}\sum_{\ell=1}^N\sum_{i=1}^{d_\lambda^G}|\psi_i^\ell(t)|^2|\gamma_i^{\lambda,\ell}|^2,$$

and then integrate both sides over T, yielding

$$\|\psi\|_{L^2(T\times G)}^2 \leq C^{-2}(1+\lambda)^{2\rho}\sum_{\ell=1}^N\sum_{i=1}^{d_\lambda^G}\int_T |\psi_i^\ell(t)|^2|\gamma_i^{\lambda,\ell}|^2 \mathrm{d}V_T(t). \tag{6.5}$$

Let us work out the last integral above. By Lemma 6.1, there are constants $\alpha,\delta > 0$, such that for every $\ell \in \{1,\dots,N\}$ and every $i \in \{1,\dots,d_\lambda^G\}$ fixed, there exists a nonempty open set $A_i^{\lambda,\ell} \subset T$ with $\mathrm{vol}(A_i^{\lambda,\ell}) \geq \delta$, such that

$$D_\ell(t,\gamma_i^{\lambda,\ell}) \geq \alpha|\gamma_i^{\lambda,\ell}|^2, \quad \forall t \in A_i^{\lambda,\ell}.$$

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Then, by Proposition 6.2, there exists $C_1 > 0$ depending on δ but not on any other parameters, such that

$$\int_T |\psi_i^\ell(t)|^2 |\gamma_i^{\lambda,\ell}|^2 \mathrm{d}V_T(t) \leq C_1 \Big(\int_{A_i^{\lambda,\ell}} |\psi_i^\ell(t)|^2 |\gamma_i^{\lambda,\ell}|^2 \mathrm{d}V_T(t) + \|\mathrm{d}_T(|\gamma_i^{\lambda,\ell}| \psi_i^\ell)\|^2_{L^2(T)} \Big)$$
$$\leq C_1 \Big(\alpha^{-1} \int_T |\psi_i^\ell(t)|^2 D_\ell(t, \gamma_i^{\lambda,\ell}) \mathrm{d}V_T(t) + B_\ell \lambda \|\mathrm{d}_T \psi_i^\ell\|^2_{L^2(T)} \Big),$$

where, by the conclusion after Lemma 6.5, $B_\ell > 0$ depends only on $\|\mathrm{L}_p^\ell\|_{\mathfrak{g}}$, $p \in \{1, \ldots, m^\ell\}$. By (6.2)

$$\sum_{i=1}^{d_\lambda^G} \int_T |\psi_i^\ell(t)|^2 |\gamma_i^{\lambda,\ell}|^2 \mathrm{d}V_T(t) \leq C_1 \Big(\alpha^{-1} \int_T \sum_{i=1}^{d_\lambda^G} |\psi_i^\ell(t)|^2 D_\ell(t, \gamma_i^{\lambda,\ell}) \mathrm{d}V_T(t) + B_\ell \lambda \sum_{i=1}^{d_\lambda^G} \|\mathrm{d}_T \psi_i^\ell\|^2_{L^2(T)} \Big)$$
$$= C_1 \Big(\alpha^{-1} \|\mathfrak{a}_\ell(t, \mathrm{X}) \psi\|^2_{L^2(T\times G)} + B_\ell \lambda \langle \Delta_T \psi, \psi \rangle_{L^2(T\times G)} \Big),$$

and hence, for some $C_2 > 0$,

$$\sum_{\ell=1}^{N} \sum_{i=1}^{d_\lambda^G} \int_T |\psi_i^\ell(t)|^2 |\gamma_i^{\lambda,\ell}|^2 \mathrm{d}V_T(t) \leq C_2 (1+\lambda) \Big(\sum_{\ell=1}^{N} \|\mathfrak{a}_\ell(t, \mathrm{X}) \psi\|^2_{L^2(T\times G)} + \langle \Delta_T \psi, \psi \rangle_{L^2(T\times G)} \Big).$$

By Lemma 6.3, there exists $C_3 > 0$, such that

$$\|\mathfrak{a}_\ell(t, \mathrm{X}) \psi\|^2_{L^2(T\times G)} \leq \Big(\|(\mathfrak{a}_\ell(t, \mathrm{X}) + \mathrm{W}_\ell) \psi\|_{L^2(T\times G)} + \|\mathrm{W}_\ell \psi\|_{L^2(T\times G)} \Big)^2$$
$$\leq 2 \Big(\|(\mathfrak{a}_\ell(t, \mathrm{X}) + \mathrm{W}_\ell) \psi\|^2_{L^2(T\times G)} + C_3^2 \langle \Delta_T \psi, \psi \rangle_{L^2(T\times G)} \Big)$$

from which we conclude that

$$\sum_{\ell=1}^{N} \sum_{i=1}^{d_\lambda^G} \int_T |\psi_i^\ell(t)|^2 |\gamma_i^{\lambda,\ell}|^2 \mathrm{d}V_T(t) \leq C_4 (1+\lambda) \Big(\sum_{\ell=1}^{N} \|(\mathfrak{a}_\ell(t, \mathrm{X}) + \mathrm{W}_\ell) \psi\|^2_{L^2(T\times G)} + \langle \Delta_T \psi, \psi \rangle_{L^2(T\times G)} \Big)$$
$$= C_4 (1+\lambda) \langle P\psi, \psi \rangle_{L^2(T\times G)},$$

the last equality following from Lemma 3.1. Plugging this back into (6.5) ends our proof. □

Now we prove Theorem 3.5.

Proof of Theorem 3.5. Let $u \in \mathscr{D}'(T \times G)$ be such that $f \doteq Pu \in \mathscr{C}^\infty(T \times G)$. Since $\tilde{P}$ is elliptic, by Corollary 4.2, we have that $\mathcal{F}_\lambda^G(u) \in \mathscr{C}^\infty(T; E_\lambda^G)$ for every $\lambda \in \sigma(\Delta_G)$, and by Proposition 6.4 — applied to $\psi = \mathcal{F}_\lambda^G(u)$ — there exist $C, \rho > 0$ and $\lambda_0 \in \sigma(\Delta_G)$, such that

$$\|\mathcal{F}_\lambda^G(u)\|_{L^2(T\times G)} \leq C^{-1} (1+\lambda)^\rho \|\mathcal{F}_\lambda^G(f)\|_{L^2(T\times G)}, \quad \forall \lambda \geq \lambda_0$$

after a suitable application of the Cauchy-Schwarz inequality. But since f is smooth, by Corollary 2.6, for every $s > 0$, there exists $C_s > 0$, such that

$$\|\mathcal{F}_\lambda^G(u)\|_{L^2(T\times G)} \leq C_s (1+\lambda)^{-s}, \quad \forall \lambda \geq \lambda_0.$$

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It is simple to see that, increasing C_s if necessary, we obtain that the last inequality holds for every $\lambda \in \sigma(\Delta_G)$. We already saw in Corollary 4.4 that the ellipticity of $\tilde{P}$ entails, for every $s > 0$, the existence of $C > 0$ and $\theta \in (0,1)$, such that (2.6) holds. Finally, Corollary 2.7 ensures smoothness of u. Furthermore, if R is as in (3.9), then it is certainly a positive semidefinite LPDO in $T \times G$, hence (6.1) implies that the same inequality holds if we exchange P for $P_0 = P + R$. The latter is also an LPDO on $T \times G$ of the same kind as P, and $\tilde{P}_0$ is clearly elliptic too. Thus, the argument above applies just as well for P_0 in place of P, proving its global hypoellipticity in $T \times G$. □

7. A class of systems

Our goal in this section is to prove Theorem 3.3. Notice that its proof would be rather simple — similar to that of Proposition 3.2 — if there were no vector fields W_ℓ in (3.1). Here, however, we are once again studying a general P defined by (3.1) in $T \times G$ and $\mathcal{L}$ denotes the system of vector fields (3.6). Our next lemma is the key to relate the condition (3.5) with the global hypoellipticity of $\mathcal{L}$ in G.

Lemma 7.1. *A distribution $u \in \mathscr{D}'(G)$ satisfies $\mathfrak{a}_\ell(t,\mathrm{X})(1_T \otimes u) \in \mathscr{C}^\infty(T \times G)$ for every $\ell \in \{1,\dots,N\}$ if and only if $\mathrm{L}u \in \mathscr{C}^\infty(G)$ for every $\mathrm{L} \in \mathcal{L}$.*

Proof. Let $u \in \mathscr{D}'(G)$ be such that $\mathfrak{a}_\ell(t,\mathrm{X})(1_T \otimes u) \in \mathscr{C}^\infty(T \times G)$ for every $\ell \in \{1,\dots,N\}$. We have

$$\mathfrak{a}_\ell(t,\mathrm{X})(1_T \otimes u) = \sum_{j=1}^{m} a_{\ell j}(t)\mathrm{X}_j u, \quad t \in T,$$

which is smooth in $T \times G$, hence for any given $t_0 \in T$

$$\mathfrak{a}_\ell(t_0)u = \sum_{j=1}^{m} a_{\ell j}(t_0)\mathrm{X}_j u \in \mathscr{C}^\infty(G), \quad \forall \ell \in \{1,\dots,N\}.$$

We conclude that $\mathrm{L}u \in \mathscr{C}^\infty(G)$ for every $\mathrm{L} \in \mathcal{L} = \{\mathfrak{a}_\ell(t_0) \ : \ t_0 \in T,\ \ell \in \{1,\dots,N\}\}$.

For the converse, suppose that $u \in \mathscr{D}'(G)$ is such that $\mathrm{L}u \in \mathscr{C}^\infty(G)$ for every $\mathrm{L} \in \mathcal{L}$. We select $\mathrm{L}_1,\dots,\mathrm{L}_r \in \mathcal{L}$ a basis for $\operatorname{span}_{\mathbb{R}} \mathcal{L}$ — this is a finite dimensional space since it is contained in $\mathfrak{g}$ — so we can write, for each $\ell \in \{1,\dots,N\}$,

$$\mathfrak{a}_\ell(t) = \sum_{j=1}^{r} \alpha_{\ell j}(t)\mathrm{L}_j, \quad t \in T,$$

where $\alpha_{\ell 1},\dots,\alpha_{\ell r} \in \mathscr{C}^\infty(T;\mathbb{R})$ are uniquely determined. We thus have

$$\mathfrak{a}_\ell(t,\mathrm{X})(1_T \otimes u) = \mathfrak{a}_\ell(t)u = \sum_{j=1}^{r} \alpha_{\ell j}(t)\mathrm{L}_j u \in \mathscr{C}^\infty(T \times G), \quad \forall \ell \in \{1,\dots,N\},$$

since $\mathrm{L}_1 u,\dots,\mathrm{L}_r u \in \mathscr{C}^\infty(G)$ by hypothesis. □

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Proposition 7.2. *Condition* (3.5) *holds if and only if* $\mathcal{L}$ *is* (GH) *in* G.

Proof. Assume first that $\mathcal{L}$ is (GH) in G, and let $u \in \mathscr{D}'(G)$ be such that $\mathfrak{a}_\ell(t,\mathrm{X})(1_T \otimes u) \in \mathscr{C}^\infty(T \times G)$ for every $\ell \in \{1,\dots,N\}$. By Lemma 7.1, we have that $\mathrm{L}u \in \mathscr{C}^\infty(G)$ for every $\mathrm{L} \in \mathcal{L}$, hence $u \in \mathscr{C}^\infty(G)$. On the other hand, if one assumes (3.5) and letting $u \in \mathscr{D}'(G)$ be such that $\mathrm{L}u \in \mathscr{C}^\infty(G)$ for every $\mathrm{L} \in \mathcal{L}$, then, by Lemma 7.1, we have that $\mathfrak{a}_\ell(t,\mathrm{X})(1_T \otimes u) \in \mathscr{C}^\infty(T \times G)$ for every $\ell \in \{1,\dots,N\}$. We conclude that $u \in \mathscr{C}^\infty(G)$. $\square$

We now prove Theorem 3.3.

Proof of Theorem 3.3. Suppose that P is (GH) in $T \times G$, and let $u \in \mathscr{D}'(G)$ be such that $\mathfrak{a}_\ell(t,\mathrm{X})(1_T \otimes u) \in \mathscr{C}^\infty(T \times G)$ for every $\ell \in \{1,\dots,N\}$. By (3.4), we have (recall that $\tilde{P}$ has no zeroth order terms, hence annihilates constants):

$$P(1_T \otimes u) = -\sum_{\ell=1}^{N} \mathfrak{a}_\ell(t,\mathrm{X})^2(1_T \otimes u) - \sum_{\ell=1}^{N}\sum_{j=1}^{m}(\mathrm{W}_\ell a_{\ell j}) \otimes (\mathrm{X}_j u).$$

The first sum in $\mathscr{C}^\infty(T \times G)$ by assumption; we claim that so is the second. Indeed, define

$$\tilde{\mathfrak{a}}_\ell(t) \doteq \sum_{j=1}^{m}(\mathrm{W}_\ell a_{\ell j})(t)\mathrm{X}_j, \quad t \in T,\ \ell \in \{1,\dots,N\}.$$

Hence, $\tilde{\mathfrak{a}}_1,\dots,\tilde{\mathfrak{a}}_N : T \to \mathfrak{g}$ are all smooth. We notice that $\operatorname{ran}\tilde{\mathfrak{a}}_\ell \subset \operatorname{span}_{\mathbb{R}} \operatorname{ran}\mathfrak{a}_\ell$ for every $\ell \in \{1,\dots,N\}$: given $t_0 \in T$ and $(U;\chi) = (U;t_1,\dots,t_n)$ a coordinate chart of T centered at t_0, we may write, in U,

$$\mathrm{W}_\ell = \sum_{k=1}^{n} b_{\ell k}(t)\frac{\partial}{\partial t_k},$$

where $b_{\ell 1},\dots,b_{\ell n} \in \mathscr{C}^\infty(U;\mathbb{R})$, hence

$$\tilde{\mathfrak{a}}_\ell(t_0) = \sum_{j=1}^{m}\sum_{k=1}^{n} b_{\ell k}(t_0)\frac{\partial a_{\ell j}}{\partial t_k}(t_0)\mathrm{X}_j = \sum_{k=1}^{n} b_{\ell k}(t_0)\lim_{h\to 0}\frac{1}{h}\Big(\mathfrak{a}_\ell(\chi^{-1}(he_k)) - \mathfrak{a}_\ell(\chi^{-1}(0))\Big)$$

certainly belongs to $\operatorname{span}_{\mathbb{R}}\operatorname{ran}\mathfrak{a}_\ell$ — since all the Newton quotients above obviously do.

We then define

$$\tilde{\mathcal{L}} \doteq \bigcup_{\ell=1}^{N} \operatorname{ran}\tilde{\mathfrak{a}}_\ell$$

which we have just proved to be contained in $\operatorname{span}_{\mathbb{R}}\mathcal{L}$. Now, since $\mathfrak{a}_\ell(t,\mathrm{X})(1_T \otimes u) \in \mathscr{C}^\infty(T \times G)$ for every $\ell \in \{1,\dots,N\}$, it follows from Lemma 7.1 that $\mathrm{L}u \in \mathscr{C}^\infty(G)$ for every $\mathrm{L} \in \mathcal{L}$, hence also for every $\mathrm{L} \in \operatorname{span}_{\mathbb{R}}\mathcal{L}$ and, in particular, for every $\mathrm{L} \in \tilde{\mathcal{L}}$; by a second application of Lemma 7.1, we conclude that $\tilde{\mathfrak{a}}_\ell(t,\mathrm{X})(1_T \otimes u) \in \mathscr{C}^\infty(T \times G)$ for every $\ell \in \{1,\dots,N\}$. It then follows that

$$P(1_T \otimes u) = -\sum_{\ell=1}^{N} \mathfrak{a}_\ell(t,\mathrm{X})^2(1_T \otimes u) - \sum_{\ell=1}^{N} \tilde{\mathfrak{a}}_\ell(t,\mathrm{X})(1_T \otimes u) \in \mathscr{C}^\infty(T \times G),$$

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and, since P is (GH) in $T \times G$, we conclude that $1_T \otimes u \in \mathscr{C}^\infty(T \times G)$, that is, $u \in \mathscr{C}^\infty(G)$, thus proving (3.5). ☐

8. Remarks and examples

We devote this section to motivate our hypotheses, to compare our results with previous ones in the literature, and, of course, to provide some examples of operators that satisfy the hypotheses of Theorem 3.5.

Let us take a look at hypothesis (1) in Theorem 3.5. The fact that $\mathfrak{a}_\ell(t_1)$ and $\mathfrak{a}_\ell(t_2)$ commute for every $t_1, t_2 \in T$ does not preclude noncommutativity of the vector fields belonging to distinct $\mathcal{L}_\ell$. In concrete examples, this prevents us from being "thrown back" to tori: more stringent hypotheses could inadvertently imply that $\mathfrak{g}$ were already commutative to start with (see, e.g., Corollary 8.8). This leads us to our first example.

Example 8.1. Choose $\mathrm{X}_1, \dots, \mathrm{X}_N \in \mathfrak{g}$, such that the Lie subalgebra generated by them is $\mathfrak{g}$. Define $\mathfrak{a}_\ell(t) \doteq a_\ell(t)\mathrm{X}_\ell$, for every $\ell \in \{1, \dots, N\}$, where each $a_\ell \in \mathscr{C}^\infty(T;\mathbb{R})$ is a nonzero function, and consider

$$P \doteq \Delta_T - \sum_{\ell=1}^{N} \Big(a_\ell(t)\mathrm{X}_\ell + \mathrm{W}_\ell \Big)^2,$$

where $\mathrm{W}_1, \dots, \mathrm{W}_N$ are skew-symmetric vector fields in T. Then condition (1) in Theorem 3.5 is satisfied: for each $\ell \in \{1, \dots, N\}$, we have

$$\mathcal{L}_\ell = \mathrm{span}_{\mathbb{R}} \{ a_\ell(t)\mathrm{X}_\ell \ : \ t \in T \},$$

and, for every $t_1, t_2 \in T$, we have $[a_\ell(t_1)\mathrm{X}_\ell, a_\ell(t_2)\mathrm{X}_\ell] = a_\ell(t_1)a_\ell(t_2)[\mathrm{X}_\ell, \mathrm{X}_\ell] = 0$. Notice how we are *not* assuming $[\mathrm{X}_\ell, \mathrm{X}_{\ell'}] = 0$ for $\ell \neq \ell'$.

Moreover, since a_ℓ is not identically zero, some nonvanishing multiple of X_ℓ belongs to $\mathcal{L}$. It follows that $\mathrm{Lie}\,\mathcal{L} = \mathfrak{g}$ (because this is the Lie algebra generated by $\mathrm{X}_1, \dots, \mathrm{X}_N$), which is evidently (GH) in G; hence, so is $\mathcal{L}$ itself, as a consequence of Lemma 5.1: condition (2) in Theorem 3.5 is thus satisfied. We conclude that P is (GH) in $T \times G$. Notice that this generalizes [1, Theorem 3].

Note that if $G = \mathbb{T}^m$, then $\mathrm{Lie}\,\mathcal{L} = \mathfrak{g}$ is possible if and only if $\mathcal{L}$ already contains m linearly independent vector fields. For a compact connected but non-Abelian Lie group G, the noncommutativity of $\mathfrak{g}$ *helps* us to reach condition (2) as N, the number of linearly independent vector fields in $\mathcal{L}$ in Example 8.1, could be much smaller than $m = \dim \mathfrak{g}$. For instance, in $G \doteq \mathrm{SU}(2)$, it is possible to find $\mathrm{X}_1, \mathrm{X}_2, \mathrm{X}_3$ three real vector fields forming a linear basis of $\mathfrak{g} = \mathfrak{su}(2)$ and such that $[\mathrm{X}_1, \mathrm{X}_2] = \mathrm{X}_3$. Choosing nonvanishing $a_1, a_2 \in \mathscr{C}^\infty(T;\mathbb{R})$ and skew-symmetric vector fields $\mathrm{W}_1, \mathrm{W}_2$ in T, we conclude that

$$P \doteq \Delta_T - (a_1(t)\mathrm{X}_1 + \mathrm{W}_1)^2 - (a_2(t)\mathrm{X}_2 + \mathrm{W}_2)^2 \tag{8.1}$$

is globally hypoelliptic in $T \times G$. It is easy, however, to construct many examples for which (8.1) is not locally hypoelliptic: If there exists an open set $U \subset M$, where $a_1 = a_2 = 0$, then $P = \Delta_T - \mathrm{W}_1^2 - \mathrm{W}_2^2$ on $U \times G$, where we can pick any distribution $u \in \mathscr{D}'(U \times G)$ that does not depend on the t variable. More generally, if there is an open set $U \subset M$, where

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a_2 vanishes (but not necessary a_1), then one can find a coordinate chart $(V;x_1,\dots,x_m)$ in G, where $\mathrm{X}_1 = \partial_{x_1}$, hence

$$P = \Delta_T - (a_1(t)\partial_{x_1} + \mathrm{W}_1)^2 - \mathrm{W}_2^2$$

in $U \times V$, which is also not locally hypoelliptic: take $u \in \mathscr{D}'(U \times V)$ depending only on $x_2,\dots,x_m$.

8.1. Relationship with the notion of simultaneous approximability for vectors

Before we provide more examples, we compare Theorem 1 with [3, Theorem 1.5], where global hypoellipticity of the same model operator was studied. Even though both results established necessary and sufficient conditions for global hypoellipticity when G is a torus, it may seem, at a first glance, that our necessary condition of $\mathcal{L}$ being (GH) in G has nothing to do with the notion of simultaneous approximability of a collection of vectors [3, Definition 1.2][1]. Note that one does not need to assume that T is a torus in order to state the notion of simultaneous approximability.

Yet, now we study the relationship between these two concepts. Still within the general setup, recall that $\mathfrak{g}$ carries an inner product $\langle\cdot,\cdot\rangle$ and select $\mathrm{X}_1,\dots,\mathrm{X}_m \in \mathfrak{g}$ a linear basis. Let $\mathfrak{a}_1,\dots,\mathfrak{a}_N$ be as in (3.3), and, for each $\ell \in \{1,\dots,N\}$, define

$$\mathcal{A}_\ell \doteq \operatorname{span}_{\mathbb{R}}\{a_{\ell 1},\dots,a_{\ell m}\} \subset C(T;\mathbb{R}).$$

Notice that the linear map $\mathrm{X} \in \mathfrak{g} \mapsto \sum_{j=1}^m \langle \mathrm{X},\mathrm{X}_j\rangle a_{\ell j} \in \mathcal{A}_\ell$ is certainly onto, with kernel precisely $\mathcal{L}_\ell^\perp$: we thus have an isomorphism $\mathcal{L}_\ell \cong \mathcal{A}_\ell$. Their dimension will be denoted by m^ℓ, and therefore there are indices $1 \leq j_1^\ell < \cdots < j_{m^\ell}^\ell \leq m$, such that

$$a_{\ell j_1^\ell},\dots,a_{\ell j_{m^\ell}^\ell} \text{ form a basis of } \mathcal{A}_\ell.$$

If we write the remaining indices as $1 \leq i_1^\ell < \cdots < i_{d^\ell}^\ell \leq m$ (where $d^\ell \doteq m - m^\ell$), then

$$a_{\ell i_q^\ell} = \sum_{p=1}^{m^\ell} \lambda_{qp}^\ell a_{\ell j_p^\ell}, \quad q \in \{1,\dots,d^\ell\},$$

where $\lambda_{qp}^\ell \in \mathbb{R}$ are uniquely determined. Thus, an $\mathrm{X} \in \mathfrak{g}$ belongs to $\mathcal{L}_\ell^\perp$ if and only if

$$\langle \mathrm{X},\mathrm{X}_{j_p^\ell}\rangle + \sum_{q=1}^{d^\ell} \lambda_{qp}^\ell \langle \mathrm{X},\mathrm{X}_{i_q^\ell}\rangle = 0, \quad \forall p \in \{1,\dots,m^\ell\},$$

meaning that X is orthogonal to

$$\mathrm{L}_p^\ell \doteq \mathrm{X}_{j_p^\ell} + \sum_{q=1}^{d^\ell} \lambda_{qp}^\ell \mathrm{X}_{i_q^\ell}, \quad p \in \{1,\dots,m^\ell\}.$$

[1] Properly adapted to the smooth setup (see condition (2) in Proposition 8.2): in that work, the authors are interested in hypoellipticity w.r.t. some classes of ultradifferentiable functions.

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That is, $\mathrm{L}_1^\ell, \ldots, \mathrm{L}_{m^\ell}^\ell$ form a basis for $\mathcal{L}_\ell$ (they are clearly linearly independent), so by Proposition 5.2 and Lemma 5.1, $\mathcal{L}$ is (GH) in G if and only if there exist $C, \rho > 0$ and $\lambda_0 \in \sigma(\Delta_G)$, such that (6.4) holds.

Now, let us see how this works on a torus. When $G = \mathbb{T}^m$, we have that $\mathrm{X}_j \doteq \partial_{x_j}$, $j \in \{1, \ldots, m\}$, form a basis of its Lie algebra $\mathfrak{g} \cong \mathbb{R}^m$ – which is a commutative Lie algebra, so the standard inner product (i.e., the one for which $\mathrm{X}_1, \ldots, \mathrm{X}_m$ is an orthonormal basis) is automatically ad-invariant, and the associated Laplace-Beltrami operator thus reads

$$\Delta_G = -\sum_{j=1}^m \mathrm{X}_j^2 = -\sum_{j=1}^m \partial_{x_j}^2.$$

Thanks to Fourier analysis, we have that $\sigma(\Delta_G) = \{|\xi|^2 \ : \ \xi \in \mathbb{Z}^m\}$ and

$$E_\lambda^G = \mathrm{span}_{\mathbb{C}}\{e^{ix\xi} \ : \ \xi \in \mathbb{Z}^m, \ |\xi|^2 = \lambda\}, \quad \forall \lambda \in \sigma(\Delta_G),$$

the exponentials actually forming an orthonormal basis of E_λ^G, hence

$$\|\mathrm{L}_p^\ell e^{ix\xi}\|_{L^2(\mathbb{T}^m)} = \left\| \Big(\partial_{x_{j_p^\ell}} + \sum_{q=1}^{d^\ell} \lambda_{qp}^\ell \partial_{x_{i_q^\ell}}\Big) e^{ix\xi} \right\|_{L^2(\mathbb{T}^m)} = \left| \xi_{j_p^\ell} + \sum_{q=1}^{d^\ell} \lambda_{qp}^\ell \xi_{i_q^\ell} \right|.$$

By (6.4), if $\mathcal{L}$ is (GH) in $G = \mathbb{T}^m$, then there exist $C, \rho > 0$ and $n_0 \in \mathbb{N}$, such that

$$\Big(\sum_{\ell=1}^N \sum_{p=1}^{m^\ell} \Big| \xi_{j_p^\ell} + \sum_{q=1}^{d^\ell} \lambda_{qp}^\ell \xi_{i_q^\ell} \Big|^2 \Big)^{\frac{1}{2}} \geq C(1 + |\xi|^2)^{-\rho}, \quad \forall \xi \in \mathbb{Z}^m, \ |\xi| \geq n_0. \tag{8.2}$$

Conversely, since every $\phi \in E_\lambda^G$ can be written as

$$\phi = \sum_{|\xi|^2 = \lambda} \phi_\xi e^{ix\xi}, \quad \phi_\xi \in \mathbb{C},$$

if (8.2) holds, then for $|\xi| \geq n_0$:

$$\sum_{\ell=1}^N \sum_{p=1}^{m^\ell} \|\mathrm{L}_p^\ell \phi\|_{L^2(\mathbb{T}^m)}^2 = \sum_{\ell=1}^N \sum_{p=1}^{m^\ell} \sum_{|\xi|^2 = \lambda} |\phi_\xi|^2 \Big| \xi_{j_p^\ell} + \sum_{q=1}^{d^\ell} \lambda_{qp}^\ell \xi_{i_q^\ell} \Big|^2 \geq C^2 (1 + \lambda)^{-2\rho} \|\phi\|_{L^2(\mathbb{T}^m)}^2,$$

so (6.4) also holds, and $\mathcal{L}$ is (GH) in $\mathbb{T}^m$.

Inequality (8.2) not only resembles the smooth version of the nonsimultaneous approximability condition in [3, Definition 1.2] but is actually equivalent to it. This is the content of the next proposition, for which statement we introduce further notation. For each $\ell \in \{1, \ldots, N\}$, assume that $d^\ell > 0$ and $m^\ell > 0$, and denote, for $\xi \in \mathbb{R}^m$,

$$\xi'_{(\ell)} \doteq (\xi_{j_1^\ell}, \ldots, \xi_{j_{m^\ell}^\ell}) \in \mathbb{R}^{m^\ell}, \quad \xi''_{(\ell)} \doteq (\xi_{i_1^\ell}, \ldots, \xi_{i_{d^\ell}^\ell}) \in \mathbb{R}^{d^\ell},$$

and also

$$v_p^\ell \doteq (\lambda_{1p}^\ell, \ldots, \lambda_{d^\ell p}^\ell) \in \mathbb{R}^{d^\ell}, \quad p \in \{1, \ldots, m^\ell\}.$$

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Proposition 8.2. *The following are equivalent:*

1. *There exist* $C,\rho > 0$ *and* $n_0 \in \mathbb{N}$, *such that* (8.2) *holds, that is*

$$\Big(\sum_{\ell=1}^{N}\sum_{p=1}^{m^\ell}\Big|\xi_{j_p^\ell} + v_p^\ell \cdot \xi''_{(\ell)}\Big|^2\Big)^{\frac{1}{2}} \geq C(1+|\xi|^2)^{-\rho}, \quad \forall \xi \in \mathbb{Z}^m,\ |\xi| \geq n_0.$$

2. *There exist* $B,M > 0$, *such that for each* $\xi \in \mathbb{Z}^m \setminus 0$, *there exist* $\ell \in \{1,\dots,N\}$ *and* $p \in \{1,\dots,m^\ell\}$, *such that*

$$|\xi_{j_p^\ell} + v_p^\ell \cdot \xi''_{(\ell)}| \geq B(1+|\xi''_{(\ell)}|)^{-M}.$$

The proof relies on standard calculations. One immediately recognizes condition (2) above as the bona fide smooth version of the Diophantine condition in [3, Definition 1.2].

In $T \times \mathbb{T}^m$, consider P as in (3.8). We say that it satisfies the *nonsimultaneous approximability condition* if one of the following holds for the family $\mathfrak{a}_1,\dots,\mathfrak{a}_N$:

- there exists $\ell \in \{1,\dots,N\}$, such that $d^\ell = 0$;
- after relabeling indices, we find $0 < N' \leq N$, such that none of $\mathfrak{a}_1,\dots,\mathfrak{a}_{N'}$ is identically zero and when we apply the procedure above, we obtain a collection $v_1^1,\dots,v_{m^1}^1,v_1^2,\dots,v_{m^{N'}}^{N'}$ satisfying one of the equivalent properties in Proposition 8.2.

Corollary 8.3. *When* $G = \mathbb{T}^m$, *our system* $\mathcal{L}$ *in* (3.6) *is* (GH) *in* G *if and only if* P *satisfies the nonsimultaneous approximability condition.*

Example 8.4. Define an LPDO P on $T \times \mathbb{T}^2$ by

$$P \doteq \Delta_T - (\partial_{x_1} + \alpha\partial_{x_2})^2 - (\beta\partial_{x_1} + \partial_{x_2})^2,$$

where $\alpha,\beta \in \mathbb{Q}$ and $\alpha\beta \neq 1$. Since both α and β are rational, it is clear, thanks to a classical result from Greenfield and Wallach [10], that neither $\mathrm{L}_1 \doteq \partial_{x_1} + \alpha\partial_{x_2}$ nor $\mathrm{L}_2 \doteq \beta\partial_{x_1} + \partial_{x_2}$ is globally hypoelliptic in $\mathbb{T}^2$. It is plain, however, that $\mathrm{L}_1,\mathrm{L}_2$ together generate the tangent space of $\mathbb{T}^2$ at every point, therefore, the system $\mathcal{L} \doteq \{\mathrm{L}_1,\mathrm{L}_2\}$ is (GH) in $\mathbb{T}^2$ and P is (GH) in $T \times \mathbb{T}^2$.

8.2. Comparison with Hörmander's condition

Back to a general compact Lie group G, with Lie algebra $\mathfrak{g}$, let $\mathfrak{h} \subset \mathfrak{g}$ be a Lie subalgebra. We regard $\mathscr{C}^\infty(T;\mathfrak{h})$ as a subset of $\mathfrak{X}(T \times G)$, the Lie algebra of all real, smooth vector fields on $T \times G$: as such, it is a Lie subalgebra of the latter. Indeed, given a basis $\mathrm{L}_1,\dots,\mathrm{L}_r$ of $\mathfrak{h}$, any $\mathfrak{a} \in \mathscr{C}^\infty(T;\mathfrak{h})$ can be written as

$$\mathfrak{a}(t) = \sum_{j=1}^{r} a_j(t)\mathrm{L}_j, \quad t \in T,$$

where $a_1,\dots,a_r \in \mathscr{C}^\infty(T;\mathbb{R})$ are uniquely determined; from this observation, our claim follows easily. Moreover, for any real vector field W in T, we have that

$$\mathfrak{a}_{\mathrm{W}} \doteq \sum_{j=1}^{r} (\mathrm{W}a_j)\mathrm{L}_j$$

also belongs to $\mathscr{C}^\infty(T;\mathfrak{h})$ by definition. One then easily sees that $\Theta \doteq \mathscr{C}^\infty(T;\mathfrak{h}) + \mathfrak{X}(T)$, the set of all vector fields Y in $T \times G$ of the form $\mathrm{Y} = \mathfrak{a}(t,\mathrm{X}) + \mathrm{W}$, where $\mathfrak{a} \in \mathscr{C}^\infty(T;\mathfrak{h})$ and $\mathrm{W} \in \mathfrak{X}(T)$, is a Lie subalgebra of $\mathfrak{X}(T\times G)$.

Now let $\mathcal{L}$ be as in (3.6) and let $\mathfrak{h} \doteq \mathrm{Lie}\,\mathcal{L} \subset \mathfrak{g}$. Given $\mathrm{Y}_1,\dots,\mathrm{Y}_N \in \Theta$, assume that for a given $(t,x) \in T \times G$, the following condition holds:

$$\exists \mathrm{Z}_1,\dots,\mathrm{Z}_\nu \in \mathfrak{X}(T), \text{ such that the set } \{\mathrm{Z}_1,\dots,\mathrm{Z}_\nu,\mathrm{Y}_1,\dots,\mathrm{Y}_N\} \text{ is of finite type at } (t,x). \tag{8.3}$$

It follows from the fact that Θ is a Lie algebra containing $\mathrm{Z}_1,\dots,\mathrm{Z}_\nu,\mathrm{Y}_1,\dots,\mathrm{Y}_N$ that

$$\Theta_{(t,x)} \doteq \{\mathrm{Y}|_{(t,x)} \ : \ \mathrm{Y} \in \Theta\} = T_{(t,x)}(T \times G),$$

hence, $(\pi_G)_*\Theta_{(t,x)} = T_xG$, where $(\pi_G)_* : T_{(t,x)}(T\times G) \to T_xG$ is the projection map.

Proposition 8.5. *If* $(\pi_G)_*\Theta_{(t,x)} = T_xG$ *for some* $(t,x) \in T \times G$*, then* $\mathrm{Lie}\,\mathcal{L} = \mathfrak{g}$.

Proof. Given $\mathrm{X} \in \mathfrak{g}$ arbitrary, there exists $\mathrm{Y} \in \Theta$, such that $(\pi_G)_*\mathrm{Y}|_{(t,x)} = \mathrm{X}|_x$. Hence, for some $a_1,\dots,a_r \in \mathscr{C}^\infty(T;\mathbb{R})$ and $\mathrm{W} \in \mathfrak{X}(T)$, we have

$$\mathrm{Y}|_{(t,x)} = \sum_{j=1}^{r} a_j(t)\mathrm{L}_j|_x + \mathrm{W}|_t \Longrightarrow \mathrm{X}|_x = \sum_{j=1}^{r} a_j(t)\mathrm{L}_j|_x.$$

As two left-invariant vector fields are the same if they match at a single point, we conclude

$$\sum_{j=1}^{r} a_j(t)\mathrm{L}_j = \mathrm{X},$$

where the left-hand side belongs to $\mathrm{Lie}\,\mathcal{L}$ for each $t \in T$ fixed. □

Since $\mathfrak{g}$ is (GH) in G, we conclude from Lemma 5.1 that:

Corollary 8.6. *If* $\mathrm{Y}_1,\dots,\mathrm{Y}_N$ *satisfy* (8.3) *at some* $(t,x) \in T \times G$*, then* $\mathcal{L}$ *is* (GH) *in* G.

Yet, simple examples show that we may have $\mathrm{Lie}\,\mathcal{L} = \mathfrak{g}$ — which is *stronger* than $\mathcal{L}$ being (GH) in G — while the finite type condition fails at every point: back to Example 8.1, if $m \geq 2$ and $a_1,\dots,a_N$ have pairwise disjoint supports, then (8.3) for $\mathrm{Y}_\ell \doteq a_\ell(t)\mathrm{X}_\ell + \mathrm{W}_\ell$, $\ell \in \{1,\dots,N\}$, fails everywhere since no X_ℓ can generate the whole $\mathfrak{g}$.

8.3. A necessary condition based on Sussmann's orbits

Let M be a compact manifold, as in Section 1. We will now show a simple result which illustrates the connection between the topology of Sussmann's orbits of a system $\mathcal{L}$ of vector fields on M — or, rather, how they are immersed into the ambient manifold

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— and the global hypoellipticity of $\mathcal{L}$ in M. This has some interesting consequences (Corollary 8.8) which better contextualize the hypotheses of Theorem 3.5.

Recall that the *orbit* of $\mathcal{L}$ through x_0 is the set of all $x \in M$ enjoying the following property: there exists a continuous curve $\gamma : [0,\delta] \to M$ (for some $\delta > 0$) with endpoints $\gamma(0) = x_0$ and $\gamma(\delta) = x$ and a partition $0 = t_0 < t_1 < \cdots < t_\kappa = \delta$, such that on each open subinterval (t_j, t_{j+1}) — for $j \in \{0, \dots, \kappa - 1\}$ — the curve γ is $\mathscr{C}^1$ and an integral curve of some $\mathrm{L}_j \in \mathcal{L}$. We denote it by $\mathrm{Orb}_{\mathcal{L}}(x_0)$. Sussmann's Orbit Theorem [19] states that the orbits of $\mathcal{L}$ are all immersed connected submanifolds of M. If we assume that $M = G$ is a compact Lie group and $\mathcal{L} \subset \mathfrak{g}$ is a system of left-invariant vector fields on G, then one has a much more precise result (see, e.g. [18, Lemma 3.4]):

1. $\mathrm{Orb}_{\mathcal{L}}(e)$ is the connected Lie subgroup of G whose Lie algebra is $\mathrm{Lie}\,\mathcal{L} \subset \mathfrak{g}$; and
2. $\mathrm{Orb}_{\mathcal{L}}(x_0) = x_0 \cdot \mathrm{Orb}_{\mathcal{L}}(e)$ for every $x_0 \in G$.

In that case, the orbits are precisely the integral manifolds of the involutive distribution

$$\mathrm{Lie}_x\,\mathcal{L} \doteq \{\mathrm{X}|_x \ : \ \mathrm{X} \in \mathrm{Lie}\,\mathcal{L}\} \subset T_x G, \quad x \in G,$$

so these results are actually a consequence of Frobenius's Theorem.

Proposition 8.7. *If $\mathcal{L}$ is* (GH) *then all of its orbits are dense in G.*

Proof. It is enough to prove that $\mathrm{Orb}_{\mathcal{L}}(e)$ is dense in G, as the remaining orbits are left translations of it. Let $H \subset G$ denote its closure. It is certainly a subgroup of G, and since it is closed, it is a Lie subgroup of G. Moreover, the set G/H is a smooth manifold with dimension $\dim G - \dim H$, which is positive if one assumes that $H \neq G$, and the canonical projection $\pi : G \to G/H$ is a smooth submersion [17, Theorem 9.22]. In that case, let $v \in \mathscr{C}^1(G/H) \setminus \mathscr{C}^\infty(G/H)$ and take $u \doteq \pi^* v \in \mathscr{C}^1(G) \setminus \mathscr{C}^\infty(G)$. Then u is annihilated by every $\mathrm{X} \in \mathfrak{g}$ tangent to H, hence, in particular, by any $\mathrm{X} \in \mathcal{L}$ since

$$\mathcal{L} \subset \mathrm{Lie}\,\mathcal{L} \subset \mathfrak{h} \doteq \text{the Lie algebra of } H.$$

Thus, $\mathcal{L}$ would not be (GH). ☐

Having in mind condition (1) in Theorem 3.5, we would like to point out in our next result that one must be really careful when assigning hypotheses to P in order to ensure its global hypoellipticity: too strong ones may inadvertently also ensure that G must have been a torus to start with!

Corollary 8.8. *If G is a noncommutative Lie group and $\mathcal{L} \subset \mathfrak{g}$ is a family of pairwise commuting vector fields, then $\mathcal{L}$ cannot be* (GH).

Proof. Notice that $\mathrm{Lie}\,\mathcal{L}$ is a commutative Lie subalgebra of $\mathfrak{g}$, hence must be contained in a maximal commutative Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$. Let then $H \subset G$ be the unique connected Lie subgroup of G whose Lie algebra is $\mathfrak{h}$. It is certainly commutative (since so is $\mathfrak{h}$), and it must be closed thanks to the maximality of $\mathfrak{h}$. Because $\mathrm{Lie}\,\mathcal{L} \subset \mathfrak{h}$, we have that every vector field in $\mathcal{L}$ is tangent to H, hence $\mathrm{Orb}_{\mathcal{L}}(e) \subset H$ so

$$\overline{\mathrm{Orb}_{\mathcal{L}}(e)} \subset \overline{H} \neq G,$$

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as we are assuming G noncommutative. In particular, $\mathrm{Orb}_{\mathcal{L}}(e)$ is not dense in G and the conclusion follows from Proposition 8.7. ☐

9. Operators with mostly constant coefficients

In this final section, we explore other results ensuring global hypoellipticity of operators P, as in (3.1). Here, we allow more general "leading terms" Q, unlike Theorem 3.5 in which we have $Q = \Delta_T$, but paying the price of more restrictive assumptions on the vector fields $\mathfrak{a}_\ell(t,\mathrm{X})$. The following one is an extension of [3, Theorem 1.9].

Theorem 9.1. *Let P in (3.1) be of the form*

$$P = Q - \sum_{\ell=1}^{N'} \Big(\mathrm{L}_\ell + \mathrm{W}_\ell\Big)^2 - \sum_{\ell=N'+1}^{N} \Big(\mathfrak{a}_\ell(t,\mathrm{X}) + \mathrm{W}_\ell\Big)^2,$$

where Q is positive semidefinite in T — i.e. $\langle Q\psi,\psi\rangle_{L^2(T)} \geq 0$ for every $\psi \in \mathscr{C}^\infty(T)$ — $\mathrm{L}_1,\ldots,\mathrm{L}_{N'} \in \mathfrak{g}$ and $\mathrm{W}_1,\ldots,\mathrm{W}_N$ are skew-symmetric and such that $\tilde{P} = Q - \mathrm{W}_1^2 - \cdots - \mathrm{W}_N^2$ is elliptic. Assume, moreover, that

1. $\mathrm{W}_1,\ldots,\mathrm{W}_{N'}$ *commute with* Δ_T *and that*
2. *the system* $\{\mathrm{Y}_\ell \doteq \mathrm{L}_\ell + \mathrm{W}_\ell \ : \ \ell = 1,\ldots,N'\}$ *is* (GH) *in* $T \times G$.

Then P is (GH) *in $T \times G$.*

Remark 9.2. Property (2) above is stronger than $\mathcal{L}$ in (3.6) being (GH) in G as it clearly implies (3.5) — which is equivalent to the latter by Proposition 7.2 — independently of the remaining assumptions.

Proof. Hypothesis (1) ensures that $\mathrm{Y}_1,\ldots,\mathrm{Y}_{N'}$ commute with the full Laplace-Beltrami operator $\Delta = \Delta_T + \Delta_G$ on $T \times G$. Therefore, hypothesis (2) implies, by means of Proposition 5.2 (see also Proposition 2.2(4) and the results in Section 2), the following: there exist $C,R,\rho > 0$, such that for all $(\mu,\lambda) \in \sigma(\Delta_T) \times \sigma(\Delta_G)$ with $\mu + \lambda \geq R$, we have

$$\Big(\sum_{\ell=1}^{N'} \|\mathrm{Y}_\ell\varphi\|^2_{L^2(T\times G)}\Big)^{\frac{1}{2}} \geq C(1+\mu+\lambda)^{-\rho}\|\varphi\|_{L^2(T\times G)}, \quad \forall \varphi \in E^T_\mu \otimes E^G_\lambda. \tag{9.1}$$

Let $u \in \mathscr{D}'(T \times G)$ be such that $f \doteq Pu \in \mathscr{C}^\infty(G)$. Since we are assuming $\tilde{P}$ elliptic in T, we have by Corollary 4.2 that $\mathcal{F}^G_\lambda(u)$ is smooth for every $\lambda \in \sigma(\Delta_G)$. As $\mathrm{Y}_1,\ldots,\mathrm{Y}_{N'}$ commute with Δ, they behave well under both the partial Fourier projection maps, that is, including $\mathcal{F}^T$, and not only $\mathcal{F}^G$:

$$\|\mathrm{Y}_\ell\mathcal{F}^G_\lambda(u)\|^2_{L^2(T\times G)} = \sum_{\mu\in\sigma(\Delta_T)} \|\mathcal{F}^T_\mu(\mathrm{Y}_\ell\mathcal{F}^G_\lambda(u))\|^2_{L^2(T\times G)} = \sum_{\mu\in\sigma(\Delta_T)} \|\mathrm{Y}_\ell(\mathcal{F}^T_\mu\mathcal{F}^G_\lambda(u))\|^2_{L^2(T\times G)} \tag{9.2}$$

for $\ell \in \{1,\ldots,N'\}$, whatever $\lambda \in \sigma(\Delta_G)$.

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Now, let $s>0$. By Corollary 4.4, there exist $C_1>0$ and $\theta\in(0,1)$, such that

$$\|\mathcal{F}_\mu^T\mathcal{F}_\lambda^G(u)\|_{L^2(T\times G)}\le C_1(1+\mu+\lambda)^{-s-2n},\quad \forall(\mu,\lambda)\in\Lambda_\theta,$$

where $n=\dim\, T$ and $\Lambda_\theta\subset\sigma(\Delta_T)\times\sigma(\Delta_G)$ is as in (2.7). We look at its complement

$$\Lambda_\theta^c=\{(\mu,\lambda)\in\sigma(\Delta_T)\times\sigma(\Delta_G)\ :\ (1+\lambda)>(1+\mu)^\theta\},$$

where it holds that $1+\mu+\lambda<(1+\lambda)^{\frac{2}{\theta}}$ since $1/\theta>1$. Therefore, thanks to (9.1), we have, for $(\mu,\lambda)\in\Lambda_\theta^c$ with $\mu+\lambda\ge R$, that

$$\|\mathcal{F}_\mu^T\mathcal{F}_\lambda^G(u)\|_{L^2(T\times G)}^2\le C^{-2}(1+\lambda)^{\frac{4\rho}{\theta}}\sum_{\ell=1}^{N'}\|\mathrm{Y}_\ell(\mathcal{F}_\mu^T\mathcal{F}_\lambda^G(u))\|_{L^2(T\times G)}^2.$$

Fixing $\lambda\in\sigma(\Delta_G)$, we have by Remark 2.5 that

$$\|\mathcal{F}_\lambda^G(u)\|_{L^2(T\times G)}^2=\sum_{\substack{\mu\in\sigma(\Delta_T)\\(\mu,\lambda)\in\Lambda_\theta}}\|\mathcal{F}_\mu^T\mathcal{F}_\lambda^G(u)\|_{L^2(T\times G)}^2+\sum_{\substack{\mu\in\sigma(\Delta_T)\\(\mu,\lambda)\in\Lambda_\theta^c}}\|\mathcal{F}_\mu^T\mathcal{F}_\lambda^G(u)\|_{L^2(T\times G)}^2 \tag{9.3}$$

in which the first sum can be bounded by

$$\begin{aligned}\sum_{\substack{\mu\in\sigma(\Delta_T)\\(\mu,\lambda)\in\Lambda_\theta}}\|\mathcal{F}_\mu^T\mathcal{F}_\lambda^G(u)\|_{L^2(T\times G)}^2&\le\sum_{\substack{\mu\in\sigma(\Delta_T)\\(\mu,\lambda)\in\Lambda_\theta}}\|\mathcal{F}_\mu^T\mathcal{F}_\lambda^G(u)\|_{L^2(T\times G)}\frac{C_1}{(1+\mu+\lambda)^{s+2n}}\\&\le\frac{C_1}{(1+\lambda)^s}\|\mathcal{F}_\lambda^G(u)\|_{L^2(T\times G)}\sum_{\mu\in\sigma(\Delta_T)}\frac{1}{(1+\mu)^{2n}},\end{aligned} \tag{9.4}$$

where the latter series converges by Weyl's asymptotic formula (1.1).

For the second sum in (9.3), we define $\Lambda_{\theta,R}^c\doteq\{(\mu,\lambda)\in\Lambda_\theta^c\ :\ \mu+\lambda\ge R\}$: it follows that

$$\sum_{\substack{\mu\in\sigma(\Delta_T)\\(\mu,\lambda)\in\Lambda_{\theta,R}^c}}\|\mathcal{F}_\mu^T\mathcal{F}_\lambda^G(u)\|_{L^2(T\times G)}^2\le C_2(1+\lambda)^{\frac{4\rho}{\theta}}\sum_{\ell=1}^{N'}\sum_{\substack{\mu\in\sigma(\Delta_T)\\(\mu,\lambda)\in\Lambda_{\theta,R}^c}}\|\mathrm{Y}_\ell(\mathcal{F}_\mu^T\mathcal{F}_\lambda^G(u))\|_{L^2(T\times G)}^2$$

which can be further bounded by

$$\begin{aligned}\sum_{\ell=1}^{N'}\sum_{\substack{\mu\in\sigma(\Delta_T)\\(\mu,\lambda)\in\Lambda_{\theta,R}^c}}\|\mathrm{Y}_\ell(\mathcal{F}_\mu^T\mathcal{F}_\lambda^G(u))\|_{L^2(T\times G)}^2&\le\sum_{\ell=1}^{N'}\|\mathrm{Y}_\ell\mathcal{F}_\lambda^G(u)\|_{L^2(T\times G)}^2\\&\le\langle Q\mathcal{F}_\lambda^G(u),\mathcal{F}_\lambda^G(u)\rangle_{L^2(T\times G)}+\sum_{\ell=1}^{N'}\|\mathrm{Y}_\ell\mathcal{F}_\lambda^G(u)\|_{L^2(T\times G)}^2\\&\le\|\mathcal{F}_\lambda^G(f)\|_{L^2(T\times G)}\|\mathcal{F}_\lambda^G(u)\|_{L^2(T\times G)},\end{aligned}$$

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where we used Proposition 2.8, Lemma 3.1, and the fact that Q is positive semidefinite. But since f is smooth, Corollary 2.6 further implies the existence of a $C_3 > 0$, such that

$$\sum_{\substack{\mu\in\sigma(\Delta_T)\\ (\mu,\lambda)\in\Lambda^c_{\theta,R}}} \|\mathcal{F}^T_\mu\mathcal{F}^G_\lambda(u)\|^2_{L^2(T\times G)} \le C_3(1+\lambda)^{-s}\|\mathcal{F}^G_\lambda(u)\|_{L^2(T\times G)}. \tag{9.5}$$

Using the fact that $\Lambda^c_\theta\setminus\Lambda^c_{\theta,R}$ is finite, it follows from (9.3), (9.4), and (9.5) that

$$\|\mathcal{F}^G_\lambda(u)\|_{L^2(T\times G)} \le C_4(1+\lambda)^{-s}, \quad \forall\lambda\in\sigma(\Delta_G),$$

for some constant $C_4 > 0$, and the smoothness of u follows from Corollary 2.7. □

The next one is very similar and generalizes [1, Theorem 2].

Theorem 9.3. *Let P in (3.1) be of the form*

$$P = Q - \sum_{\ell=1}^{N'}\left(\mathrm{L}_\ell\right)^2 - \sum_{\ell=N'+1}^{N}\left(\mathfrak{a}_\ell(t,\mathrm{X}) + \mathrm{W}_\ell\right)^2,$$

where Q is positive semidefinite, $\mathrm{L}_1,\dots,\mathrm{L}_{N'}\in\mathfrak{g}$ and $\mathrm{W}_{N'+1},\dots,\mathrm{W}_N$ are skew-symmetric and such that $\tilde{P} = Q - \mathrm{W}^2_{N'+1} - \cdots - \mathrm{W}^2_N$ is elliptic. Assume, moreover, that the system $\{\mathrm{L}_1,\dots,\mathrm{L}_{N'}\}$ is (GH) *in G. Then P is* (GH) *in $T\times G$.*

Proof. By Proposition 5.2, there exist $C,\rho > 0$ and $\lambda_0\in\sigma(\Delta_G)$, such that the basic inequality (5.1) holds for $\{\mathrm{L}_1,\dots,\mathrm{L}_{N'}\}$ in G. In particular, for arbitrary $\psi\in\mathscr{C}^\infty(T)$ and $\phi\in E^G_\lambda$ with $\lambda\ge\lambda_0$, we have

$$\sum_{\ell=1}^{N'}\|\mathrm{L}_\ell(\psi\otimes\phi)\|^2_{L^2(T\times G)} = \sum_{\ell=1}^{N'}\|\psi\|^2_{L^2(T)}\|\mathrm{L}_\ell\phi\|^2_{L^2(G)} \ge C^2(1+\lambda)^{-2\rho}\|\psi\otimes\phi\|^2_{L^2(T\times G)}.$$

Selecting an orthonormal basis $\psi^\mu_1,\dots,\psi^\mu_{d^T_\mu}$ of E^T_μ, we may write any $\varphi\in E^T_\mu\otimes E^G_\lambda$ as

$$\varphi = \sum_{j=1}^{d^T_\mu}\psi^\mu_j\otimes\tilde{\varphi}_j, \quad \tilde{\varphi}_j\in E^G_\lambda,$$

and since the terms in the sum above are pairwise orthogonal in $L^2(T\times G)$, we have

$$\|\varphi\|^2_{L^2(T\times G)} = \sum_{j=1}^{d^T_\mu}\|\psi^\mu_j\otimes\tilde{\varphi}_j\|^2_{L^2(T\times G)},$$

hence also, in particular

$$\|\mathrm{L}_\ell\varphi\|^2_{L^2(T\times G)} = \sum_{j=1}^{d^T_\mu}\|\mathrm{L}_\ell(\psi^\mu_j\otimes\tilde{\varphi}_j)\|^2_{L^2(T\times G)}.$$

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We conclude that for $(\mu,\lambda) \in \sigma(\Delta_T) \times \sigma(\Delta_G)$ with $\lambda \geq \lambda_0$ and $\varphi \in E^T_\mu \otimes E^G_\lambda$, we have

$$\sum_{\ell=1}^{N'} \|\mathrm{L}_\ell \varphi\|^2_{L^2(T\times G)} \geq \sum_{j=1}^{d^T_\mu} C^2(1+\lambda)^{-2\rho} \|\psi^\mu_j \otimes \tilde{\varphi}_j\|^2_{L^2(T\times G)} = C^2(1+\lambda)^{-2\rho}\|\varphi\|^2_{L^2(T\times G)}. \tag{9.6}$$

Let $u \in \mathscr{D}'(T\times G)$ be such that $f \doteq Pu \in \mathscr{C}^\infty(G)$, so again, $\mathcal{F}^G_\lambda(u) \in \mathscr{C}^\infty(T; E^G_\lambda)$ for every $\lambda \in \sigma(\Delta_G)$. For each $\ell \in \{1,\dots,N'\}$, since L_ℓ is a left-invariant vector field on G, and as such commutes with Δ_G, we have that $\mathrm{Y}_\ell \doteq \mathrm{L}_\ell$ commutes with Δ, so again (9.2) holds. Therefore, for $(\mu,\lambda) \in \sigma(\Delta_T)\times\sigma(\Delta_G)$ with $\lambda \geq \lambda_0$, we have, by (9.6) and (9.2),

$$\|\mathcal{F}^T_\mu \mathcal{F}^G_\lambda(u)\|^2_{L^2(T\times G)} \leq C^{-2}(1+\lambda)^{2\rho} \sum_{\ell=1}^{N'} \|\mathcal{F}^T_\mu(\mathrm{L}_\ell \mathcal{F}^G_\lambda(u))\|^2_{L^2(T\times G)},$$

so summing both sides over $\mu \in \sigma(\Delta_T)$ yields

$$\|\mathcal{F}^G_\lambda(u)\|^2_{L^2(T\times G)} \leq C^{-2}(1+\lambda)^{2\rho}\|\mathcal{F}^G_\lambda(f)\|_{L^2(T\times G)}\|\mathcal{F}^G_\lambda(u)\|_{L^2(T\times G)}$$

for every $\lambda \geq \lambda_0$, where we proceed as in the previous theorem; as such, we conclude smoothness of u, keeping in mind the finiteness of the set $\{\lambda \in \sigma(\Delta_G) \,:\, \lambda < \lambda_0\}$. □

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