RANDOM WALKS ON RANDOM TREES

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1. Introduction

Let T denote one of the n^{n-2} trees with n labelled nodes that is rooted at a given node x (see [6] or [8] as a general reference on trees). If i and j are any two nodes of T, we write $i \sim j$ if they are joined by an edge in T. We want to consider random walks on T; we assume that when we are at a node i of degree d the probability that we proceed to node j at the next step is d_i^{-1} if $i \sim j$ and zero otherwise. Our object here is to determine the first two moments of the first return and first passage times for random walks on T when T is a specific tree and when T is chosen at random from the set of all labelled trees with certain properties.

2. Some lemmas

We shall need the following results due to Clarke [2], Cayley [1], and Meir and Moon [4]; we adopt the convention that $(x)_0 = 1$ and $(x)_s = x(x-1) \cdots (x-s+1)$ for positive integers s.

LEMMA 2.1. *If $1 \leqq k \leqq n-1$ and $C(n,k)$ denotes the number of trees with n labelled nodes in which a given node has degree k, then*

$$C(n,k) = \binom{n-2}{k-1}(n-1)^{n-k-1}.$$

LEMMA 2.2. *If $0 \leqq k \leqq n-1$ and $T(n,k)$ denotes the number of trees with n labelled nodes that contain a given path of length k, then*

$$T(n,k) = (k+1)n^{n-k-2}.$$

LEMMA 2.3. *If $1 \leqq k \leqq n-1$ and $P(n,k)$ denotes the probability that the distance between two given nodes in a tree with n labelled nodes is k, then $P(n,k) = (k+1)n^{-k}(n-2)_{k-1}$.*

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3. Some identities

Riordan [9; §1.5] has established a number of identities involving Abel sums of the type

$$A_N(r,s;p,q) = \sum_{M=0}^{N} \binom{N}{M}(M+r)^{M+p}(N-M+s)^{N-M+q}.$$

In what follows we shall use the identities

$$(3.1)\qquad A_N(r,s;0,-1) = s^{-1}(r+s+N)^N,$$

$$(3.2)\qquad A_N(r,s;1,-1) = s^{-1}\sum_{v=0}^{N}(v+r)(N)_v(r+s+N)^{N-v},$$

$$(3.3)\qquad A_N(r,s;0,0) = \sum_{v=0}^{N}(N)_v(r+s+N)^{N-v},$$

and

$$(3.4)\qquad A_N(r,s;1,0) = \sum_{v=0}^{N}\left\{\binom{v+r+1}{2}-\binom{r}{2}\right\}(N)_v(r+s+N)^{N-v}.$$

We shall also use the identities

$$(3.5)\qquad \sum_{v=1}^{N-K} v(N-K)_v N^{-v} = (N-K) - K\sum_{v=1}^{N-K}(N-K)_v N^{-v}$$

and

$$(3.6)\qquad \sum_{v=1}^{N-K} v^2(N-K)_v N^{-v} = (N+K^2)\sum_{v=1}^{N-K}(N-K)_v N^{-v} - K(N-K),$$

where $0 \leqq K \leqq N$. The first of these follows easily if we use the fact that $v(N-K)_v = (N-K)(N-K)_v - (N-K)_{v+1}$; the case $K=0$ of the second was proved in [7] and the general case can be proved by induction on K.

Many of out results will involve binomial sums that are not easy to evaluate. Their asymptotic behavior, however, can be determined, subject to some restrictions, by approximating them by an integral of the type

$$I_h = \int_0^\infty x^h e^{-x^2/2}\,dx;$$

in particular, we shall frequently use the fact that if $k = o(n^{\frac{1}{2}})$, then

$$\sum_{v=1}^{n-k}(n-k)_v n^{-v} \sim (\tfrac{1}{2}\pi n)^{\frac{1}{2}} \text{ as } n \text{ and } k \text{ tend to infinity}$$

(see [5]). We shall state these asymptotic results without proof; the details of the missing arguments are quite similar to those given, for example, in [7] and [5].

4. First return times: the first moment

If y is any node of T let T_y denote the subtree determined by those nodes z of T such that the unique path in T from the root x to z passes through y. Let v_y denote the degree of y in T_y and let w_y denote the number of nodes in T_y; notice that $T_x = T$, $w_x = n$, and $v_x = d_x$ but that $v_y = d_y - 1$ if $y \neq x$. If one begins a random walk on the tree T_y at the node y, let λ_y denote the number of steps before one returns to y for the first time; and if one begins a random walk on the tree T at the node y, where $y \neq x$, let γ_{yx} denote the number of steps before one reaches node x for the first time. Finally, we let $M(\theta)$ denote the i-th factorial moment of the random variable θ, for $i = 1, 2$; for convenience, we write $M_i(x)$ and $M_i(yx)$ instead of $M_i(\lambda_x)$ and $M_i(\gamma_{yx})$.

LEMMA 4.1. *If $y \sim x$, then*

$$M_1(yx) = 1 + v_y M_1(y)$$

and

$$M_2(yx) = v_y M_2(y) + 2v_y M_1(y) M_1(yx).$$

PROOF. If $f_i = \Pr\{\lambda_y = i\}$ for $i = 0, 1, \ldots$ and $g_i = \Pr\{\gamma_{yx} = i\}$ for $i = 1, 2, \ldots$ let $P(s; y) = \sum_i f_i s^i$ and $P(s; yx) = \sum_i g_i s^i$. (Notice that $f_i = 0$ if i is odd; we, let $f_0 = 1$ if $w_y = 1$, otherwise $f_0 = 0$.) We remark that it is easy to see that $P(1; y) = P(1; yx) = 1$.

In any random walk from y to x on T, either we go directly from y to x on the first step, with probability $p = (v_y + 1)^{-1}$, or we commence a random walk on T_y, with probability $q = 1 - p$, and return to y before starting, in effect another random walk on T that eventually proceeds to x. This observation implies that

$$P(s; yx) = ps + qP(s; y) P(s; yx).$$

The required results now follow easily from this relation and the fact that the first and second factorial moments of λ_y and γ_{yx} are given by the first and second derivatives of $P(s; y)$ and $P(s; yx)$ evaluated at $s = 1$. (We remark that the above equation and the relation between the probability and binomial moment generating functions of a variable can be used to derive a simple relation between the binomial moments of λ_x and γ_{yx}.)

THEOREM 4.1. $v_x M_1(x) = 2(w_x - 1)$.

PROOF. If we begin a random walk on T_x at node x, then the first step must be to one of the v_x nodes y such that $y \sim x$; consequently,

$$\lambda_x = v_x^{-1} \sum_{y \sim x} (1 + \gamma_{yx}).$$

This implies, in view of the first part of Lemma 4.1, that

$$v_x M_1(x) = \sum_{y\sim x} (2 + v_y M_1(y)) = 2v_x + \sum_{y\sim x} v_y M_1(y).$$

If we iterate the last expression we eventually obtain a contribution of two for every node of T_x other than x, as required. (This theorem also follows easily from a fundamental result on Markov chains; see for example ,[3; p. 79, 120].)

THEOREM 4.2. *If $n > 1$ and $E(x,n)$ denotes the expected value of λ_x over all the n^{n-2} trees T, then $E(x,n) = 2n(1-(1-n^{-1})^{n-1})$.*

COROLLARY 4.2.1. $E(x;n) \sim 2(1-e^{-1})n$.

PROOF. It follows from Lemma 2.1 and Theorem 4.1 that

$$E(x;n) = \frac{1}{n^{n-2}} \sum_{k=1}^{n-1} \binom{n-1}{k-1} (n-1)^{n-k-1} \{2(n-1)k^{-1}\}.$$

If we apply the binomial theorem to this sum we obtain the stated result.

5. First return times: the second moment

We now derive a formula for $M_2(x) - M_1(x)$; the variance $V(x)$ of λ_x can then be determined by using the formula

$$V(x) = M_2(x) + M_1(x) - M_1^2(x).$$

THEOREM 5.1. $v_x(M_2(x) - M_1(x)) = 8 \sum_{y\neq x} w_y(w_y - 1)$.

PROOF. We first observe that

$$\lambda_x(\lambda_x - 2) = v_x^{-1} \sum_{y\sim x} (\gamma_{yx}+1)(\gamma_{yx}-1) ;$$

this follows, as before, from the fact that a random walk from x must go to one of the v_x nodes y such that $y \sim x$. This implies, in view of the second part of Lemma 4.1, that

$$v_x(M_2(x) - M_1(x)) = \sum_{y\sim x} (v_y M_2(y) + 2v_y M_1(y) M_1(yx) + M_1(yx) - 1).$$

Now $v_y M_1(y) = 2(w_y - 1)$ and $M_1(yx) = 2w_y - 1$, by Theorem 4.1 and Lemma 4.1, so this can be rewritten as

$$v_x(M_2(x) - M_1(x)) = \sum_{y\sim x} (v_y(M_2(y) - M_1(y)) + 8w_y(w_y - 1)).$$

If we iterate this last expression we eventually obtain the stated result.

THEOREM 5.2 *If $1 \leqq k \leqq n-1$ and $D(x;n,k)$ denotes the expected value of $\lambda_x(\lambda_x - 2)$ over all the $C(n,k)$ trees T in which $v_x = k$, then*

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$$kD(x;n,k) = 4(n-1)\sum_{v=1}^{n-k-1} v(v+3)(n-k-1)_v(n-1)^{-v}.$$

PROOF. It follows from Theorem 5.1 that

$$kD(x;n,k) = 8kM_2(w_y) + 8(n-k-1)M_2(w_z) \tag{1}$$

where y and z denote nodes that are, and are not joined to x in a random tree T in which $v_x = k$. If $1 \leqq w \leqq n-k$, then there are

$$\binom{n-2}{w-1} C(n-w,k-1)w^{w-2}$$

such trees for which $w_y = w$. The nodes other than y in T_y can be selected in $\binom{n-2}{w-1}$ ways and there are w^{w-2} possibilities for the tree T_y; there are $C(n-w, k-1)$ ways to form a tree on the $n-w$ nodes not in T_y in which node x is joined to $k-1$ other nodes. If we now join x and y we obtain a tree of the required type. A slight modification of this argument shows that if $1 \leqq w \leqq n-k-1$ then there are

$$\binom{n-2}{w-1} C(n-w,k)(n-w-1)w^{w-2}$$

trees of the required type for which $w_z = w$; the factor $(n-w-1)$ arises from the fact that z is joined to any one of the $n-w-1$ nodes other than x that are not in T_z.

The probability that a given node y is joined to x in a random tree T in which $v_x = k$ is $k(n-1)^{-1}$ and the probability that a given node z is not joined to x is $(n-k-1)(n-1)^{-1}$. It follows, therefore, from these considerations and Lemma 2.1, that

$$\begin{aligned} M_2(w_y) &= \frac{n-1}{kC(n,k)} \sum_{w=1}^{n-k} \binom{n-2}{w-1} C(n-w,k-1)w^{w-1}(w-1) \\ &= \frac{k-1}{k(n-1)^{n-k-2}} \sum_{w=1}^{n-k} \binom{n-k-1}{w-1} w^{w-1}(w-1)(n-w-1)^{n-k-w-1} \end{aligned}$$

and

$$\begin{aligned} M_2(w_z) &= \frac{(n-1}{(n-k-1)C(n,k)} \sum_{w=1}^{n-k-1} \binom{n-2}{w-1} C(n-w,k)w^{w-1}(w-1)(n-w-1) \\ &= \frac{1}{(n-1)^{n-k-2}} \sum_{w=1}^{n-k-1} \binom{n-k-2}{w-1} w^{w-1}(w-1)(n-w-1)^{n-w-k-1}. \end{aligned}$$

The expression for $M_2(w_y)$ can be simplified by using identities (3.1) and (3.2) with $N=n-k-1$, $M=w-1$, $r=1$, and $s=k-1$; the expression for $M_2(w_z)$ can be simplified by using identities (3.3) and (3.4) with $N=n-k-2$, $M=w-1$, $r=1$, and $s=k$. If we substitute these expressions in the right hand side of equation (1) and simplify we obtain the stated result.

COROLLARY 5.2.1. *If* $1 \leqq k \leqq n-1$, *then*

$$kD(x;n,k) = 4(n-1)\{(n-1+k(k-3))\sum_{v=1}^{n-k-1}(n-k-1)_v(n-1)^{-v}-(k-3)(n-k-1)\}.$$

PROOF. This follows from Theorem 5.2 and identities (3.5) and (3.6).

COROLLARY 5.2.2. *If* $k=o(n^{\frac{1}{2}})$ *then* $kD(x;n,k) \sim (8\pi)^{1/2}n^{5/2}$.

THEOREM 5.3 *If* $n>1$ *and* $D(x;n)$ *denotes the expected value of* $\lambda_x(\lambda_x-2)$ *over all the* n^{n-2} *trees* T, *then*

$$D(x;n)=\frac{4}{n^{n-2}}\sum_{v=1}^{n-2}v(v+3)(n-1)_v\{n^{n-1-v}-(n-1)^{n-1-v}\}.$$

PROOF. The definition of $D(x;n)$ implies that

$$D(x;n)=\frac{1}{n^{n-2}}\sum_{k=1}^{n-1}C(n,k)D(x;n,k).$$

If substitute the expressions for $C(n,k)$ and $D(x;n,k)$ given by Lemma 2.1 and Theorem 5.2, interchange the order of summation, and use the binomial theorem to evaluate the resulting inner sum, we obtain the required formula for $D(x;n)$.

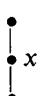
COROLLARY 5.3.1. *If* $n>1$ *then*

$$D(x;n)=4n^2(1-2(1-n^{-1})^n+\frac{4}{n^{n-2}}\sum_{v=0}^{n-1}(n-1)_v\{(n-2)n^{n-1-v}-(n-1)^{n-v}\}.$$

PROOF. This follows from Theorem 5.3 and identities (3.5) and (3.6); it can also be deduced from Corollary 5.2.1 in the same way as Theorem 5.3 was deduced from Theorem 5.2.

COROLLARY 5.3.2. $D(x;n) \sim (8\pi)^{1/2}(1-e^{-1})n^{5/2}$.

The formulas established in the last two sections may be checked when $n=3$ by considering examples in Figure 1.

tree	frequency	$\Pr\{\lambda_x=i\}$	$M_1(x)$	$M_2(x)-M_1(x)$
	2/3	$\begin{cases}(\frac{1}{2})^j, i=2j\\ 0, i=2j+1\end{cases}$	4	16
	1/3	$\begin{cases}1, \ i=2\\ 0, \ i\neq 2\end{cases}$	2	0

$D(x;3,1)=16$ $\quad D(x;3,2)=0$ $\quad D(x;3)=10\frac{2}{3}$

Fig. 1. First Return Times When $n=3$.

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6. First passage times: the first moment

Let $d(x, y)$ denote the distance between x and some other node y in the tree T. If $d(x, y) = k$, where $1 \leqq k \leqq n-1$, there will be no loss of generality if we assume the nodes of T are labelled so that $x = 0$, $y = k$, and the interior nodes of the path from x to y are $1, 2, \cdots, k-1$ when $k \geqq 2$. Let R_k denote the tree T_k and if $1 \leqq i \leqq k-1$ let R_i denote the subtree determined by the nodes of T_i that do not belong to T_{i+1}; let n_i denote the number of nodes in R_i.

THEOREM 6.1. *If* $d(x, y) = k$, *then* $M_1(yx) = 2(n_1 + 2n_2 + \cdots + kn_k) - k$.

PROOF. We first observe that $\gamma_{yx} = \gamma_{k,k-1} + \cdots + \gamma_{21} + \gamma_{10}$.

Lemma 4.1 and Theorem 4.1 imply that $M_1(i, i-1) = 2m_i - 1$ for $1 \leqq i \leqq k$. Hence $M_1(yx) = 2(m_1 + m_2 + \cdots + m_k) - k$. The result follows from the fact that $m_i = n_i + n_{i+1} + \cdots + n_k$ for $1 \leqq i \leqq k$.

THEOREM 6.2. *If* $1 \leqq k \leqq n-1$ *and* $E(y, x; n, k)$ *denotes the expected value* γ_{yx} *over all the* $P(n, k)n^{n-2}$ *trees* T *in which* $d(x, y) = k$, *then* $E(y, x; n, k) = k(n-1)$.

PROOF. If $d(x, y) = k$, then it is easy to show either by a direct argument or by appealing to symmetry that $M_1(n_i) = n(k+1)^{-1}$ for $1 \leqq i \leqq k$. The result now follows from Theorem 6.1.

THEOREM 6.3. *If* $x \neq y$ *and* $E(y, x; n)$ *denotes the expected value of* γ_{yx} *over all the* n^{n-2} *trees* T, *then*

$$E(y, x; n) = n \sum_{v=2}^{n} (n)_v n^{-v}.$$

COROLLARY 6.3.1 $E(y, x; n) \sim (\frac{1}{2}\pi)^{1/2} n^{3/2}$.

PROOF. It follows from Lemma 2.3 and Theorem 6.2 that

$$E(y, x; n) = \sum_{k=1}^{n-1} P(n, k) E(y, x; k, n) = \sum_{k=1}^{n-1} k(k+1)(n)_{k+1} n^{-k-1}.$$

The result now follows from identities (3.5) and (3.6) if we first replace $k+1$ by v.

7. First passage times: the variance

Let us suppose, as in the first part of §6, that $d(x, y) = k$. If $1 \leqq i \leqq k$, let

$$W_i = \Sigma' \, w_z(w_z - 1)$$

where the sum is over all nodes z of the subtree R_i defined in § 6 other than the node i. We now derive a formula for the variance $V(yx)$ of γ_{yx}.

THEOREM 7.1. *If $d(x, y) = k$, then*

$$V(yx) = 8 \sum_{i=1}^{k} iW_i + 4 \sum_{i=1}^{k} (2i-1)w_i(w_i - 1).$$

PROOF. Lemma 4.1 and Theorem 4.1 imply that

$$V(i, i-1) = v_i(M_2(i) - M_1(i)) + 4w_i(w_i - 1)$$

for $1 \leqq i \leqq k$; therefore,

$$V(yx) = \sum_{i=1}^{k} \{v(M_2(i) - M_1(i)) + 4w_i(w_i - 1)\}$$

since the variables $\gamma_{k,k-1}, \cdots, \gamma_{21}, \gamma_{10}$ are independent. It follows from Theorem 5.1 that

$$v_i(M_2(i) - M_1(i)) = 8(W + \cdots + W_k) + 8(w_{k+1}(w_{k+1} - 1) + \cdots + w_k(w_k - 1))$$

for $1 \leqq i \leqq k$. If we subtitute this expression for $v_i(M_2(i) - M_1(i))$ in the formula for $V(yx)$ and simplify we obtain the stated result. (The formula in Theorem 7.1 would look a bit tidier if we included the term $w_i(w_i - 1)$ in the definition of W_i; the form we have given, however, is somewhat more convenient to use in proving the next result.)

THEOREM 7.2. *If $1 \leqq k \leqq n-1$ and $C(y, x; n, k)$ denotes the variance of γ_{yx} over all the $P(n,k)^{n-2}$ trees in which $d(x, y) = k$, then*

$D(y, x; n, k)$

$$= \frac{2}{3}kn \sum_{v=1}^{n-k-1} (3v^2 + 2(k+2)v + (k^2 + k - 5))(n-k-1)_v n^{-v} + \frac{2}{3}k^2(k-1)n.$$

PROOF. We first derive a formula for $M_2(w_z)$ over all the $T(n, k)$ trees T that contain a given path of length k joining nodes x and y where z is any given node not in this path.

If $1 \leqq w \leqq n - k - 1$, then there are

$$\binom{n-k-2}{w-1} w^{w-2}(n-w)T(n-w, k)$$

such trees for which $w_z = w$. The nodes other than z in T_z can be selected in $\binom{n-k-2}{w-1}$ ways and there are w^{w-2} possibilities for the tree T_z; there are $T(n-w, k)$ ways to form a tree on the $n - w$ nodes in T_z that contain the given path of length k joining x and h; and if we now join z to any one of the $n - w$

nodes in this tree we obtain a tree T of the required type. It follows, therefore, from these considerations and Lemma 2.2, that

$$M_2(w_z) = \frac{1}{T(n,k)} \sum_{w=1}^{n-k-1} \binom{n-k-2}{w-1} w^{w-1}(w-1)(n-w)T(n-w,k)$$

$$= \frac{1}{n^{n-k-2}} \sum_{w=1}^{n-k-1} \binom{n-k-2}{w-1} w^{w-1}(w-1)(n-w)^{n-w-k-1}.$$

If $d(x, y) = k$, then the expected number of nodes other than i in any given subtree R_i of T is $(n-k-1)(k+1)^{-1}$. Consequently,

$$M_1(W_i) = (n-k-1)(k+1)^{-1}M_2(w_z)$$

and this implies that

$$M_1\left(\sum_{i=1}^{k} iW_i\right) = \tfrac{1}{2}k(n-k-1)M_2(w_z). \tag{2}$$

If we use identities (3.3) and (3.4) with $N = n-k-2$, $M = w-1$, $r = 1$, and $s = k+1$ to simplify the expression for $M_2(w_z)$, we find that

$$M_1\left(\sum_{i=1}^{k} iW_i\right) = \tfrac{1}{4}\,kn \sum_{v=1}^{n-k-1} (v-1)(v+2)(n-k-1)_v n^{-v}.$$

We now derive a formula for $M_2(w_i)$ over all the trees T that contain a given path of length k joining nodes x and y where i is the i-th node after x in this path. If $k+1-i \leqq w \leqq n-i$, then there are

$$\binom{n-k-1}{w-(k+1-i)} T(w, k-i)T(n-w, i-1).$$

such treees for which $w_i = w$. The last $k+1-i$ nodes of the path joining x and y are certainly in T_i and the remaining $w-(k+1-i)$ nodes in T_i can be selected in $\binom{n-k-1}{w-(k+1-i)}$ ways; there are $T(w, k-i)$ ways to form a tree T_i on w nodes that contains the subpath from i to k of the path from x to y and there are $T(n-w, i-1)$ ways to form a tree on the remaining $n-w$ nodes that contains the subpath from x to $i-1$ of the path from x to y; if we now join nodes i and $i-1$ we obtain a tree T of the required type. It follows, therefore, from these considerations and Lemma 2.2, that if $1 \leqq i \leqq k$, then

$$M_2(w_i) = \frac{1}{T(n,k)} \sum_{w=k+1-i}^{n-i} \binom{n-k-1}{w-(k+1-i)} T(w, k-i)T(n-w, i-1)w(w-1)$$

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$$= \frac{i(k+1-i)}{(k+1)n^{n-k-2}} \sum_{M=0}^{N} \binom{N}{M} (M+k+1-i)^{M}(M+k-i)(N-M+i)^{N-M-1}$$

where $N = n-k-1$ and $M = w-(k+1-i)$. If we use identities (3.1) and (3.2) with $r = k+1-i$ and $s = i$ to simplify the sum, we find that

$$M_2(w_i) = \frac{n(k+1-i)}{k+1} \sum_{v=0}^{n-k-1} (v+k+1-i)(n-k-1)_v n^{-v}$$
$$-n(k+1-i)(k+1)^{-1}.$$

It is easy to show by induction that

$$\sum_{i=1}^{k} (2i-1)(k+1-i) = k(k+1)(2k+1)/6$$

and

$$\sum_{i=1}^{k} (2i-1)(k+1-i)^2 = k(k+1)(k^2+k+1)/6.$$

These identities and the formula for $M_2(w_i)$ imply that

$$M_1\left\{\sum_{i=1}^{k} (2i-1)w_i(w_i-1)\right\} \tag{3}$$
$$= \tfrac{1}{6}kn \sum_{v=1}^{n-k-1} \{(2k+1)v+(k^2+k+1)\}(n-k-1)_v n^{-v} + \tfrac{1}{6}k^2(k-1)n.$$

The stated formula for $D(y,x;n,k)$ now follows from Theorem 7.1 and equations (2) and (3).

COROLLARY 7.2.1. *If* $1 \leqq k \leqq n-1$ *and*

$$H(n,k) = \sum_{v=1}^{n-k-1} (n-k-1)_v n^{-v}$$

then

$$D(y,x;n,k) = \tfrac{2}{3}kn\{[3n+(2k-3)(k+2)]H(n,k)-(k-1)(n-2k-1)\}.$$

PROOF. This follows from Theorem 7.2 and identities (3.5) and (3.6).

COROLLARY 7.2.2. *If* $k = o(n^{1/2})$, *then* $D(y,x;n,k) \sim (2\pi)^{1/2}\, kn^{5/2}$.

THEOREM 7.3. *If* $x \neq y$ *and* $D(y,x;n)$ *denotes the variance of* γ_{yx} *over all the* n^{n-2} *trees* T, *then*

$$D(y,x;n) = \frac{1}{30}\,\frac{n}{(n-1)} \sum_{v=3}^{n} v(v-1)(v-2)(8v^2+19v-49)(n)_v n^{-v}.$$

PROOF. The definition of $D(y,x;n)$ implies that

$$D(y,x;n)=\sum_{k=1}^{n-1} P(n,k)D(y,x;n,k)\,.$$

If we subtitute the expressions for $P(n,k)$ and $D(y,x;n,k)$ given by Lemma 2.3 and Corollary 7.2.1, interchange the order of summation, and evaluate the resulting inner sum, we obtain the stated formula for $D(y,x;n)$.

COROLLARY 7.3.1. $S(y,x;n) \sim \frac{32}{15}n^3$

The formulas established in the last two sections may be checked when $n = 3$ by considering the trees in Figure 2.

tree	frequency	$Pr\{\gamma_{yx} = i\}$	$M_1(yx)$	$V(yx)$
	1/3	$\begin{cases} 1, & i=1 \\ 0, & i \neq 1 \end{cases}$	1	0
	1/3	$\begin{cases} (\frac{1}{2})^j, & i=2j+1 \\ 0, & i=2j \end{cases}$	3	8
	1/3	$\begin{cases} 0, & i=2j+1 \\ (\frac{1}{2})^j, & i=2j \end{cases}$	4	8

$E(y,x;3,1) = 2$ $\quad E(y,x;3,2) = 4$ $\quad E(y,x;3) = 2\frac{2}{3}$

$D(y,x:3,1) = 4$ $\quad D(y,x:3,2) = 8$ $\quad D(y,x:3) = 5\frac{1}{3}$

Fig. 2. First Passage Times When $n = 3$.

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