THE TOEPLITZ OPERATOR PROOF
OF NONCOMMUTATIVE BOTT PERIODICITY

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Abstract

We adapt the Toeplitz operator proof of Bott periodicity to give a short direct proof of Bott periodicity for the representable K-theory of σ-C^*-algebras. We further show how the use of this proof and the right definitions simplifies the derivation of the basic properties of representable K-theory.

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Introduction

In this paper, we give a short and direct proof of Bott periodicity for the representable K-theory of σ-C^*-algebras [12], using the Toeplitz operator method of [2]. We use definitions for representable K-theory which first appeared in [13] and which are closely related to the usual definition of K-theory for C^*-algebras, and our proof is not much more complicated than it would be for C^*-algebras. Thus, this paper can also be regarded as an exposition of the Toeplitz operator proof of Bott periodicity for the K-theory of C^*-algebras. We also include direct proofs that our definitions of representable K-theory agree with the usual K-theory of C^*-algebras, and show how the definitions

we use simplify the proofs of the basic properties of representable K-theory.

The literature contains two elementary proofs of Bott periodicity for the K-theory of compact spaces, one involving approximation arguments on clutching functions [3] and one by Atiyah using Toeplitz operators [2]. The first of these proofs generalizes nicely to the K-theory of Banach algebras: see the exposition in [19, Section 9]. The resulting proof, however, seems unnatural for C^*-algebras, because it uses approximation arguments on invertible elements that take one out of the group of unitary elements of a C^*-algebra. The Toeplitz operator proof, on the other hand, can be done for C^*-algebras using only unitaries and projections. This proof is well known, but the C^*-algebra case has apparently never been published. The closest approximation we know in the literature is due to Cuntz [5]. That version is short, slick, and applies to a more general situation, but obscures the essential simplicity of the ideas involved. Section 3 of this paper gives essentially the noncommutative version of Atiyah's Toeplitz operator proof, including a remark about the simplification possible when one considers C^*-algebras instead of σ-C^*-algebras.

We now turn to a σ-C^*-algebras. Two proofs of Bott periodicity for representable K-theory have appeared: one in Weidner's thesis [20] (see also [21]), which actually applies much more generally but uses much complicated machinery, and one in [12], which is somewhat indirect, using the Milnor $\varprojlim^1$-sequence to reduce to the case of C^*-algebras. The approximation proof in [19, Section 9] does not generalize to σ-C^*-algebras, because the group of invertible elements in a unital σ-C^*-algebra is not open (see [11, Proposition 1.14]). However, as we show, the Toeplitz operator proof, which uses only unitaries and projections, generalizes very nicely to the σ-C^*-algebra case, provided that one uses the right definitions.

The definitions we use are taken from [13]. We believe that these definitions are the right ones to use, and as evidence for that belief we offer their obvious connection with the usual definitions of K-theory for C^*-algebras, the fact that they work well in the proof given here, the ease with which the main properties of representable K-theory are derived from them, and the fact that they generalize to Fréchet algebras ([16]). We show in Section 4 of this paper how the development of [12] is simplified if one uses the definitions of this paper.

The proof of Bott periodicity given here also serves two other purposes. One is to provide a proof of Bott periodicity which generalizes to Fréchet algebras. As demonstrated in [16], the proof given here does so very nicely; as far as we know, no other proof does. For the second purpose, recall that the main theorem of [14] asserted the existence of homotopy equivalences

$\Omega_\infty U_{nc} \simeq P$ and $\Omega_\infty P \simeq U_{nc}$ involving the C^∞ loop algebras of the representing algebras P and U_{nc} for K-theory. (The connection between this result and Bott periodicity is explained in [15].) The proof given there was very indirect, relying eventually on the Bott periodicity theorem in [12]. We plan, in a future paper, to give direct proofs of these homotopy equivalences, and our proposed direct proofs are closely modelled on the proofs in this paper. Partly because of this application, we avoid Atiyah's trick ([1, Section 1]) in our proof, giving instead at the appropriate point a direct calculation.

For readers interested only in C^*-algebras, the logical prerequisites of this paper are only the standard definitions of $K_0(A)$ and $K_1(A)$ in terms of projections and unitaries. (See for example Blackadar's book [4].) For readers interested in σ-C^*-algebras, some of their basic properties are needed; see [11, Sections 1, 3, and 5]. We also need the σ-C^*-algebra versions of two basic lemmas on unitaries and projections, which appear as isolated lemmas in two of our earlier papers; these are stated at the end of Section 1.

This paper is organized as follows. In Section 1, we state the definitions of K-theory that we use, prove from scratch that they agree with the usual groups for a C^*-algebra, and state several necessary lemmas. Sections 2 and 3 contain the proofs of the "easy" and "hard" halves of Bott periodicity, including, in Section 3, an explanation of the simplifications that appear in the C^*-algebra case. Section 4 shows how to simplify the derivation of the basic properties of representable K-theory, including the equivariant case, by using the approach of the first three sections.

1. Preliminaries

In this section, we give our definitions of representable K-theory, and prove from scratch that they give the usual groups for C^*-algebras. We also give the analogs for σ-C^*-algebras of several standard lemmas on C^*-algebras.

Throughout this paper, K is the algebra of compact operators on a fixed separable infinite dimensional Hilbert space, and A^+ is the unitization of a σ-C^*-algebra A. If A is unital, then $U(A)$ is the unitary group and $U_0(A)$ is the path component of 1 in $U(A)$.

1.1 DEFINITION. Let A be a σ-C^*-algebra. Then $RK_0(A)$ is the set of homotopy classes in the set of projections $p \in M_2((K \otimes A)^+)$ such that $p - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in M_2(K \otimes A)$. Also, $RK_1(A)$ is the set of homotopy classes in the set of unitaries $u \in (K \otimes A)^+$ such that $u - 1 \in K \otimes A$.

In both sets, we define addition to be direct sum, followed for RK_0 by

conjugation by the unitary

$$c = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

followed in both cases by an identification of $M_2(K)$ with K.

If $\varphi: A \to B$ is a homomorphism of $\sigma\text{-}C^*$-algebras, then we define $\varphi_*: RK_i(A) \to RK_i(B)$ by $\varphi([p]) = [(\mathrm{id}_{M_2(K)} \otimes \varphi)^+(p)]$ for $i = 0$ and $\varphi([u]) = [(\mathrm{id}_K \otimes \varphi)^+(u)]$ for $i = 1$.

The definition of RK_0 is taken from [13, Definition 2.2]; note that c is chosen so as to conjugate $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ to $1 \oplus 0$. The definition of RK_1 is suggested by [13, Definition 3.3], with modifications to make it more closely resemble the definition of RK_0. Note that we do not need separate definitions for nonunital A. This simplifies the bookkeeping in some proofs.

1.2 REMARK. We will use $K_0(A)$ and $K_1(A)$ to refer to the K-theory of a C^*-algebra A, defined as in, for example, [4, Sections 5.5 and 8.1]. In the definition of $K_0(A)$, we will use projections and either unitary equivalence or homotopy as appropriate (see [4, Section 4.6]).

It follows from [13] and [12] that RK_* as defined here is the same as in our earlier papers. It therefore follows that $RK_i(A) \cong K_i(A)$ for C^*-algebras A. We will, however, give a selfcontained proof in this section. Note that the definition of $RK_1(A)$ is very close to the usual definition of $K_1(A)$ for A unital. The definition of $RK_0(A)$ is not quite so close to the usual definition of $K_0(A)$, but the examples in [12, Section 4] show that one can't expect to do better.

For completeness, we show here that $RK_0(A)$ and $RK_1(A)$ really are functorial abelian groups.

1.3 LEMMA. $RK_0(A)$ and $RK_1(A)$ are abelian semigroups with identities $[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}]$ and $[1]$. The map φ_* is a well defined semigroup homomorphism which depends only on the homotopy class of φ.

PROOF. It is immediate that addition in both sets is well defined. (It does not depend on the choice of the isomorphism $M_2(K) \cong K$, because any two are homotopic.) That the identities are as claimed is proved by using a homotopy from an isomorphism $K \to M_2(K)$ to the inclusion $a \mapsto a \oplus 0$ of K in $M_2(K)$. Associativity is proved similarly. Commutativity is obtained via the standard homotopy from $a \oplus b$ to $b \oplus a$. The statements about φ_* are trivial.

1.4 PROPOSITION. $RK_0(A)$ is a functorial homotopy invariant abelian group.

PROOF. It only remains to prove that inverses exist. We claim that $-[p] = \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} (1-p) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right]$. Eliminating the conjugation by c in the definition of addition, we see that it suffices to find a homotopy of projections from $p \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} (1-p) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ to $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, differing from the latter by elements of $M_4(K \otimes A)$. Equivalently, we produce a homotopy from $p \oplus (1-p)$ to $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, differing from the latter by elements of $M_4(K \otimes A)$.

Let $k = \pi/2$ and define

$$
w_t = \left[\begin{array}{cc|cc} \cos(kt) & 0 & \sin(kt) & 0 \\ 0 & \cos(kt) & 0 & \sin(kt) \\ \hline -\sin(kt) & 0 & \cos(kt) & 0 \\ 0 & -\sin(kt) & 0 & \cos(kt) \end{array} \right],
$$

which is a unitary path in $M_4(\mathbb{C})$ from $w_0 = 1$ to

$$
w_1 = \left[\begin{array}{cc|cc} 0 & & 1 & 0 \\ & & 0 & 1 \\ \hline 1 & 0 & & \\ 0 & 1 & & 0 \end{array} \right].
$$

Then $w_t(0 \oplus (1-p))w_t^*$ is orthogonal to $p \oplus 0$ for all t, and similarly with $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ in place of p. Choose a unitary path $t \mapsto x_t$ in $M_4(\mathbb{C})$ with $x_0 = 1$ and

$$(*) \quad x_t \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \oplus 0 + w_t \left[0 \oplus \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right] w_t^* \right] x_t^* = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

(See the C^*-algebra version of Lemma 1.9 below.) Define

$$q_t = x_t[p \oplus 0 + w_t(0 \oplus (1-p))w_t^*]x_t^*.$$

Then $q_0 = p \oplus (1-p)$, $q_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ (using $(*)$ at $t = 1$), and it follows from $(*)$ that $q_t - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in M_4(K \otimes A)$ for all t. Therefore $t \mapsto q_t$ is the required homotopy.

1.5 PROPOSITION. $RK_1(A)$ is a functorial homotopy invariant abelian group, in which $[u] + [v] = [uv]$.

PROOF. We only have to show that $[u]+[v] = [uv]$ for $u, v \in U((K \otimes A)^+)$ with $u - 1$, $v - 1 \in K \otimes A$. It suffices to prove that $[u \oplus v] = [uv \oplus 1]$, since we already know that $[1]$ is the identity. The standard construction yields a

homotopy $t \mapsto z_t$ from $u \oplus v$ to $uv \oplus 1$ which satisfies $z_t - 1 \in M_2(K \otimes A)$ for all t.

We now prove the isomorphisms $RK_i(A) \cong K_i(A)$ for C^*-algebras.

1.6 PROPOSITION. *For any* C^*-*algebra* A, *there is a natural isomorphism* $RK_0(A) \cong K_0(A)$.

PROOF. Let p_0 denote the projection $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, in the 2×2 matrices over any unital C^*-algebra. Stability and the split exact sequence

$$0 \to K \otimes A \to (K \otimes A)^+ \xrightarrow{\pi} \mathbb{C} \to 0$$

give an isomorphism $K_0(A) \cong \mathrm{Ker}(\pi_*) \subset K_0((K \otimes A)^+)$. Define $\Phi : RK_0(A) \to \mathrm{Ker}(\pi_*)$ by $\Phi([p]) = [p] - [p_0]$. Then Φ is well defined because homotopic projections are equivalent, and the image of Φ really is in $\mathrm{Ker}(\pi_*)$ because $M_2(\pi)(p) = p_0 \in M_2(\mathbb{C})$. We now show that Φ is a homomorphism. Let $\iota : \mathbb{C} \to M_2(\mathbb{C})$ be the inclusion in the upper left corner. Note that ι determines a homomorphism $(\iota \otimes \mathrm{id}_A)^* : RK_0(A) \to RK_0(M_2(A))$ and an isomorphism $(\iota \otimes \mathrm{id}_{K \otimes A})^* : K_0(K \otimes A) \to K_0(M_2(K \otimes A))$ such that $(\iota \otimes \mathrm{id}_{K \otimes A})^* \circ \Phi_A = \Phi_{M_2(A)} \circ (\iota \otimes \mathrm{id}_A)^*$. Therefore it suffices to show that

$(*)$ $\Phi_{M_2(A)} \circ (\iota \otimes \mathrm{id}_A)^*([p_1] + [p_2]) = (\iota \otimes \mathrm{id}_{K \otimes A})^*(\Phi([p_1]) + \Phi([p_2]))$.

Let $\varphi : M_2(K) \to K$ be an isomorphism, and let c be as in Definition 1.1, so that $[p_1] + [p_2] = [M_2((\varphi \otimes \mathrm{id}_A)^+)(c(p_1 \oplus p_2)c^*)]$. Now $(\iota \otimes \mathrm{id}_K) \circ \varphi$ is homotopic to $\mathrm{id}_{M_2(K)}$.   Therefore the left hand side of $(*)$ is $[c(p_1 \oplus p_2)c^*] - [c(p_0 \oplus p_0)c^*]$. Working in $M_4((K \otimes A)^+)$, we see that this is equal to $[p_1 \oplus p_2] - [p_0 \oplus p_0]$, which in turn is equal to the left hand side of $(*)$.

To check that Φ is surjective, represent an arbitrary class $\eta \in K_0(A)$ as $[p] - [p_0]$, where $p = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} \in M_2(M_n(A^+))$ and $p - p_0 \in M_2(M_n(A))$. (See the remark following [4, Definition 5.5.1].) Then $e = \begin{pmatrix} p_{11} \oplus 1 & p_{12} \\ p_{21} & p_{22} \end{pmatrix}$ is a projection in $M_2((K \otimes A)^+)$ with $e - p_0 \in M_2(K \otimes A)$, and $\Phi([e]) = \eta$.

Finally, we prove that Φ is injective. Let $\Phi([p]) = 0$. Then in some $M_n(M_2((K \otimes A)^+))$, there is a homotopy $t \mapsto q_t$ from $q_0 = p \oplus p_0 \oplus \cdots \oplus p_0$ to $q_1 = p_0 \oplus \cdots \oplus p_0$ (n summands in both expressions). Let $\pi : M_{2n}((K \otimes A)^+) \to M_{2n}(\mathbb{C})$ be the quotient map. Find a continuous unitary path $t \mapsto u_t$ in $M_{2n}(\mathbb{C})$ with $u_0 = 1$ and $u_t \pi(q_t) u_t^* = \pi(q_0)$. (Note that $\pi(q_0) = \pi(q_1)$ since $p - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in M_2(K \otimes A)$.) Replace q_t by $u_t q_t u_t^*$, regarding u_t as being in $M_{2n}((K \otimes A)^+)$. This does not change either q_0 or

q_1, but now $q_t - p_0 \oplus \cdots \oplus p_0 \in M_{2n}(K \otimes A)$ for all t. This homotopy now shows that $[p] + (n-1)[p_0] = 0$ in $RK_0(A)$. So $[p] = 0$.

1.7 PROPOSITION. *For any C^*-algebra A, there is a natural isomorphism $RK_1(A) \cong K_1(A)$.*

PROOF. See the remarks following [4, Definition 8.1.1].

The following two lemmas are the analogs for σ-C^*-algebras of two lemmas which are important in the K-theory of C^*-algebras. We state them here for ease of reference, but refer to our earlier papers for the proofs. It should be noted that the proofs do not depend on the theory developed in those papers.

1.8 LEMMA ([12, LEMMA 1.11]). *Let $A \to B$ be a surjective homomorphism of unital σ-C^*-algebras. Then $U_0(A) \to U_0(B)$ is surjective.*

1.9 LEMMA ([13, LEMMA 2.4]). *Let A be a unital σ-C^*-algebra, and let $t \mapsto p_t$ be a continuous path of projections in A. Then there is a continuous path $t \mapsto u_t$ of unitaries in A such that $u_0 = 1$ and $u_t p_0 u_t^* = p_t$. If $p_t - p_0$ is in an ideal I for all t, then we can take $u_t - 1 \in I$.*

Finally, we record here a fact which has been used implicitly elsewhere and will be needed later in this paper. Call a σ-C^*-algebra nuclear if it is the inverse limit of a system of nuclear C^*-algebras with surjective maps, as in the remarks following [11, Proposition 3.3].

1.10 LEMMA. *Let $0 \to I \to A \to B \to 0$ be an exact sequence of σ-C^*-algebras, and let D be a σ-C^*-algebra. If A or D is nuclear, then $0 \to I \otimes D \to A \otimes D \to B \otimes D \to 0$ is exact.*

PROOF. Write the exact sequence $0 \to I \to A \to B \to 0$ as an inverse limit of exact sequences $0 \to I_n \to A_n \to B_n \to 0$ of C^*-algebras, with all maps in the inverse systems being surjective ([11, Proposition 5.3(2)]). Also write $D = \varprojlim D_n$ with all maps surjective. Then each sequence $0 \to I_n \otimes D_n \to A_n \otimes D_n \to B_n \otimes D_n \to 0$ is exact, since A_n or D_n is nuclear. Also the maps $I_{n+1} \otimes D_{n+1} \to I_n \otimes D_n$, etc., are all surjective, so by [11, Proposition 5.3(2)], the inverse limit $0 \to I \otimes D \to A \otimes D \to B \otimes D \to 0$ of these sequences is exact.

2. The Isomorphism of $K_0(SA)$ with $K_1(A)$

In this section we prove the easy half of Bott periodicity. No Toeplitz operators will appear here.

In the K-theory of C^*-algebras, this part is usually obtained from the exact sequence

$$(*) \qquad K_1(I) \to K_1(B) \to K_1(B/I) \xrightarrow{\partial} K_0(I) \to K_0(B) \to K_0(B/I)$$

corresponding to an exact sequence $0 \to I \to B \to B/I \to 0$. One sets $I = SA = C_0((0, 1)) \otimes A$, $B = CA = C_0([0, 1)) \otimes A$, and $B/I = A$, and observes that $K_0(B) = K_1(B) = 0$ because B is contractible. (See for example [19, Section 8].) Here we will follow the approach used in the proof of [4, Theorem 8.2.2], and give a direct proof that $RK_0(SA) \cong RK_1(A)$. An examination of the connecting homomorphism in $(*)$ shows, however, that our proof is essentially the same as the one in [19].

2.1 DEFINITION. Let A be a σ-C^*-algebra. Define $\lambda: RK_1(A) \to RK_0(SA)$ as follows. Given $u \in U((K \otimes A)^+)$ with $u - 1 \in K \otimes A$, choose some continuous path $t \mapsto v(t)$ in $U(M_2((K \otimes A)^+))$ with $v(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $v(1) = \begin{pmatrix} u & 0 \\ 0 & x \end{pmatrix}$ for some x, and $v(t) - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in M_2(K \otimes A)$ for all t. (For example, we could take

$$(*)$$
$$v(t) = \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(kt) & \sin(kt) \\ -\sin(kt) & \cos(kt) \end{pmatrix} \begin{pmatrix} u^* & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(kt) & \sin(kt) \\ -\sin(kt) & \cos(kt) \end{pmatrix}^*,$$

with $k = \pi/2$, giving $x = u^*$.) Define $p(t) = v(t) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} v(t)^*$. As we show in Lemma 2.3 below, p defines a class in $RK_0(SA)$, and we take $\lambda([u])$ to be that class.

Define $\nu: RK_0(SA) \to RK_1(A)$ as follows. Let $p \in M_2((K \otimes SA)^+)$ be a projection such that $p - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in M_2(K \otimes SA)$. Regard p as a function from $[0, 1]$ to $M_2((K \otimes A)^+)$, with $p(0) = p(1) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $p(t) - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in M_2(K \otimes A)$. Use Lemma 1.9 to find a unitary path $t \mapsto v(t)$ in $M_2((K \otimes A)^+)$ such that $v(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $v(t) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} v(t)^* = p(t)$, and $v(t) - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in M_2(K \otimes A)$. Then $v(1) = \begin{pmatrix} u & 0 \\ 0 & x \end{pmatrix}$ for some unitaries $u, x \in (K \otimes A)^+$, and we define $\nu([p])$ to be the class of u in $RK_1(A)$.

The following lemma is needed to prove that λ is well defined.

2.2 LEMMA. Let A be a pro-C^*-algebra, let $u \in U((K \otimes A)^+)$ satisfy $u - 1 \in K \otimes A$, and let $t \mapsto v_0(t)$ and $t \mapsto v_1(t)$ be two unitary paths

in $M_2((K \otimes A)^+)$ with $v_i(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $v_i(1) = \begin{pmatrix} u & 0 \\ 0 & x_i \end{pmatrix}$, and $v_i(t) - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in$ $M_2(K \otimes A)$. Then there exists a continuous function $(s, t) \mapsto v_s(t) \in$ $U(M_2((K \otimes A)^+))$ with $v_s(t)$ as given for $s = 0$ and $s = 1$, with $v_s(1) = \begin{pmatrix} u & 0 \\ 0 & x_s \end{pmatrix}$, and with $v_s(t) - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in M_2(K \otimes A)$ for all $s, t \in [0, 1]^2$.

PROOF. Multiplying $v_s(t)$ by $v_0(t)^*$ for $s = 0, 1$, we reduce to the case in which $u = 1$ and $v_0(t) = 1$ for all t. Now let $\varphi: M_2(K \otimes A)^+ \to (K \otimes A)^+$ be an isomorphism determined by an isomorphism $\varphi_0: M_2(K) \to K$. Note that $\mathrm{id}_{M_2(K)}$ is homotopic to $k \mapsto 0 \oplus \varphi_0(k)$, and furthermore that the homotopy can be chosen to send $0 \oplus K$ into $0 \oplus K$ (as subsets of $M_2(K)$) at every stage. This homotopy gives a homotopy $(s, t) \mapsto \overline{v}_s(t)$ from $t \mapsto v_1(t)$ to $t \mapsto 1 \oplus \varphi(v_1(t))$. The condition on $0 \oplus K$ ensures that $\overline{v}_s(1)$ has the form $1 \oplus \overline{x}_s$ for every s. (Thus, $s \mapsto \overline{x}_s$ is a homotopy from x_1 to 1.) The other conditions of the lemma are trivially satisfied for the homotopy $s \mapsto \overline{v}_s$, and thus we may assume, in addition to the above, that $v_1(t) = 1 \oplus w(t)$ for some unitary function w with $w(0) = 1$ and $w(t) - 1 \in K \otimes A$. Now we simply set $u_s(t) = 1 \oplus w(st)$, a homotopy which obviously satisfies the condition required and gives $v_0 = 1$ as required.

2.3 LEMMA. *The maps λ and ν of Definition 2.1 are well defined natural group homomorphisms.*

PROOF. The only issue is well definedness. In the definition of λ, an easy calculation proves that $p(t) - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ is in $M_2(K \otimes A)$ for all t, and equal to 0 for $t = 0, 1$. It follows that $p \in M_2((K \otimes SA)^+)$ and $p - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in$ $M_2(K \otimes SA)$, so that p defines a class in $RK_0(SA)$. Next, if v_0 and v_1 are two choices for v, let $v_s(t)$ be as in the previous lemma, and observe that the homotopy $s \mapsto p_s = v_s \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} v_s^*$ shows that $\lambda([u])$ does not depend on the choice of v. Finally, if $s \mapsto u_s$ is a homotopy, then we can regard it as an element $u \in U([K \otimes C([0, 1]) \otimes A]^+)$ such that $u - 1 \in K \otimes C([0, 1]) \otimes A$. Choose v as in the definition, for this u; then the homotopy $s \mapsto p_s$, with $p_s(t) = v_s(t) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} v_s(t)^*$, shows that $\lambda([u_0]) = \lambda([u_1])$. This completes the proof that λ is well defined.

Now we show that ν is well defined. It is clear that $u - 1 \in K \otimes A$ in the definition, so that u defines an element of $RK_1(A)$. Now let v_0 and v_1 be two different choices for v, with $v_i(1) = \begin{pmatrix} u_i & 0 \\ 0 & x_i \end{pmatrix}$. Clearly $v_0(t)^* v_1(t)$ commutes with $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, and so has the form $t \mapsto y(t) \oplus z(t)$. Then $t \mapsto u_0 y(t)$ is a homotopy from u_0 to u_1; thus, $[u_0] = [u_1]$, and $\nu([p])$ does not depend

on v. That $\nu([p])$ depends only on the homotopy class of p is now proved just as for λ.

2.4 THEOREM. *There is a natural isomorphism* $RK_0(SA) \cong RK_1(A)$.

PROOF. We only have to prove that $\lambda \circ \nu$ and $\nu \circ \lambda$ are the identity maps. To do so, simply use the same v in both parts of Definition 2.1.

3. The Isomorphism of $K_1(SA)$ with $K_0(A)$

Here we prove the more difficult half of Bott periodicity, using Toeplitz operators. In this section we take the suspension SA to be $C_0(S^1 - \{1\}) \otimes A$, regarded as a subalgebra of $C(S^1) \otimes A$ in the obvious way, where S^1 is the unit circle in $\mathbb{C}$. Also, z will denote the function $z(\zeta) = \zeta$, regarded as an element of $C(S^1)$ or of $L^2(S^1)$.

3.1 DEFINITION. Let A be a σ-C^*-algebra. Define $\beta: RK_0(A) \to RK_1(SA)$ as follows. Set $p_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in M_2((K \otimes A)^+)$. For $p \in M_2((K \otimes A)^+)$ with $p - p_0 \in M_2(K \otimes A)$, define $u \in U(M_2(K \otimes SA)^+)$ by $u(\zeta) = (1 - p + \zeta p)(1 - p_0 + \zeta p_0)^*$. Identify $M_2(K \otimes SA)^+$ with $(K \otimes SA)^+$ via some isomorphism $M_2(K) \to K$, and define $\beta([p])$ to be the class of u in $RK_1(SA)$.

Before defining the inverse map, we introduce some notation. Let μ_0 be the representation of $C(S^1)$ on $L^2 = L^2(S^1)$ by multiplication operators, and let μ denote the obvious extension to a homomorphism from $C(S^1) \otimes (K \otimes A)^+$ to the multiplier algebra $M(K(L^2) \otimes (K \otimes A)^+)$. Let q_0 be the projection from L^2 to the Hardy space H^2, which is the closed linear span of the functions z^n for $n \geq 0$. Also let $q = q_0 \otimes 1$ be the corresponding projection in $M(K(L^2) \otimes (K \otimes A)^+)$. Using some isomorphism of $(H^2)^\perp$ with H^2, we identify $K(L^2) \otimes (K \otimes A)^+$ with $M_2(K(H^2) \otimes (K \otimes A)^+)$, and also make the corresponding identification of the multiplier algebras. Thus q becomes $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Let

$$\bar{\mu}: C(S^1) \otimes (K \otimes A)^+ \to M_2(M(K(H^2) \otimes (K \otimes A)^+))$$

be the composition of μ with the identification just made.

Now define $\gamma: RK_1(SA) \to RK_0(A)$ as follows. Let $v \in U((K \otimes SA)^+)$ with $v - 1 \in K \otimes SA$. View $(K \otimes SA)^+$ as a subalgebra of $C(S^1) \otimes (K \otimes A)^+$. Define $p = \bar{\mu}(v)^* \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \bar{\mu}(v)$. As we prove in Lemma 3.2 below,

$p \in M_2([K(H^2) \otimes K \otimes A]^+)$ and $p - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in M_2(K(H^2) \otimes K \otimes A)$. Identify $K(H^2) \otimes K \otimes A$ with $K \otimes A$ via an isomorphism $K(H^2) \otimes K \to K$, and let $\gamma([v])$ be the resulting class of p in $RK_0(A)$.

3.2 LEMMA. *The maps β and γ of Definition 3.1 are well defined natural group homomorphisms.*

PROOF. Again, the only issue is well definedness. We do β first. Using the notation of the definition, we first have to check that $u(\zeta) - 1 \in M_2(K \otimes A)$ for $\zeta \in S^1$, and $u(1) = 1$. The last is clear; for the first, a calculation shows that

$$u(\zeta) - 1 = (\zeta - 1)p(p - p_0) + (1 - \bar{\zeta})(p - p_0)p_0,$$

which is in $M_2(K \otimes A)$ because $p - p_0$ is. It is now clear that a homotopy of projections yields a corresponding homotopy of unitaries, so that β is in fact well defined.

We now turn to γ. We first have to check, in the notation of the definition, that $p \in M_2([K(H^2) \otimes K \otimes A]^+)$ and that $p - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in M_2(K(H^2) \otimes K \otimes A)$. The first relation will follow from the second, and the second is equivalent to $\mu(v)^* q \mu(v) - q \in K(L^2) \otimes K \otimes A$. Since $\mu(1) = 1$, we get

(∗) $\mu(v)^* q \mu(v) - q = \mu(v)^* [q \mu(v - 1) - \mu(v - 1)q]$.

Now for $f \in C(S^1)$ it is a standard fact that $q_0 \mu_0(f) - \mu_0(f)q_0 \in K(L^2)$. (To prove this: a calculation shows that $q_0 \mu_0(z^n) - \mu_0(z^n)q_0$ is an operator of rank $|n|$, so the general result follows by continuity.) It follows that $q\mu(a) - \mu(a)q \in K(L^2) \otimes K \otimes A$ for $a \in C(S^1) \otimes K \otimes A$. Taking $a = v - 1$, we see that the right-hand side of (∗) is in $K(L^2) \otimes K \otimes A$, as desired.

Next, observe that $[p]$ does not depend on the choices of the isomorphisms $(H^2)^\perp \cong H^2$ or $K(H^2) \otimes K \cong K$, since in both cases, any two isomorphisms are homotopic. Finally, the construction of p also converts a homotopy of unitaries into a homotopy of projections, showing that $\gamma([v])$ as defined depends only on $[v]$.

3.3 REMARK. We will show in the next two lemmas that $\gamma = \beta^{-1}$. Since some of the calculations necessary in the σ-C^*-algebra case slightly obscure the essential simplicity of the proof, we describe the C^*-algebra case here.

Let A be a unital C^*-algebra, and let $p \in A$ be a projection. Then the unitary in the definition of β is the direct sum of a (large) identity and the unitary $u(\zeta) = 1 - p + \zeta p$ in $(SA)^+$. We have $\mu(u) = 1 \otimes (1-p) + \mu_0(z) \otimes p$, where $\mu_0(z) \in L(L^2)$ is the bilateral shift (multiplication by z). Thus,

$\mu(u)^* q \mu(u) = q + e \otimes p$, where $e \in L(L^2)$ is the rank one projection onto the space $\mathbb{C} \cdot z^{-1}$. Therefore $\gamma([u]) = [e \otimes p] = [p]$, that is, $\gamma \circ \beta([p]) = [p]$. It is easy enough to replace A by $M_n(A)$, and to extend to classes of the form $[p_1] - [p_2]$ (it is convenient to take $p_1 \perp p_2$), so as to conclude in the C^*-algebra case that $\gamma \circ \beta$ is the identity. (One uses the usual unitization method to deal with the nonunital case.)

To see the connection with Toeplitz operators, consider the Toeplitz operator $t = q\mu(u)q$ associated to u, for some unitary $u \in (SA)^+$. It is Fredholm in the appropriate sense because u is invertible. If $u = 1 - p + zp$ as above, then $t = q_0 \otimes (1 - p) + s \otimes p$, where $s = q_0 \mu_0(z)q_0$ is the unilateral shift. Thus, t is a partial isometry with index $[1 - t^*t] - [1 - tt^*] = -[p]$. (See [18] for the appropriate notion of index.) It is easily shown in general that $\gamma([u]) = -\text{index}(t) \in K_0(A)$.

In the C^*-algebra case, the proof of Bott periodicity is now most easily completed by adapting Atiyah's trick from [1, Section 1]. The first thing to notice is that the internal products appearing in [1] are unnecessary, and that the external product $K_0(A) \times K_0(B) \to K_0(A \otimes_{\min} B)$ is easily defined by $([p], [q]) \mapsto [p \otimes q]$ for unital A, B, and extended to the nonunital case in the usual way. It follows that any natural map respects external products. One then proves that $\lambda \circ \beta \colon K_0(A) \to K_0(S^2 A)$ is given by the external product with the class $(\lambda \circ \beta)([1]) \in K_0(S^2\mathbb{C})$, which is quite easy in the context of the usual definitions of K-theory for $\mathbb{C}^*$-algebras. Atiyah's argument now shows that the known relation $(\gamma \circ \nu) \circ (\lambda \circ \beta) = \text{id}$ for any A, combined with the nautrality of $\lambda \circ \beta$ and $\gamma \circ \nu$, implies that $(\lambda \circ \beta) \circ (\gamma \circ \nu)$ is also the identity.

In the σ-C^*-algebra case, the factor $(1 - p_0 + \zeta p_0)$ complicates the calculations, because we cannot require that p commute with p_0. (The algebra $C(X)$ in [12, Example 4.9] is a counterexample.) The argument for $\gamma \circ \beta = \text{id}$ is, however, essentially the same. In the argument for $\beta \circ \gamma = \text{id}$, the formulas for external products are not so nice, and provide an additional complication. We find it simpler to give a direct proof. This proof is also a more useful model for a direct proof of the corresponding part of the main result of [14], mentioned in the introduction.

3.4 REMARK. We should also say something about the connection between our proof and that of Cuntz in [5, Section 4]. Take his E to be K_0, and interpret E_{-1} as K_1, defined in terms of unitaries. Take the connecting homomorphism ∂ in the exact sequence $(*)$ of Section 2 to be as in [4, Definition 8.3.1]. We will describe the resulting isomorphism ∂ in [5, Theorem 4.4]. For simplicity, take A to be unital, and consider only the classes of

unitaries $u \in (SA)^+$. Then

$$\partial([u]) = \left[v \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} v^* \right] - \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right] \in K_0((K \otimes A)^+),$$

where $v \in M_2(T \otimes A)$ is a unitary whose image in $(SA)^+$ is $u \oplus u^*$, and T, as in [5], is the Toeplitz algebra, that is, the C^*-algebra of the unilateral shift. With the identifications made in our definition of γ, we can take $v = \bar{\mu}(u)$. Thus $\partial = -\gamma$.

Cuntz proves directly that $E_*(T_0) = 0$, where T_0 is a certain codimension one ideal in T. The proof is not as transparent as our approach, but it is short and actually proves a good deal more. In fact, Cuntz's proof applies essentially as it stands to functors on the category of σ-C^*-algebras satisfying his axioms (E1), (E2), (E3) ([5, Section 2]). (One needs Lemma 1.10 here.) In particular, his proof gives Bott periodicity immediately for the theory RK_* as defined in [12]. But it is not clear how to generalize it to Fréchet algebras, while the proof given here does generalize ([16]).

3.5 LEMMA. *The maps of Definition* 3.1 *satisfy* $\gamma \circ \beta = \mathrm{id}$.

PROOF. Let $p_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ as usual, and let $p \in M_2((K \otimes A)^+)$ be a projection with $p - p_0 \in M_2(K \otimes A)$. Let $u = (1 - p + zp)(1 - p_0 + zp_0)^*$ be the unitary representing $\beta([p])$ as in Definition 3.1; we postpone the application of the isomorphism $M_2(K) \to K$ used there until the end of the proof.

We now compute $\mu(u)^* q \mu(u)$. For this we need some notation. Let $e_{ij} \in L(L^2)$ be the rank one partial isometry sending z^j to z^i and vanishing on $(\mathbb{C}z^j)^\perp$, and abbreviate e_{ii} to e_i. Then

$$\mu(u)^* q \mu(u) =$$
$$[1 \otimes (1 - p_0) + \mu(z)(1 \otimes p_0)] \cdot [1 \otimes (1 - p) + \mu(z)^*(1 \otimes p)] \cdot q$$
$$\cdot [1 \otimes (1 - p) + \mu(z)(1 \otimes p)] \cdot [1 \otimes (1 - p_0) + \mu(z)^*(1 \otimes p_0)]$$
$$= [1 \otimes (1 - p_0) + \mu(z)(1 \otimes p_0)]$$
$$\cdot [q + e_{-1} \otimes p] \cdot [1 \otimes (1 - p_0) + \mu(z)^*(1 \otimes p_0)]$$
$$= q - e_0 \otimes p_0 + e_{-1} \otimes (1 - p_0)p(1 - p_0) + e_{-1,0} \otimes (1 - p_0)pp_0$$
$$+ e_{0,-1} \otimes p_0 p(1 - p_0) + e_0 \otimes p_0 p p_0.$$

The first step is the application of μ to u, the second is obtained by multiplying together the middle three terms, and the third by multiplying out and identifying $\mu(z)$ as the bilateral shift. Now choose an isomorphism of $(H^2)^\perp$ with H^2; we take the one sending z^{-i} to z^{i-1} for $i \geq 1$. Thus, the isomorphism $K(L^2) \to M_2(K(H^2))$ sends e_0 to $\begin{pmatrix} e_0 & 0 \\ 0 & 0 \end{pmatrix}$, e_{-1} to $\begin{pmatrix} 0 & 0 \\ 0 & e_0 \end{pmatrix}$, $e_{0,-1}$ to

$\begin{pmatrix} 0 & e_0 \\ 0 & 0 \end{pmatrix}$, and $e_{-1,0}$ to $\begin{pmatrix} 0 & 0 \\ e_0 & 0 \end{pmatrix}$. We further remember that $p_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, and we write $p = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}$. Writing elements of $M_2(K(H^2) \otimes M_2(K) \otimes A$ as 4×4 matrices, with 2×2 blocks representing elements of $K(H^2) \otimes M_2(K) \otimes A$, we get

$$\mu(u)^* q \mu(u) = \begin{bmatrix} (1-e_0) \otimes 1 + e_0 \otimes p_{11} & 0 \mid 0 & e_0 \otimes p_{12} \\ 0 & 1 \mid 0 & 0 \\ \hline 0 & 0 \mid 0 & 0 \\ e_0 \otimes p_{21} & 0 \mid 0 & e_0 \otimes p_{22} \end{bmatrix} .$$

This projection is homotopic within the appropriate class to

$$f = \begin{bmatrix} (1-e_0) \otimes 1 + e_0 \otimes p_{11} & 0 \mid e_0 \otimes p_{12} & 0 \\ 0 & 1 \mid 0 & 0 \\ \hline e_0 \otimes p_{21} & 0 \mid e_0 \otimes p_{22} & 0 \\ 0 & 0 \mid 0 & 0 \end{bmatrix}$$

$$= (1-e_0) \otimes \begin{bmatrix} 1 & & 0 \\ & 1 & \\ & & 0 \\ 0 & & 0 \end{bmatrix} + e_0 \otimes \begin{bmatrix} p_{11} & 0 \mid p_{12} & 0 \\ 0 & 1 \mid 0 & 1 \\ p_{21} & 0 \mid p_{21} & 0 \\ 0 & 0 \mid 0 & 0 \end{bmatrix} .$$

Let H be the space on which K acts, and set $H_1 = e_0 H^2 \otimes H$ and $H_2 = [(1-e_0)H^2 \otimes H] \oplus [H^2 \otimes H]$. Then f is recognizable as the direct sum of a copy of p in $M_2((K(H_1) \otimes A)^+)$ and a copy of $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ in $M_2((K(H_2) \otimes A)^+)$. Since $H_1 \cong H_2 \cong H$ and $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ is the identity of $RK_0(A)$, we have proved that $[\mu(u)^* q \mu(u)] = [p]$, that is, that $\gamma \circ \beta([p]) = [p]$.

3.6 LEMMA. *The maps of Definition* 3.1 *satisfy* $\beta \circ \gamma = \mathrm{id}$.

PROOF. Let $v \in (K \otimes SA)^+$ be a unitary with $v - 1 \in K \otimes SA$. Then $\beta \circ \gamma([v])$ is the class of the unitary

$$\zeta \mapsto u(\zeta) = [1 - \mu(v)^* q \mu(v) + \zeta \mu(v)^* q \mu(v)] \cdot [1 - q + \zeta q]^* .$$

We may disregard the identification $K(L^2) \cong M_2(K(H^2)) \cong K$, and we therefore regard u as an element of $B = C(S^1) \otimes M(K(L^2) \otimes (K \otimes A)^+)$. We rewrite it as

$$u = [1 \otimes \mu(v)]^* \cdot [1 - 1 \otimes q + z \otimes q] \cdot [1 \otimes \mu(v)] \cdot [1 - 1 \otimes q + z \otimes q]^*$$
$$= \eta_0(v)^* \psi_0(v),$$

where $\eta_0, \psi_0 \colon C(S^1) \otimes (K \otimes A)^+ \to B$ are given by

$$\eta_0(b) = 1 \otimes \mu(b) \quad \text{and} \quad \psi_0(b) = [1 - 1 \otimes q + z \otimes q]\eta_0(b)[1 - 1 \otimes q + z \otimes q]^* .$$

We now want to construct certain homotopies of homomorphisms from $C(S^1) \otimes (K \otimes A)^+$ to B. It follows from [11, Proposition 3.3] (which is standard for C^*-algebras) and the fact that $C(S^1)$ is the univeral C^*-algebra generated by a unitary (namely z) that unital homomorphisms $\varphi: C(S^1) \otimes (K \otimes A)^+ \to B$ are in one to one correspondence with pairs consisting of a unitary $w \in B$ and a unital homomorphism $\overline{\varphi}: (K \otimes A)^+ \to B$ such that w commutes with all $\overline{\varphi}(a)$. (The tensor product in [10, Proposition 3.3] is the maximal tensor product, but, since $C(S^1)$ is nuclear, all tensor products are the same.) The relation between φ, w, and $\overline{\varphi}$ is of course $w = \varphi(z \otimes 1)$ and $\overline{\varphi}(a) = \varphi(1 \otimes a)$.

We will write elements of $B = C(S^1) \otimes M(K(L^2) \otimes (K \otimes A)^+)$ as infinite matrices relative to the basis of L^2 consisting of the power of z, with lines between the -1 and 0 columns and between the -1 and 0 rows to indicate the indexing. The entries are of course elements of $C(S^1) \otimes (K \otimes A)^+$. The homomorphism $\overline{\eta}_0$ obtained from the previous paragraph is just $a \mapsto \eta_0(1 \otimes a)$, and the unitary is the bilateral shift

$$
\eta_0(z \otimes 1) = 1 \otimes \mu(z \otimes 1) =
\begin{bmatrix}
\ddots & & & | & & 0 \\
 & 1 & 0 & | & & \\
 & & 1 & 0 & | & \\
\hline
 & & & 1 & | & 0 \\
 & & & | & 1 & 0 \\
0 & & & | & & 1 & \ddots
\end{bmatrix}.
$$

One checks readily that $\overline{\psi}_0 = \overline{\eta}_0$, and that the corresponding unitary is

$$
\psi_0(z \otimes 1) =
\begin{bmatrix}
\ddots & & & | & & 0 \\
 & 1 & 0 & | & & \\
 & & 1 & 0 & | & \\
\hline
 & & & z & | & 0 \\
 & & & | & 1 & 0 \\
0 & & & | & & 1 & \ddots
\end{bmatrix}.
$$

Define a unitary c_t for $t \in [0, 1]$ by

$$
c_t =
\begin{bmatrix}
\ddots & & & | & & 0 \\
 & 1 & & | & & \\
\hline
 & & \cos(kt) & | & \sin(kt) & \\
 & & -\sin(kt) & | & \cos(kt) & \\
 & & & | & & 1 & \\
0 & & & | & & & \ddots
\end{bmatrix},
$$

with $k = \pi/2$. Clearly c_t commutes with all $\mu(1 \otimes a)$, so we can define homomorphisms η_t and ψ_t by $\eta_t(z \otimes 1) = \eta_0(z \otimes 1)c_t$ and $\eta_t(1 \otimes a) = \eta_0(1 \otimes a)$, and by $\psi_t(z \otimes 1) = \psi_0(z \otimes 1)c_t$ and $\psi_t(1 \otimes a) = \eta_0(1 \otimes a)$. These agree with the earlier definitions of η_0 and ψ_0 because $c_0 = 1$.

We now verify an important property, namely that

$$(*) \quad \eta_t(b) - \psi_t(b) \in C_0(S^1 - \{1\}) \otimes K(L^2) \otimes (K \otimes A) \quad \text{for } b \in C(S^1) \otimes (K \otimes A).$$

First observe that

$$\eta_t(z \otimes 1) - \psi_t(z \otimes 1) = [\eta_0(z \otimes 1) - \psi_0(z \otimes 1)]c_t.$$

The first factor is in the ideal $I = C_0(S^1 - \{0\}) \otimes K(L^2) \otimes (K \otimes A)^+$, whence so is the product. It follows that $\eta_t(z^n \otimes 1) - \psi_t(z^n \otimes 1) \in I$ for all n. Furthermore, for $a \in K \otimes A$ we have

$$\eta_t(1 \otimes a) = \psi_t(1 \otimes a) = \eta_0(1 \otimes a) \in \mathbb{C} \cdot 1 \otimes \mathbb{C} \cdot 1 \otimes (K \otimes A).$$

Therefore

$$\eta_t(z^n \otimes a) - \psi_t(z^n \otimes a) = [\eta_t(z^n \otimes 1) - \psi_t(z^n \otimes 1)]\eta_0(1 \otimes a)$$
$$\in C_0(S^1 - \{0\}) \otimes K(L^2) \otimes (K \otimes A),$$

as desired. As a consequence, we obtain

$$\eta_t(v)^*\psi_t(v) - 1 = \eta_t(v)^*[\psi_t(v-1) - \eta_t(v-1)] \in C_0(S^1 - \{0\}) \otimes K(L^2) \otimes (K \otimes A),$$

that is, $\eta_t(v)^*\psi_t(v) \in (K(L^2) \otimes K \otimes SA)^+$ with $\eta_t(v)^*\psi_t(v) - 1 \in K(L^2) \otimes K \otimes SA$. We have thus constructed a homotopy from $\eta_0(v)^*\psi_0(v) = u$ to $\eta_1(v)^*\psi_1(v)$.

Now compute

$$\eta_1(z \otimes 1) = \eta_0(z \otimes 1)c_1 = \begin{bmatrix} \ddots & & & & | & & & 0 \\ & 1 & 0 & & | & & & \\ & & 1 & 0 & | & & & \\ \hline & & & 0 & | & 1 & & \\ & & & -1 & | & 0 & 0 & \\ & & & & | & & 1 & 0 \\ 0 & & & & | & & & 1 & \ddots \end{bmatrix}$$

and

$$\psi_1(z \otimes 1) = \psi_0(z \otimes 1)c_1 = \begin{bmatrix} \ddots & & & | & & & 0 \\ & 1 & 0 & | & & & \\ & & 1 & 0 & | & & \\ \hline & & & 0 & | & z & \\ & & -1 & | & 0 & 0 & \\ & & & | & & 1 & 0 \\ 0 & & & | & & & 1 & \ddots \end{bmatrix}.$$

Let e_0 be the projection on the constant functions in L^2, and let $\bar{e}_0 = 1 \otimes e_0 \otimes 1$. Then $\bar{e}_0$ commutes with $\eta_1(z \otimes 1)$ and $\psi_1(z \otimes 1)$, also with all $\eta_0(1 \otimes a)$, and hence with all $\eta_1(b)$ and $\psi_1(b)$. Furthermore, $(1 - \bar{e}_0)\psi_1(b)(1 - \bar{e}_0) = (1 - \bar{e}_0)\eta_1(b)(1 - \bar{e}_0)$, while $\bar{e}_0\eta_1(f \otimes a)\bar{e}_0 = f(1) \otimes e_0 \otimes a$ and $\bar{e}_0\psi_1(f \otimes a)\bar{e}_0 = f \otimes e_0 \otimes a$. (One checks these formulas on $z \otimes 1$ and on $1 \otimes a$.) Therefore

$$\eta_1(v)^* \psi_1(v) = \begin{bmatrix} \ddots & & | & & & 0 \\ & 1 & | & & & \\ & & 1 & | & & \\ \hline & & | & v & & \\ & & | & & 1 & \\ & & | & & & 1 \\ 0 & & | & & & & \ddots \end{bmatrix}$$

which we can regard as $v \oplus 1$. Thus,

$$\gamma \circ \beta([v]) = [\eta_0(v)^*\psi_0(v)] = [\eta_1(v)^*\psi_1(v)] = [v \oplus 1] = [v],$$

as desired.

3.7 THEOREM. *There is a natural isomorphism* $RK_1(SA) \cong RK_0(A)$.

PROOF. According to the previous two lemmas, we have $\gamma \circ \beta = \mathrm{id}$ and $\beta \circ \gamma = \mathrm{id}$.

4. A Simplified Construction of Representable K-Theory

In [12], we defined $RK_0(A)$, for A unital, to be the set of homotopy classes in the unitary group of the stable outer multiplier algebra $M(K \otimes A)/(K \otimes A)$. This definition was motivated by the definition of representable K-theory of a space as the set of homotopy classes of maps to the Fredholm operators

(see for example [7]), and by Mingo's work [9], which shows that this defini-
tion gives the correct answer for C^*-algebras. The definition was extended
to nonunital σ-C^*-algebras by considering the unitization, and to RK_1 by
suspending. The proofs of the basic properties of the theory required some
messy lemmas involving multiplier algebras. (See [12, Section 1]). The proof
of the Milnor $\varprojlim^1$-sequence was messy, and the proof of Bott periodic-
ity was obtained by reduction to the C^*-algebra case. Furthermore, in the
construction of equivariant representable K-theory, done in [12, Section 5],
additional messy lemmas were needed and the group was assumed to be sec-
ond countable.

In this section, we show how to develop the basic properties of repre-
sentable K-theory from Definition 1.1. This development is much cleaner
and more straightforward, and much more closely parallels the usual pre-
sentation of K-theory for Banach algebras or C^*-algebras in, for example,
[19] and [4, Sections 5, 8, and 9]. This method also generalizes cleanly to
the equivariant case, for an arbitrary compact group. Most of the work has
already been done, either here or elsewhere. All that remains is to prove
one lemma, slightly modify several proofs, and put the pieces together in the
proper order.

4.1 LEMMA. *Let* $0 \to I \overset{\iota}{\to} A \overset{\pi}{\to} B \to 0$ *be an exact sequence of σ-C^*-
algebras. Then there is a natural exact sequence*

(*) $\quad \cdots \to RK_0(S^2A) \overset{\pi_*}{\to} RK_0(S^2B) \to RK_0(SI) \overset{\iota_*}{\to} RK_0(SA) \overset{\pi_*}{\to} RK_0(SB)$
$\to RK_0(I) \overset{\iota_*}{\to} RK_0(A) \overset{\pi_*}{\to} RK_0(B).$

PROOF. We only have to prove exactness at $RK_0(A)$; the rest follows from
the homotopy invariance of RK_0 by a standard argument, given, for example,
in [8, Lemma 5 of Section 7]. (Also see [12, Corollary 2.5].) It is also trivial
to check that $\pi_* \circ \iota_* = 0$. So we prove $\mathrm{Ker}(\pi_*) \subset \mathrm{Im}(\iota_*)$.

Let $p \in M_2((K \otimes A)^+)$ be a projection satisfying $p - \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \in M_2(K \otimes A)$
and $\pi_*([p]) = 0$. Then there is a homotopy $t \mapsto q_t$ from $q_0 = \pi(p)$ to
$q_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ in $M_2((K \otimes B)^+)$ such that $q_t - \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \in M_2(K \otimes B)$. By Lemma
1.9, there is a unitary path $t \mapsto v_t$ in $M_2((K \otimes B)^+)$ with $v_1 = 1$, $v_t q_t v_t^* = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, and $v_t - 1 \in M_2(K \otimes B)$. Now use Lemma 1.8 to lift v to a unitary
path $t \mapsto u_t$ in $M_2((K \otimes A)^+)$ with $u_1 = 1$ and $u_t - 1 \in M_2(K \otimes A)$. The
homotopy $t \mapsto u_t p u_t^*$ shows that $[p] = [u_0 p u_0^*]$. Furthermore, $\pi(u_0 p u_0^*) = v_0 q_0 v_0^* = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, so $u_0 p u_0^* - \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \in M_2(K \otimes I)$. Thus, $[u_0 p u_0^*]$ is an element
of $RK_0(I)$ with $\iota_*([u_0 p u_0^*]) = [p]$.

4.2 Outline of the development of representable K-theory for σ-C^*-algebras.
Step 1. Prove the two basic lemmas on projections and unitaries stated here as Lemmas 1.8 and 1.9. Note that the profs of these lemmas are reasonably simple, and do not depend on other material in the papers from which they are taken.

Step 2. Define $RK_0(A)$ and $RK_1(A)$ for a σ-C^*-algebra A, and prove that they are homotopy invariant functorial abelian groups, and agree with the usual K-groups on C^*-algebras, as was done here in Section 1.

Step 3. Prove Lemma 4.1, the one-sided long exact sequence for RK_0.

Step 4. Prove Bott periodicity, $RK_i(SA) \cong RK_{1-i}(A)$, as in Sections 2 and 3. The part of the exact sequence shown in $(*)$ of Lemma 4.1 is then just enough to give the usual six term exact sequence

$$
\begin{array}{ccccc}
RK_0(I) & \longrightarrow & RK_0(A) & \longrightarrow & RK_0(B) \\
\uparrow & & & & \downarrow \\
RK_1(B) & \longleftarrow & RK_1(A) & \longleftarrow & RK_1(I).
\end{array}
$$

Step 5. Prove the Mayer-Vietoris sequence: if A is a σ-C^*-algebra, and J_0 and J_1 are closed ideals in A, then there is a natural six term exact sequence

$$RK_0(A/(J_0 \cap J_1)) \to RK_0(A/J_0) \oplus RK_0(A/J_1) \to RK_0(A/(J_0 + J_1))$$

$$\uparrow \qquad\qquad\qquad\qquad \downarrow$$

$$RK_1(A/(J_0 + J_1)) \leftarrow RK_1(A/J_0) \oplus RK_1(A/J_1) \leftarrow RK_1(A/(J_0 \cap J_1)),$$

with the usual formulas for the maps. (See [12, Proposition 2.8].) Since we already have the six term exact sequence $(**)$, the proof of [17, Theorem 4.1] is directly applicable. Alternatively, the proof used for [12, Proposition 2.8] now gives the entire sequence without truncation, and [12, Lemma 2.6] becomes trivial.

Step 6. Prove countable additivity: $RK_i(\prod_n A_n) \cong \prod_n RK_i(A_n)$, for σ-C^*-algebras A_n [12, Proposition 3.1]. The proof is even simpler than the proof given there.

Step 7. Prove the Milnor $\varprojlim^1$-sequence ([12, Theorem 3.2]). The proof is simpler than there. With $A = \varprojlim A_n$, and other notation as in [12], the first part of that proof gives the exact sequence

$$0 \to \varprojlim{}^1 RK_{1-i}(A_n) \to RK_i(T) \to \varprojlim RK_i(A_n) \to 0$$

for $i = 0$ and $i = 1$, and the next part gives $RK_i(T) \cong RK_i(A)$, again for both $i = 0$ and $i = 1$. The rest of the proof in [12], consisting of special

arguments for the case $i = 0$, is now superfluous. In particular, nothing resembling the proof of [12, Lemma 2.6] is ever needed.

Step 8. Prove stability: $RK_i(K(H) \otimes A) \cong RK_i(A)$ for any Hilbert space H. This is essentially immediate from the definition if H is finite-dimensional or separable infinite dimensional. For nonseparable H, an attempt to use the usual sort of approximation argument seems to require methods similar to those of [13]. The simplest thing to do is to choose a rank one projection $e \in K(H)$, observe that $a \mapsto e \otimes a$ defines an isomorphism on the K-theory of C^*-algebras, and use the Milnor $\varprojlim^1$-sequence to conclude that this is also true for σ-C^*-algebras.

We now consider equivariant K-theory. For the rest of this section, we let G be an arbitrary, but fixed, compact group. We assume that G acts continuously on all σ-C^*-algebras ([12, Definition 5.1]), and that all homomorphisms are equivariant. We use [12, Lemma 5.2] without comment. The superscript G indicates the fixed point algebra. G always acts by conjugation by the regular representation on $K(L^2(G))$, trivially on K, $C(S^1)$, etc., and via the diagonal action on tensor products. For equivariant K-theory of C^*-algebras, we refer to [10].

4.3 DEFINITION. Let A be a σ-C^*-algebra with a continuous action of G. Then $RK_0^G(A)$ is the set of homotopy classes of G-invariant projections $p \in M_2([K \otimes K(L^2(G)) \otimes A]^+)$ such that $p - \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \in M_2(K \otimes K(L^2(G)) \otimes A)$. Also, $RK_1^G(A)$ is the set of homotopy classes in the set of G-invariant unitaries $u \in [K \otimes K(L^2(G)) \otimes A]^+$ such that $u - 1 \in K \otimes K(L^2(G)) \otimes A$. Additoin and induced maps are defined as for $RK_0(A)$ and $RK_1(A)$, using the isomorphism $M_2(K) \cong K$ exactly as before. We further let $R(G)$ be the representation ring of G (see [18]), and define an $R(G)$-module structure on $RK_i^G(A)$ as follows. If V is a finite dimensional unitary representation space of G, then $V \otimes L^2(G)$ is isomorphic to the direct sum of $\dim(V)$ copies of $L^2(G)$, so that $K \otimes K(L^2(G)) \cong K \otimes K(L^2(G)) \otimes K(V)$. Define $[V][p] = [1_V \otimes p]$, where 1_V is the identity operator on V, and $1_V \otimes p$ defines a class in $RK_0^G(A)$ by the isomorphism just obtained. Similarly define $[V][u] = [1_V \otimes u]$ in $RK_1^G(A)$.

4.4 THEOREM. (Compare [12, Theorem 5.8].) RK_*^G *is a functor from* σ-C^*-*algebras to* $R(G)$-*module satisfying the following properties.*

(1) *Homotopy invariance: if* φ *and* ψ *are* G-*homotopic homomorphisms, then* $\varphi_* = \psi_*$.

(2) *Long exact sequence: if* $0 \to I \to A \to B \to 0$ *is an equivariant short exact sequence of* σ-C^*-*algebras, then there is a natural six term exact*

sequence

$$\begin{array}{ccc}
RK_0^G(I) \longrightarrow RK_0^G(A) \longrightarrow RK_0^G(B) \\
\uparrow \qquad\qquad\qquad\qquad \downarrow \\
RK_1^G(B) \longleftarrow RK_1^G(A) \longleftarrow RK_1^G(I).
\end{array}$$

(3) *Bott periodicity*: *there are natural isomorphisms* $RK_i^G(SA) \cong RK_{i-1}^G(A)$ *for every* A.

(4) *More Bott periodicity*: *if* V *is a finite dimensional complex vector space on which* G *acts via a unitary representation, then there is a natural isomorphism* $RK_i^G(C_0(V) \otimes A) \cong RK_i^G(A)$ *for every* A.

(5) *Stability*: *if* V *is any Hilbert space on which* G *acts via a unitary representation, then there is a natural isomorphism* $RK_i^G(K(V) \otimes A) \cong RK_i^G(A)$ *for every* A.

(6) *Milnor* $\varprojlim^1$-*sequence*: *if* $A = \varprojlim A_n$ *equivariantly, then there is a natural short exact sequence*

$$0 \to \varprojlim {}^1 RK_{1-i}^G(A_n) \to RK_i^G(A) \to \varprojlim RK_i^G(A_n) \to 0.$$

(7) *Green-Julg-Rosenberg Theorem*: *there are natural isomorphisms* $RK_i^G(A) \cong RK_i(C^*(G, A))$ *for every* A.

(8) *Agreement with K-theory*: *if* A *is a* C^*-*algebra, then there are natural isomorphisms* $RK_i^G(A) \cong K_i^G(A)$.

(9) *Exterior invariance*: *if two actions of* G *on* A *are exterior equivalent* ([10, *Definition* 2.7.1]), *then* $RK_i^G(A)$ *is the same for both actions.*

PROOF. We immediately see that, as a group, $RK_i^G(A)$ has been essentially defined to be $RK_i([K(L^2(G)) \otimes A]^G)$. Thus, what we already know about RK_i gives us the fact that each $RK_i^G(A)$ is a functorial abelian group and parts (1), (2), (3), and (6). We also get (5) this way, because if V is any Hilbert space on which G acts via a unitary representation, then $V \otimes L^2(G)$ is isomorphic to a Hilbert direct sum of $\dim(V)$ copies of $L^2(G)$. For part (8), an examination of the definition of $K_0^G(A)$ in terms of projections for a unital C^*-algebra A ([10, Section 2.4]) shows that is also has been essentially defined to be $K_0([K(L^2(G)) \otimes A]^G)$. (Note that $L^2(G)$ contains every irreducible representation of G.) This gives (8) for unital algebras and $i = 0$, and the rest follows by taking unitizations and suspensions.

Next observe that the multiplications by $[V] \in R(G)$ are just the induced maps on representable K-theory coming from $a \mapsto 1_V \otimes a$, regarded as a homomorphism from $[K(L^2(G)) \otimes A]^G$ to $[K(V) \otimes K(L^2(G)) \otimes A]^G$. We

omit the routine verification that these do in fact define an $R(G)$-module structure on $RK_i^G(A)$. Naturality in parts (1), (2), (3), (5), and (6) now ensures that the maps there are also all $R(G)$-module homomorphisms, and comparison with [10, Chapter 2] does the same for (8).

For (7), we observe that noncommutative duality theory, as summarized in [6, Section 1], implies that $[K(L^2(G)) \otimes A]^G$ is naturally isomorphic to $C^*(G, A)$ if A is a C^*-algebra, and therefore also (by taking inverse limits) if A is a σ-C^*-algebra. With the $R(G)$-module structure on $RK_i(C^*(G, A))$ defined as in [10, Section 2.7], the resulting isomorphism $RK_i^G(A) \cong RK_i(C^*(G, A))$ is an $R(G)$-module homomorphism for C^*-algebras, and hence also for σ-C^*-algebras.

We now get (9) by observing that crossed products by exterior equivalent actions are naturally isomorphic when A is a C^*-algebra, and thus when A is a σ-C^*-algebra. Only (4) remains; to prove that, recall that the isomorphism $\beta: K_i^G(A) \to K_i^G(C_0(V) \otimes A)$ is given by the external product with a class $\eta \in K_0(C_0(V))$. Write $\eta = [p] - [q]$ with p and q projections in some $K(W) \otimes C_0(V)^+$, and let $\varphi, \psi: A \to K(W) \otimes C_0(V)^+ \otimes A$ be $\varphi(a) = p \otimes a$ and $\psi(a) = q \otimes a$. Note that the images of p and q in $K(W)$ are equivalent and hence G-homotopic. Therefore, if $\pi: K(W) \otimes C_0(V)^+ \otimes A \to K(W) \otimes A$ is the quotient map associated with the unitization $C_0(V)^+$, we have $\pi_* \circ \varphi_* = \pi_* \circ \psi_*$, so that $\varphi_* - \psi_*$ defines a map $RK_i^G(A) \to RK_i^G(K(W) \otimes C_0(V) \otimes A)$. This map is natural, and (using stability) is the isomorphism β when A is a C^*-algebra, and thus by the Milnor $\varprojlim^1$-sequence, $\varphi_* - \psi_*$ is an isomorphism for any σ-C^*-algebra A.

4.5 REMARK. We never used the original Julg's Theorem (see [10, Section 2.6]) in our proof of the corresponding part (7) of the previous theorem. In fact, our methods give an immediate proof of Julg's theorem for C^*-algebras. One notices, for unital A, that $K_0^G(A)$ is essentially defined to be $K_0([K(L^2(G)) \otimes A]^G)$. (This is best seen in [10, Section 2.4].) Noncommutative duality theory, as in [6, Section 1], gives $C^*(G, C^*(G, A, \alpha), \delta) \cong K(L^2(G)) \otimes A$ equivariantly, where α is the action of G on A and δ is the dual coaction. It is trivial to check that for any coaction δ of a compact group G on a C^*-algebra B, one has $C^*(G, B, \delta)^G \cong B$. So $[K(L^2(G)) \otimes A]^G \cong C^*(G, A)$, and $K_0^G(A) \cong K_0(C^*(G, A))$.

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