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FACTORIZATION OF THE CANONICAL BASES FOR HIGHER-LEVEL FOCK SPACES

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Abstract The level l Fock space admits canonical bases $\mathcal{G}_e$ and $\mathcal{G}_\infty$. They correspond to $\mathcal{U}_v(\widehat{\mathfrak{sl}}_e)$ and $\mathcal{U}_v(\mathfrak{sl}_\infty)$-module structures. We establish that the transition matrices relating these two bases are unitriangular with coefficients in $\mathbb{N}[v]$. Restriction to the highest-weight modules generated by the empty l-partition then gives a natural quantization of a theorem by Geck and Rouquier on the factorization of decomposition matrices which are associated to Ariki–Koike algebras.

Keywords: Kashiwara–Lusztig canonical basis; Fock space, decomposition matrices; cyclotomic Hecke algebras

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1. Introduction

In the classification of finite complex reflection groups by Shephard and Todd [**29**], there is a single infinite family of groups $G(lp, p, n)$ parametrized by the triples $(l, p, n) \in \mathbb{N}^3$ and 34 other 'exceptional' groups. If $p = 1$, the group $G(l, 1, n)$ is the wreath product of the cyclic group of order l with the symmetric group S_n. It generalizes both the Weyl group of type A_{n-1} (corresponding to the case $l = 1$) and the Weyl group of type B_n ($l = 2$). We may associate to $G(l, 1, n)$ its Hecke algebra over the ring $A := \mathbb{C}[q^{\pm 1}, Q_1^{\pm 1}, \dots, Q_l^{\pm 1}]$, where $(q, Q_1, \dots, Q_l)$ is an $(l + 1)$-tuple of indeterminates. This algebra can be seen as a deformation of the group algebra of $G(l, 1, n)$ and has applications to the modular representation theory of finite reductive groups (see, for example, [**26**]). As an A-algebra, it has the set of generators $\{T_0, \dots, T_{n-1}\}$ such that the defining relations are

$$\prod_{i=1}^{l}(T_0 - Q_i) = 0, \quad (T_i - q)(T_i + 1) = 0, \quad i = 1, \dots, n - 1,$$

and the braid relations of type B_n. We denote this algebra by $\mathcal{H}_A$. If we extend the scalars of $\mathcal{H}_A$ to $K = \mathbb{C}(q, Q_1, \dots, Q_l)$, the field of fractions of A, we obtain the algebra $\mathcal{H}_K := K \otimes_A \mathcal{H}_A$, whose representation theory is well understood. For example, we know how to classify the irreducible representations, their dimensions, etc. [**2**, **15**]. The theory is far more difficult in the modular case. Let $\theta\colon A \to \mathbb{C}$ be a ring homomorphism and let $\mathcal{H}_{\mathbb{C}} := \mathbb{C} \otimes_A \mathcal{H}_A$ be the associated Hecke algebra. Due to results of Dipper and Mathas [**7**], one can reduce various important problems to the case where

$$\theta(q) = \eta_e := \exp\left(\frac{2\mathrm{i}\pi}{e}\right)$$

is an eth root of unity, for $e \in \mathbb{Z}_{\geqslant 2}$, and $\theta(Q_j) = \eta_e^{s_j}$, for $j = 1, \dots, l$, where $(s_1, \dots, s_l) \in \mathbb{Z}^l$. An important object of study in the modular case is the decomposition map. As $\mathcal{H}_A$ is a cellular algebra [**13**], the decomposition map may be defined as follows. Let $V_K \in \mathrm{Irr}(\mathcal{H}_K)$. Then there exists a specific $\mathcal{H}_A$-module V_A, which is called a *cell module*, such that $V_K = K \otimes_A V_A$. We can then associate to V_K the $\mathcal{H}_{\mathbb{C}}$-module $V_{\mathbb{C}} = \mathbb{C} \otimes_A V_A$. This gives a well-defined map between Grothendieck groups $R_0(\mathcal{H}_K)$ of finitely generated $\mathcal{H}_K$-modules and $R_0(\mathcal{H}_{\mathbb{C}})$ of finitely generated $\mathcal{H}_{\mathbb{C}}$-modules. We denote the decomposition map by

$$d_\theta\colon R_0(\mathcal{H}_K) \to R_0(\mathcal{H}_{\mathbb{C}}).$$

We denote the associated decomposition matrix by D_e. It is known that we may choose V_A more general than the cell module and the decomposition map is still well defined [**10**].

There exist algorithms to compute the map d_θ, but it remains difficult to describe it in general. One useful tool here is a result by Geck and Rouquier [**12**], which gives information on the matrix D_e by factorizing the decomposition map. Let $\theta^q\colon A \to \mathbb{C}(q)$ be the specialization map defined as $\theta^q(Q_i) = q^{s_i}$ for $i = 1, \dots, l$. Denote by

$$\mathcal{H}_{\mathbb{C}(q)} := \mathbb{C}(q) \otimes_A \mathcal{H}_A$$

the associated Hecke algebra. As above, we have the decomposition map

$$d_{\theta^q}\colon R_0(\mathcal{H}_K) \to R_0(\mathcal{H}_{\mathbb{C}(q)})$$

and the associated decomposition matrix D_∞. Then [**12**, Proposition 2.12] implies the following.

Theorem 1.1 (Geck–Rouquier). *There exists a unique $\mathbb{Z}$-linear map*

$$d^\theta_{\theta^q}\colon R_0(\mathcal{H}_{\mathbb{C}(q)}) \to R_0(\mathcal{H}_{\mathbb{C}})$$

such that the following diagram commutes:

$$\begin{tikzcd}
R_0(\mathcal{H}_K) \arrow[rr, "d_\theta"] \arrow[dr, "d_{\theta^q}"'] & & R_0(\mathcal{H}_{\mathbb{C}}) \\
& R_0(\mathcal{H}_{\mathbb{C}(q)}) \arrow[ur, "d^\theta_{\theta^q}"'] &
\end{tikzcd}$$

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Thus, we have the factorization $D_e = D_\infty \cdot D_\infty^e$ of the decomposition matrices, where D_∞^e is the decomposition matrix for $d_{\theta q}^\theta$. We shall call D_∞^e the *relative decomposition matrix*. This result shows that a part of the representation theory of $\mathcal{H}_\mathbb{C}$ depends not on e but only on the representation theory of $\mathcal{H}_{\mathbb{C}(q)}$, which is 'easier' to understand (for example, there are closed formulae for the entries of D_∞ when $l = 2$ [**24**]). An example of its application is that one may give an explicit relationship among various classifications of simple modules arising from the theory of canonical basic sets in type B_n [**18**].

In view of Fock space theory, which is now standard in the study of Hecke algebras, Theorem 1.1 naturally leads to several questions. As noted above, there is an algorithm for computing the decomposition matrices of $\mathcal{H}_\mathbb{C}$ and $\mathcal{H}_{\mathbb{C}(q)}$. This algorithm relies on the first author's proof [**1**] of the Lascoux–Leclerc–Thibon (LLT) conjecture [**23**]. His theorem asserts that D_e (respectively, D_∞) is equal to the evaluation at $v = 1$ of the matrix $D_e(v)$ (respectively, $D_\infty(v)$) that is obtained by expanding the canonical basis in a highest-weight $\mathcal{U}_v(\widehat{\mathfrak{sl}}_e)$-module (respectively, $\mathcal{U}_v(\mathfrak{sl}_\infty)$-module) into linear combination of the standard basis of a Fock space. Thus, Theorem 1.1 implies the existence of a matrix D_∞^e such that $D_e(1) = D_\infty(1) \cdot D_\infty^e$. The entries of $D_e(v)$ and $D_\infty(v)$ are known to be in $\mathbb{N}[v]$, i.e. polynomials with non-negative integer coefficients. Hence, it is natural to ask the following questions.

(Q_1) Does the matrix D_∞^e have a natural quantization? Namely, is there a matrix $D_\infty^e(v)$ with entries in $\mathbb{N}[v]$ such that

$$D_e(v) = D_\infty(v) \cdot D_\infty^e(v)?$$

(Q_2) If $D_\infty^e(v)$ is known to exist, find a practical algorithm to compute $D_\infty^e(v)$.

In other words, we ask whether the matrix of the canonical basis for $\mathcal{U}_v(\widehat{\mathfrak{sl}}_e)$-modules factorizes through the matrix of the canonical basis for $\mathcal{U}_v(\mathfrak{sl}_\infty)$-modules.

Integrable highest-weight $\mathcal{U}_v(\widehat{\mathfrak{sl}}_e)$-modules and $\mathcal{U}_v(\mathfrak{sl}_\infty)$-modules are realized as irreducible components of Fock spaces of higher level. By [**30**], these Fock spaces also admit canonical bases. So the above questions also make sense for the matrices $\Delta_e(v)$ and $\Delta_\infty(v)$ that are associated to the canonical bases of the whole Fock space. Thus, instead of (Q_1), we ask whether there exists a matrix $\Delta_\infty^e(v)$ with entries in $\mathbb{N}[v]$ such that

$$\Delta_e(v) = \Delta_\infty(v) \cdot \Delta_\infty^e(v).$$

The matrix $\Delta_\infty^e(v)$ is expected to have several interpretations. Observe that recent conjectures and results [**3**–**5**] show that $D_e(v)$ and $D_\infty(v)$ should be interpreted as graded decomposition matrices of Hecke algebras. $D_\infty^e(v)$ might also be interpreted as a graded analogue of D_∞^e in this setting. According to conjectures of Yvonne [**31**,**33**] and Rouquier [**28**, § 6.4], $\Delta_e(1)$ and $\Delta_\infty(1)$ are expected to be decomposition matrices of a generalized η_e and q-Schur algebras, respectively. Thus, $\Delta_\infty^e(v)$ might have a similar meaning to $D_\infty^e(v)$ as well.

In another direction, we interpret the factorization $D_e = D_\infty \cdot D_\infty^e$ in the context of parabolic BGG categories from the previous section. This second interpretation should

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also have a graded version that is independent of the first (note that Hecke algebras are not positively graded).

We answer affirmatively to questions (Q_1) and (Q_2) for $\Delta^e_\infty(v)$. We first show the existence of the matrices $D^e_\infty(v)$ and $\Delta^e_\infty(v)$ with entries in $\mathbb{Z}[v]$. In fact, $D^e_\infty(v)$ is a submatrix of $\Delta^e_\infty(v)$ and we provide an efficient algorithm for computing it (and thus an algorithm for computing D^e_∞). Then we prove that the entries of $\Delta^e_\infty(v)$ are in $\mathbb{N}[v]$. More precisely, we show that they can be expressed as sums of products of structure constants of the affine Hecke algebras of type A with respect to the Kazhdan–Lusztig basis and its generalization by Grojnowski and Haiman [**14**].

Let us briefly summarize the main ingredients of our proofs. The Fock space theory developed in [**20**] and the notion of canonical bases for these Fock spaces introduced in [**30**] make apparent strong connections between the representation theories of $\mathcal{U}_v(\widehat{\mathfrak{sl}}_e)$ and $\mathcal{U}_v(\mathfrak{sl}_\infty)$. They permit us to prove the existence of a matrix $\Delta^e_\infty(v)$ with entries in $\mathbb{Z}[v]$ such that $\Delta_e(v) = \Delta_\infty(v) \cdot \Delta^e_\infty(v)$. This factorization can be regarded as an analogue, at the level of canonical bases, of the compatibility of the crystal graph structures established in [**19**]. It is achieved by introducing a new partial order on the set of l-partitions that does not depend on e. This order differs from that used in [**30**] and has the property that $\Delta_e(v)$ and $\Delta_\infty(v)$ are simultaneously unitriangular. The compatibility between the $\mathcal{U}_v(\widehat{\mathfrak{sl}}_e)$ and $\mathcal{U}_v(\mathfrak{sl}_\infty)$-module structures on the Fock space then implies the factorization $\Delta_e(v) = \Delta_\infty(v) \cdot \Delta^e_\infty(v)$. To show the positivity, recall that the coefficients of the matrices $\Delta_\infty(v)$ and $\Delta_e(v)$ are expressed by parabolic Kazhdan–Lusztig polynomials of the affine Hecke algebras of type A [**30**]. We see in a simpler manner than [**30**] how the parabolic Kazhdan–Lusztig polynomials are related to the entries of $\Delta_\infty(v)$ and $\Delta_e(v)$ for a fixed pair of l-partitions. The positivity result then follows from this and the positivity of the structure constants of the affine Hecke algebra.

2. Background on Fock spaces and canonical bases

We refer the interested reader to [**1**, **21**] for a detailed review of canonical and crystal basis theory. [**11**, § 7] also gives a good survey of modular representation theory of Hecke algebras. Let v be an indeterminate, let $e > 1$ be an integer and let $\mathcal{U}_v(\widehat{\mathfrak{sl}}_e)$ be the quantum group of type $A^{(1)}_{e-1}$. It is an associative $\mathbb{Q}(v)$-algebra with Chevalley generators e_i, f_i, t_i, t_i^{-1} for $i \in \mathbb{Z}/e\mathbb{Z}$ and ∂. We refer the reader to [**30**, § 2.1] for a precise definition. The bar involution '$\bar{\cdot}$' is the ring automorphism of $\mathcal{U}_v(\widehat{\mathfrak{sl}}_e)$ such that $\bar{v} = v^{-1}$, $\bar{\partial} = \partial$ and

$$\bar{e}_i = e_i, \quad \bar{f}_i = f_i, \quad \bar{t}_i = t_i^{-1} \quad \text{for } i \in \mathbb{Z}/e\mathbb{Z}.$$

We denote by $\mathcal{U}'_v(\widehat{\mathfrak{sl}}_e)$ the subalgebra generated by $\{e_i, f_i, t_i, t_i^{-1} \mid i \in \mathbb{Z}/e\mathbb{Z}\}$. By slight abuse of notation, we identify the elements of $\mathbb{Z}/e\mathbb{Z}$ with their corresponding labels in $\{0, \dots, e-1\}$ when there is no risk of confusion. Write $\{\Lambda_0, \dots, \Lambda_{e-1}\}$ for the set of fundamental weights of $\mathcal{U}_v(\widehat{\mathfrak{sl}}_e)$, and write δ for the null root. Let $l \in \mathbb{Z}_{\geqslant 1}$ and consider $\boldsymbol{s} = (s_1, \dots, s_l) \in \mathbb{Z}^l$, which we call a *multi-charge*. We set

$$\mathfrak{s} = (s_1 (\operatorname{mod} e), \dots, s_l (\operatorname{mod} e)) \in (\mathbb{Z}/e\mathbb{Z})^l \quad \text{and} \quad \Lambda_{\mathfrak{s}} := \Lambda_{s_1 (\operatorname{mod} e)} + \cdots + \Lambda_{s_l (\operatorname{mod} e)}.$$

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Similarly, let $\mathcal{U}_v(\mathfrak{sl}_\infty)$ be the quantum group of type A_∞. It is an associative $\mathbb{Q}(v)$-algebra with Chevalley generators E_j, F_j, T_j and T_j^{-1} for $j \in \mathbb{Z}$. We use the same symbol '$\bar{\cdot}$' to denote its bar-involution, which is the ring automorphism of $\mathcal{U}_v(\mathfrak{sl}_\infty)$ such that $\bar{v} = v^{-1}$ and

$$\bar{E}_j = E_j, \quad \bar{F}_j = F_j, \quad \bar{T}_j = T_j^{-1} \quad \text{for } j \in \mathbb{Z}.$$

Write $\{\omega_j, j \in \mathbb{Z}\}$ for its set of fundamental weights. To $\boldsymbol{s} = (s_1, \ldots, s_l) \in \mathbb{Z}^l$, we associate the dominant weight $\Lambda_{\boldsymbol{s}} := \omega_{s_1} + \cdots + \omega_{s_l}$.

2.1. Fock spaces

Let $\Pi_{l,n}$ be the set of l-partitions with rank n, that is, the set of sequences $\boldsymbol{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(l)})$ of partitions such that

$$|\boldsymbol{\lambda}| = |\lambda^{(1)}| + \cdots + |\lambda^{(l)}| = n.$$

Set $\Pi_l = \cup_{n \geqslant 0} \Pi_{l,n}$. We also write $\Pi = \cup_{n \geqslant 0} \Pi_{1,n}$ for short. The *Fock space* $\mathcal{F}$ of level l is a $\mathbb{Q}(v)$-vector space which has the set of all l-partitions as the given basis, so that we write

$$\mathcal{F} = \bigoplus_{\boldsymbol{\lambda} \in \Pi_l} \mathbb{Q}(v)\boldsymbol{\lambda}.$$

The Fock space $\mathcal{F}$ may be endowed with a structure of $\mathcal{U}_v(\widehat{\mathfrak{sl}}_e)$ and $\mathcal{U}_v(\mathfrak{sl}_\infty)$-modules. Let $\boldsymbol{\lambda}$ be an l-partition (identified with its Young diagram). Then, the *nodes* of $\boldsymbol{\lambda}$ are the triples $\gamma = (a, b, c)$, where $c \in \{1, \ldots, l\}$ and a, b are the row and column indices of the node γ in $\lambda^{(c)}$, respectively. The *content* of γ is the integer $c(\gamma) = b - a + s_c$ and the *residue* $\operatorname{res}(\gamma)$ of γ is the element of $\mathbb{Z}/e\mathbb{Z}$ such that

$$\operatorname{res}(\gamma) \equiv c(\gamma) (\operatorname{mod} e). \tag{2.1}$$

For $i \in \mathbb{Z}/e\mathbb{Z}$, we say that γ is an i-node of $\boldsymbol{\lambda}$ when $\operatorname{res}(\gamma) \equiv i (\operatorname{mod} e)$. Similarly, for $j \in \mathbb{Z}$, we say that γ is a j-node of $\boldsymbol{\lambda}$ when $c(\gamma) = j$. We say that a node γ is *removable* when $\gamma = (a, b, c) \in \boldsymbol{\lambda}$ and $\boldsymbol{\lambda} \setminus \{\gamma\}$ is an l-partition and *addable* when $\gamma = (a, b, c) \notin \boldsymbol{\lambda}$ and $\boldsymbol{\lambda} \cup \{\gamma\}$ is an l-partition.

Let $i \in \mathbb{Z}/e\mathbb{Z}$. Following the convention of [**30**], we define a total order on the set of i-nodes of $\boldsymbol{\lambda}$. Consider two nodes $\gamma_1 = (a_1, b_1, c_1)$ and $\gamma_2 = (a_2, b_2, c_2)$ in $\boldsymbol{\lambda}$. We define the order by

$$\gamma_1 \prec_{\boldsymbol{s}} \gamma_2 \iff \begin{cases} c(\gamma_1) < c(\gamma_2), \\ c(\gamma_1) = c(\gamma_2) \text{ and } c_1 < c_2. \end{cases}$$

Let $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ be two l-partitions of rank n and $n+1$ such that $[\boldsymbol{\mu}] = [\boldsymbol{\lambda}] \cup \{\gamma\}$, where γ is an i-node. Define

$$\begin{aligned} N_i^{\succ}(\boldsymbol{\lambda}, \boldsymbol{\mu}) = \sharp\{&\text{addable } i\text{-nodes } \gamma' \text{ of } \boldsymbol{\lambda} \text{ such that } \gamma' \succ_{\boldsymbol{s}} \gamma\} \\ &- \sharp\{\text{removable } i\text{-nodes } \gamma' \text{ of } \boldsymbol{\mu} \text{ such that } \gamma' \succ_{\boldsymbol{s}} \gamma\}, \end{aligned} \tag{2.2}$$

$$\begin{aligned} N_i^{\prec}(\boldsymbol{\lambda}, \boldsymbol{\mu}) = \sharp\{&\text{addable } i\text{-nodes } \gamma' \text{ of } \boldsymbol{\lambda} \text{ such that } \gamma' \prec_{\boldsymbol{s}} \gamma\} \\ &- \sharp\{\text{removable } i\text{-nodes } \gamma' \text{ of } \boldsymbol{\mu} \text{ such that } \gamma' \prec_{\boldsymbol{s}} \gamma\}, \end{aligned} \tag{2.3}$$

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$$N_i(\boldsymbol{\lambda}) = \sharp\{\text{addable } i\text{-nodes of } \boldsymbol{\lambda}\} - \sharp\{\text{removable } i\text{-nodes of } \boldsymbol{\lambda}\},$$

$$M_0(\boldsymbol{\lambda}) = \sharp\{0\text{-nodes of } \boldsymbol{\lambda}\}.$$

Theorem 2.1 (Jimbo *et al.* [20]). *Let $\boldsymbol{s} \in \mathbb{Z}^l$. The Fock space $\mathcal{F}$ has a structure of an integrable $\mathcal{U}_v(\widehat{\mathfrak{sl}}_e)$-module $\mathcal{F}_e^{\boldsymbol{s}}$ defined by*

$$\begin{aligned}
e_i\boldsymbol{\lambda} &= \sum_{\operatorname{res}([\boldsymbol{\lambda}]/[\boldsymbol{\mu}])=i} v^{-N_i^{\prec}(\boldsymbol{\mu},\boldsymbol{\lambda})}\boldsymbol{\mu},\\
f_i\boldsymbol{\lambda} &= \sum_{\operatorname{res}([\boldsymbol{\mu}]/[\boldsymbol{\lambda}])=i} v^{N_i^{\succ}(\boldsymbol{\lambda},\boldsymbol{\mu})}\boldsymbol{\mu},\\
t_i\boldsymbol{\lambda} &= v^{N_i(\boldsymbol{\lambda})}\boldsymbol{\lambda},\\
\partial\boldsymbol{\lambda} &= -(\Delta + M_0(\boldsymbol{\lambda}))\boldsymbol{\lambda},
\end{aligned}$$

*for $i \in \mathbb{Z}/e\mathbb{Z}$, where Δ is the rational number defined in [**20**, Theorem 2.1]. The module structure on $\mathcal{F}_e^{\boldsymbol{s}}$ depends on $\boldsymbol{s}$ and e.*

We may consider $\mathcal{F}$ as a $\mathcal{U}'_v(\widehat{\mathfrak{sl}}_e)$-module by restriction. We denote it by the same $\mathcal{F}_e^{\boldsymbol{s}}$ by abuse of notation.

Let $j \in \mathbb{Z}$. For l-partitions $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ of rank n and $n+1$ such that $[\boldsymbol{\mu}] = [\boldsymbol{\lambda}] \cup \{\gamma\}$, where γ is a j-node, we define $N_j^{\succ}(\boldsymbol{\lambda},\boldsymbol{\mu})$, $N_j^{\prec}(\boldsymbol{\lambda},\boldsymbol{\mu})$ and $N_j(\boldsymbol{\lambda})$ as in (2.2) except that we consider j-nodes for $e = \infty$ instead of i-nodes for e finite.

Theorem 2.2 (Jimbo *et al.* [20]). *Let $\boldsymbol{s} \in \mathbb{Z}^l$. The Fock space $\mathcal{F}$ has a structure of an integrable $\mathcal{U}_v(\mathfrak{sl}_\infty)$-module $\mathcal{F}_\infty^{\boldsymbol{s}}$ defined as*

$$\begin{aligned}
E_j\boldsymbol{\lambda} &= \sum_{c([\boldsymbol{\lambda}]/[\boldsymbol{\mu}])=j} v^{-N_j^{\prec}(\boldsymbol{\mu},\boldsymbol{\lambda})}\boldsymbol{\mu},\\
F_j\boldsymbol{\lambda} &= \sum_{c([\boldsymbol{\mu}]/[\boldsymbol{\lambda}])=j} v^{N_j^{\succ}(\boldsymbol{\lambda},\boldsymbol{\mu})}\boldsymbol{\mu},\\
T_j\boldsymbol{\lambda} &= v^{N_j(\boldsymbol{\lambda})}\boldsymbol{\lambda},
\end{aligned}$$

for $j \in \mathbb{Z}$. The module structure on $\mathcal{F}_e^{\boldsymbol{s}}$ depends on $\boldsymbol{s}$.

The following result is implicit in [**20**, Proposition 3.5].

Proposition 2.3. *The $\mathcal{U}'_v(\widehat{\mathfrak{sl}}_e)$ and $\mathcal{U}_v(\mathfrak{sl}_\infty)$-module structures $\mathcal{F}_e^{\boldsymbol{s}}$ and $\mathcal{F}_\infty^{\boldsymbol{s}}$ are compatible in the sense that we may write the action of e_i, f_i and t_i for $i \in \mathbb{Z}/e\mathbb{Z}$ as follows:*

$$\begin{aligned}
e_i &= \sum_{j\in\mathbb{Z},\, j\equiv i (\operatorname{mod} e)} \Big(\prod_{r\geqslant 1} T_{j-re}^{-1}\Big) E_j,\\
f_i &= \sum_{j\in\mathbb{Z},\, j\equiv i (\operatorname{mod} e)} \Big(\prod_{r\geqslant 1} T_{j+re}\Big) F_j,\\
t_i &= \prod_{j\in\mathbb{Z},\, j\equiv i (\operatorname{mod} e)} T_j.
\end{aligned}$$

Remark 2.4. The infinite sums and products in the proposition reduce, in fact, to finite ones since the number of nodes in $\boldsymbol{\lambda}$ is finite.

The empty multi-partition $\boldsymbol{\emptyset}$ is a highest-weight vector in $\mathcal{F}_e^{\boldsymbol{s}}$ and $\mathcal{F}_\infty^{\boldsymbol{s}}$ of weight $\Lambda_{\mathfrak{s}}$ and $\Lambda_{\boldsymbol{s}}$, respectively. We then define $V_e(\boldsymbol{s})$ and $V_\infty(\boldsymbol{s})$ as the highest-weight modules $\mathcal{U}'_v(\widehat{\mathfrak{sl}}_e) \cdot \boldsymbol{\emptyset}$ and $\mathcal{U}_v(\mathfrak{sl}_\infty) \cdot \boldsymbol{\emptyset}$, respectively. Observe that the module structure on $V_e(\boldsymbol{s})$ really depends on $\boldsymbol{s}$ and not only on its class $\mathfrak{s}$ modulo e. By the previous proposition, it follows that $V_\infty(\boldsymbol{s})$ is endowed with the structure of a $\mathcal{U}'_v(\widehat{\mathfrak{sl}}_e)$-module and $V_e(\boldsymbol{s})$ coincides with the $\mathcal{U}'_v(\widehat{\mathfrak{sl}}_e)$-submodule of $V_\infty(\boldsymbol{s})$ generated by the highest-weight vector $\boldsymbol{\emptyset}$.

2.2. Uglov's canonical bases

We now briefly recall Uglov's plus canonical basis of the Fock spaces. Let $\mathbb{A}(v)$ be the ring of rational functions which have no pole at $v = 0$. Set

$$\mathcal{L} := \bigoplus_{n \geqslant 0} \bigoplus_{\boldsymbol{\lambda} \in \Pi_{l,n}} \mathbb{A}(v)\boldsymbol{\lambda} \quad \text{and} \quad \mathcal{B} := \{\boldsymbol{\lambda} (\operatorname{mod} v\mathcal{L}) \mid \boldsymbol{\lambda} \in \Pi_l\}.$$

Theorem 2.5 (Foda *et al.* [9]). *The pair $(\mathcal{L}, \mathcal{B})$ is a crystal basis for $\mathcal{F}_e^{\boldsymbol{s}}$ and $\mathcal{F}_\infty^{\boldsymbol{s}}$.*

Note that, although the crystal lattice $\mathcal{L}$ and the basis $\mathcal{B}$ of $\mathcal{L}/v\mathcal{L}$ are the same for $\mathcal{F}_e^{\boldsymbol{s}}$ and $\mathcal{F}_\infty^{\boldsymbol{s}}$, the induced crystal structures $\mathcal{B}_e$ and $\mathcal{B}_\infty$ on $\mathcal{B}$ do not coincide. The crystal structure $\mathcal{B}_e$ is obtained as follows. Let $\boldsymbol{\lambda}$ be an l-partition, and let $i \in \mathbb{Z}/e\mathbb{Z}$. We consider the set of addable and removable i-nodes of $\boldsymbol{\lambda}$. We read the nodes in the increasing order with respect to $\prec_{\boldsymbol{s}}$, and let w_i be the resulting word of the nodes. If a removable i-node appears just before an addable i-node, we delete both and continue the same procedure as many times as possible. In the end, we reach a word $\tilde{w}_i$ of nodes such that the first p nodes are addable and the last q nodes are removable, for some $p, q \in \mathbb{N}$. If $p > 0$, let γ be the rightmost addable i-node in $\tilde{w}_i$. The node γ is called the *good* i-node of $\boldsymbol{\lambda}$. Then the crystal $\mathcal{B}_e$ may be read off from its crystal graph as follows.

Vertices: l-partitions whose nodes are coloured with residues.

Edges: $\boldsymbol{\lambda} \xrightarrow{i} \boldsymbol{\mu}$ if and only if $\boldsymbol{\mu}$ is obtained by adding a good i-node to $\boldsymbol{\lambda}$.

We denote by $B_e(\boldsymbol{s})$ the connected component of $\mathcal{B}_e$ that contains the highest-weight vertex $\boldsymbol{\emptyset}$. We may identify $B_e(\boldsymbol{s})$ with the crystal graph of $V_e(\boldsymbol{s})$. The crystal graph of $\mathcal{B}_\infty$ is obtained in a similar manner. We use j-nodes ($j \in \mathbb{Z}$), for $e = \infty$, instead of i-nodes, for e finite. We may also identify the crystal graph of $V_\infty(\boldsymbol{s})$ with $B_\infty(\boldsymbol{s})$, the connected component of $\mathcal{B}_\infty$ which contains the highest-weight vertex $\boldsymbol{\emptyset}$.

Let $e \in \mathbb{Z}_{\geqslant 2} \cup \{\infty\}$. We set

$$\mathcal{U}_v(\mathfrak{g}) = \begin{cases} \mathcal{U}'_v(\widehat{\mathfrak{sl}}_e) & \text{if } e < \infty, \\ \mathcal{U}_v(\mathfrak{sl}_\infty) & \text{if } e = \infty \end{cases}$$

for ease of notation. We define a $\mathbb{Z}[v]$-lattice $\mathcal{L}_{\mathbb{Z}}$ of $\mathcal{L}$ by

$$\mathcal{L}_{\mathbb{Z}} := \bigoplus_{n \geqslant 0} \bigoplus_{\boldsymbol{\lambda} \in \Pi_{l,n}} \mathbb{Z}[v]\boldsymbol{\lambda}.$$

In [**30**], Uglov introduced a bar-involution '$\bar{\cdot}$' on $\mathcal{F}_e^{\boldsymbol{s}}$, which is defined as

$$\overline{u \cdot f} = \bar{u} \cdot \bar{f} \quad \text{for } u \in \mathcal{U}_v(\mathfrak{g}) \text{ and } f \in \mathcal{F}_e^{\boldsymbol{s}}, \qquad \bar{\boldsymbol{\emptyset}} = \boldsymbol{\emptyset}.$$

Such a bar-involution is easier to define for the Fock space $\mathcal{F}_\infty^{\boldsymbol{s}}$, as is explained in [**4**, § 3.9]. In the two cases, this leads to the following theorem definition.

Theorem 2.6 (Uglov [30]). *Let $\boldsymbol{s} \in \mathbb{Z}^l$ and $e \in \mathbb{Z}_{\geqslant 2} \cup \{\infty\}$. There exists a unique basis*

$$\mathcal{G}_e(\boldsymbol{s}) = \{G_e(\boldsymbol{\lambda}, \boldsymbol{s}) \mid \boldsymbol{\lambda} \in \Pi_l\}$$

of $\mathcal{F}_e^{\boldsymbol{s}}$ such that the basis elements are characterized by the following two conditions.

(i) $\overline{G_e(\boldsymbol{\lambda}, \boldsymbol{s})} = G_e(\boldsymbol{\lambda}, \boldsymbol{s})$.

(ii) $G_e(\boldsymbol{\lambda}, \boldsymbol{s}) \equiv \boldsymbol{\lambda} (\operatorname{mod} v\mathcal{L}_{\mathbb{Z}})$.

The basis $\mathcal{G}_e(\boldsymbol{s})$ is called the *plus canonical basis* of $\mathcal{F}_e^{\boldsymbol{s}}$. It strongly depends on $e \in \mathbb{Z}_{\geqslant 2} \cup \{\infty\}$. The purpose of the next theorem is to identify the Kashiwara–Lusztig canonical basis of $V_e(\boldsymbol{s})$ with a subset of $\mathcal{G}_e(\boldsymbol{s})$.

Theorem 2.7 (Uglov [30]). *Let $\boldsymbol{s} \in \mathbb{Z}^l$ and $e \in \mathbb{Z}_{\geqslant 2} \cup \{\infty\}$. Define*

$$\mathcal{G}_e^{\circ}(\boldsymbol{s}) = \mathcal{G}_e(\boldsymbol{s}) \cap V_e(\boldsymbol{s}).$$

Then $\mathcal{G}_e^{\circ}(\boldsymbol{s})$ coincides with the canonical basis of the irreducible highest-weight $\mathcal{U}_v(\mathfrak{g})$-module $V_e(\boldsymbol{s})$. Moreover, $G_e(\boldsymbol{\lambda}, \boldsymbol{s}) \in \mathcal{G}_e^{\circ}(\boldsymbol{s})$ if and only if $\boldsymbol{\lambda} \in B_e(\boldsymbol{s})$.

3. Compatibility of canonical bases

In this section we prove that each $G_e(\boldsymbol{\lambda}, \boldsymbol{s})$ may be expanded into a $\mathbb{Z}[v]$-linear combination of the canonical basis $\mathcal{G}_\infty(\boldsymbol{s})$. A crucial observation for the proof is that we may define a partial order on multi-partitions that is independent of e. Then, the transition matrix becomes unitriangular with respect to the partial order.

3.1. Some combinatorial preliminaries

A 1-*runner abacus* is a subset A of $\mathbb{Z}$ such that $-k \in A$ and $k \notin A$ for all sufficiently large $k \in \mathbb{N}$. To visualize a 1-runner abacus, we view $\mathbb{Z}$ as a horizontal runner and place a bead on the kth position for each $k \in A$. Thus, the runner is full of beads on the far left and has no beads on the far right. For $l \geqslant 1$, an *l-runner abacus* is an l-tuple of 1-runner abaci. Let $\mathcal{A}^l$ be the set of l-runner abaci. To each pair of an l-partition $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(l)})$ and a multi-charge $\boldsymbol{s} = (s_1, \dots, s_l) \in \mathbb{Z}^l$, we associate the l-runner abacus

$$a(\boldsymbol{\lambda}, \boldsymbol{s}) := \{(\lambda_i^{(d)} + s_d + 1 - i, d) \mid i \geqslant 1,\ 1 \leqslant d \leqslant l\},$$

which is a subset of $\mathbb{Z} \times [1, l]$. One checks easily that the map

$$(\boldsymbol{\lambda}, \boldsymbol{s}) \in \Pi_l \times \mathbb{Z}^l \mapsto a(\boldsymbol{\lambda}, \boldsymbol{s}) \in \mathcal{A}^l$$

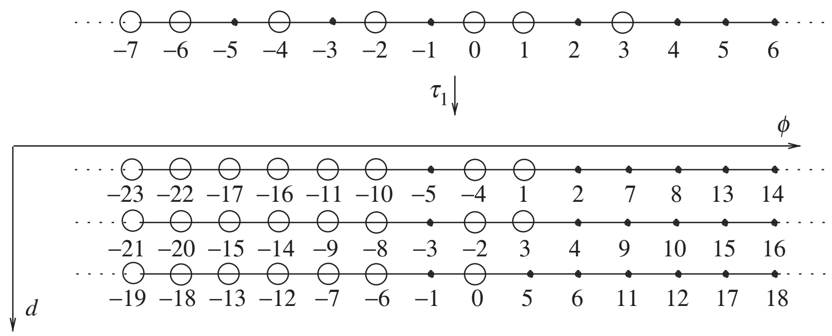

Figure 1. Computation of the bijection τ_l using abaci.

is bijective. To describe the embedding of Fock spaces into the space of semi-infinite wedge products and then cut semi-infinite wedge products to finite wedge products, we need to introduce a bijective map

$$\tau_l \colon \Pi \times \mathbb{Z} \cong \mathcal{A} \to \mathcal{A}^l \cong \Pi_l \times \mathbb{Z}^l.$$

Definition 3.1. Let $\tau_l \colon \mathbb{Z} \to \mathbb{Z} \times [1, l]$ be the bijective map defined as

$$k \mapsto (\phi(k), d(k)),$$

where $k = c(k) + e(d(k) - 1) + elm(k)$ such that

$$c(k) \in [1, e], \qquad d(k) \in [1, l], \qquad m(k) \in \mathbb{Z}$$

and $\phi(k) = c(k) + em(k)$. Then we define $\tau_l : \Pi \times \mathbb{Z} \cong \mathcal{A} \to \mathcal{A}^l \cong \Pi_l \times \mathbb{Z}^l$ as

$$A \mapsto \tau_l(A) = \{(\phi(k), d(k)) \mid k \in A\} \in \mathcal{A}^l \quad \text{for } A \in \mathcal{A}.$$

Remark 3.2.

(i) If $(\boldsymbol{\lambda}, \boldsymbol{s}) = \tau_l(\lambda, s)$, then $s = s_1 + \cdots + s_l$.

(ii) To read off the multi-charge $\boldsymbol{s} = (s_1, \dots, s_l)$ from the l-runner abacus, we proceed as follows. If the left-adjacent position of a bead on a runner is vacant, we move the bead to the left to occupy the vacant position, and we repeat this procedure as many times as possible. Then, s_d is the column number of the rightmost bead of the dth runner.

Example 3.3. Let $e = 2$ and $l = 3$. Then the preimage of

$$(\boldsymbol{\lambda}, \boldsymbol{s}) = (((1 \cdot 1), (1 \cdot 1), (1)), (0, 0, -1))$$

is $(\lambda, s) = ((4 \cdot 3 \cdot 3 \cdot 2 \cdot 1), -1)$.

Now, $(\lambda, s) = \tau_l^{-1}(\boldsymbol{\lambda}, \boldsymbol{s})$ has the 1-runner abacus

$$a(\lambda, s) = \{(k_i := \lambda_i + s + 1 - i) \mid i \geqslant 1\},$$

and the semi-infinite sequence $(k_1, k_2, \dots)$ defines a semi-infinite wedge product.

We fix r sufficiently large such that

$$\lambda_i = 0 \quad \text{for } i \geqslant r. \tag{3.1}$$

Then (λ, s) is determined by the finite sequence $\boldsymbol{k} := (k_1, \dots, k_r)$. For example, $((4 \cdot 3 \cdot 3 \cdot 2 \cdot 1), -1)$ is determined by $\boldsymbol{k} = (3, 1, 0, -2, -4, -6, -7)$. We write $\boldsymbol{k} = \tau_l^{-1}(\boldsymbol{\lambda}, \boldsymbol{s})$ by abuse of notation. Then they give the wedge basis in the space of finite wedge products Λ^r, which will be introduced in a different guise in § 5.2.

We read the beads $\tau_l(k_1), \dots, \tau_l(k_r)$ on the l-runner abacus $a(\boldsymbol{\lambda}, \boldsymbol{s})$ from right to left, starting with the lth runner, and obtain a permutation $w(\boldsymbol{k}) = (w_1, \dots, w_r)$ of $\boldsymbol{k}$. In our example, we have $w(\boldsymbol{k}) = (0, -6, -7, 3, -2, 1, -4)$.

Definition 3.4. Let $\tau_l(w_i) = (\zeta_i, b_i)$ for $1 \leqslant i \leqslant r$, that is, ζ_i and b_i are the column number and the row number of the bead $\tau_l(w_i)$ on the l-runner abacus $a(\boldsymbol{\lambda}, \boldsymbol{s})$, respectively. Then we define

$$\zeta(\boldsymbol{\lambda}) = (\zeta_1, \dots, \zeta_r) \quad \text{and} \quad b(\boldsymbol{\lambda}) = (b_1, \dots, b_r).$$

Example 3.5. In our example, we have

$$\zeta(\boldsymbol{\lambda}) = (0, -2, -3, 1, 0, 1, 0) \quad \text{and} \quad b(\boldsymbol{\lambda}) = (3, 3, 3, 2, 2, 1, 1).$$

We will need $\zeta(\boldsymbol{\lambda})$ and $b(\boldsymbol{\lambda})$ when we express $\Delta^e_{\boldsymbol{\lambda},\boldsymbol{\mu}}(v)$ in Kazhdan–Lusztig polynomials. In this respect, the following remark is important.

Remark 3.6. Suppose that we have fixed $\boldsymbol{\lambda}$ and $\boldsymbol{s}$. Assume that e and e' are two positive integers. Then $\boldsymbol{k} = \tau_l^{-1}(\boldsymbol{\lambda}, \boldsymbol{s})$ does not coincide in general for distinct e and e'. Nevertheless, one can choose r such that $\zeta(\boldsymbol{\lambda})$ and $b(\boldsymbol{\lambda})$ for e *coincide* with those for e'. For this to hold, it suffices that the r beads are the same for e and e'. Thus, it suffices to choose r as in (3.1) such that $1 - k_r$ is divisible by e and e'. If we divide the l-runner abacus into *cells* with height l and width e (respectively, e') so that the initial cell contains exactly the locations labelled by $1, 2 \dots, el$ (respectively, $1, 2 \dots, e'l$), the finite sequence ends at the upper-left corner of a far-left cell for both e and e'. In our running example, if we want to make $\zeta(\boldsymbol{\lambda})$ and $b(\boldsymbol{\lambda})$ coincide for $e = 2$ and $e' = 3$, we read all the beads with labels greater than or equal to -17 in Figure 1.

Let $P = \mathbb{Z}^r$ and let W be the affine symmetric group that is the semi-direct product of the symmetric group S_r and the normal subgroup P. W acts on $\beta = (\beta_1, \dots, \beta_r) \in P$ on the right by

$$\begin{aligned} \beta \cdot s_i &= (\beta_1 \dots, \beta_{i+1}, \beta_i, \dots, \beta_r) && \text{for } 1 \leqslant i \leqslant r-1, \\ \beta \cdot \mu &= \beta + e\mu && \text{for } \mu \in P. \end{aligned}$$

Then

$$A^r = \{a = (a_1, \dots, a_r) \in P \mid 1 \leqslant a_1 \leqslant \dots \leqslant a_r \leqslant e\}$$

is a fundamental domain for the action. We denote the stabilizer of $a \in A^r$ by ${}_aW$. It is clear that ${}_aW$ is a subgroup of S_r. Let w_a be the maximal element of ${}_aW$. We denote

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by ${}^{a}W$ and ${}^{a}S_r$ the set of minimal length coset representatives in ${}_{a}W \setminus W$ and ${}_{a}W \setminus S_r$, respectively.

In a similar manner, W acts on $\beta = (\beta_1, \dots, \beta_r) \in P$ on the left by

$$\begin{aligned} s_i \cdot \beta &= (\beta_1 \dots, \beta_{i+1}, \beta_i, \dots, \beta_r) && \text{for } 1 \leqslant i \leqslant r-1, \\ \mu \cdot \beta &= \beta + l\mu && \text{for } \mu \in P. \end{aligned}$$

Then

$$B^r = \{b = (b_1, \dots, b_r) \in P \mid l \geqslant b_1 \geqslant \dots \geqslant b_r \geqslant 1\}$$

is a fundamental domain for the action. We denote the stabilizer of $b \in B^r$ by W_b, its maximal element by w_b, and the set of minimal length coset representatives in W/W_b and S_r/W_b by W^b and S_r^b, respectively.

Write $k = c(k) + e(d(k) - 1) + elm(k)$ and $\phi(k) = c(k) + em(k)$ for $k \in \mathbb{Z}$, as before, and define

$$\begin{aligned} c(\boldsymbol{k}) &= (c(k_1), \dots, c(k_r)), \\ d(\boldsymbol{k}) &= (d(k_1), \dots, d(k_r)), \\ m(\boldsymbol{k}) &= (m(k_1), \dots, m(k_r)), \\ \phi(\boldsymbol{k}) &= (\phi(k_1), \dots, \phi(k_r)) \end{aligned}$$

for $\boldsymbol{k} = \tau_l^{-1}(\boldsymbol{\lambda}, \boldsymbol{s}) \in \mathbb{Z}^r$. Then,

- there exist $a(\boldsymbol{k}) \in A^r$ and $u(\boldsymbol{k}) \in {}^{a(\boldsymbol{k})} S_r$ such that $c(\boldsymbol{k}) = a(\boldsymbol{k}) \cdot u(\boldsymbol{k})$ and
- there exist $b(\boldsymbol{k}) \in B^r$ and $v(\boldsymbol{k}) \in S_r^{b(\boldsymbol{k})}$ such that $d(\boldsymbol{k}) = v(\boldsymbol{k}) \cdot b(\boldsymbol{k})$.

It is clear that $b(\boldsymbol{k}) = b(\boldsymbol{\lambda})$. We define $\zeta(\boldsymbol{k}) := \phi(\boldsymbol{k}) \cdot v(\boldsymbol{k})$. Then, comparing it with

$$b(\boldsymbol{k}) = v(\boldsymbol{k})^{-1} \cdot d(\boldsymbol{k}) = d(\boldsymbol{k}) \cdot v(\boldsymbol{k}),$$

we have $\zeta(\boldsymbol{k}) = \zeta(\boldsymbol{\lambda})$. In what follows we will use the notation $b(\boldsymbol{\lambda})$ and $\zeta(\boldsymbol{\lambda})$. From the definitions, we have

$$\zeta(\boldsymbol{\lambda}) = a(\boldsymbol{k}) \cdot u(\boldsymbol{k})v(\boldsymbol{k}) + e(m(\boldsymbol{k}) \cdot v(\boldsymbol{k})),$$

which shows that $\zeta(\boldsymbol{\lambda})$ belongs to $a(\boldsymbol{k})W$.

Example 3.7. With $\boldsymbol{k} = (3, 1, 0, -2, -4, -6, -7)$, $e = 2$ and $l = 3$, we obtain

$$\begin{aligned} c(\boldsymbol{k}) &= (1, 1, 2, 2, 2, 2, 1), \\ d(\boldsymbol{k}) &= (2, 1, 3, 2, 1, 3, 3), \\ m(\boldsymbol{k}) &= (0, 0, -1, -1, -1, -2, -2), \\ \phi(\boldsymbol{k}) &= (1, 1, 0, 0, 0, -2, -3), \\ a(\boldsymbol{k}) &= (1, 1, 1, 2, 2, 2, 2), \\ b(\boldsymbol{k}) &= b(\boldsymbol{\lambda}) = (3, 3, 3, 2, 2, 1, 1), \\ \zeta(\boldsymbol{k}) &= \zeta(\boldsymbol{\lambda}) = (0, -2, -3, 1, 0, 1, 0). \end{aligned}$$

3.2. Ordering multi-partitions

Now we introduce the dominance order in a general setting. Let $k \in \mathbb{N}$ and

$$\boldsymbol{u} = (u_1, \dots, u_k) \in \mathbb{Q}^k, \qquad \boldsymbol{v} = (v_1, \dots, v_k) \in \mathbb{Q}^k.$$

Then we write $\boldsymbol{u} \rhd \boldsymbol{v}$ if $\boldsymbol{u} \neq \boldsymbol{v}$ and

$$\sum_{s=1}^{a} u_s \geqslant \sum_{s=1}^{a} v_s \quad \text{for } a = 1, \dots, k.$$

We fix a decreasing sequence $1 > \alpha_1 > \alpha_2 > \cdots > \alpha_l > 0$ of rational numbers. Then, for each $\boldsymbol{\lambda} \in \Pi_{l,n}$, we read the rational numbers

$$\lambda_j^{(i)} - j + s_i - \alpha_i \quad \text{for } j = 1, \dots, n + s_i \text{ and } i = 1, \dots, l,$$

in decreasing order and denote the resulting sequence by $\gamma(\boldsymbol{\lambda}) \in \mathbb{Q}^k$, where

$$k = \sum_{i=1}^{l} s_i + nl.$$

Note that one can recover $\boldsymbol{\lambda}$ from $\gamma(\boldsymbol{\lambda}) = (\gamma_1, \dots, \gamma_k)$. Hence, if $\gamma(\boldsymbol{\lambda}) = \gamma(\boldsymbol{\mu})$, then $\boldsymbol{\lambda} = \boldsymbol{\mu}$. This follows from the fact that, for all $i \in [1, l]$, the set

$$\{\gamma_k - s_i + \alpha_i \mid \gamma_k - [\gamma_k] = \alpha_i\}$$

is the set of β-numbers of λ^i.

Definition 3.8. Let $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \Pi_{l,n}$. Then we write $\boldsymbol{\lambda} \succ \boldsymbol{\mu}$ if $\gamma(\boldsymbol{\lambda}) \rhd \gamma(\boldsymbol{\mu})$.

One can check that this defines a partial order which depends on the choice of α but does not depend on e. This is a crucial remark in view of the following result.

Theorem 3.9.

(i) *For each $\boldsymbol{\lambda} \in \Pi_{l,n}$, there exist polynomials $\Delta^e_{\boldsymbol{\lambda},\boldsymbol{\mu}}(v) \in \mathbb{Z}[v]$ for $\boldsymbol{\mu} \in \Pi_{l,n}$ such that we have the unitriangular expansion*

$$G_e(\boldsymbol{\lambda},\boldsymbol{s}) = \boldsymbol{\lambda} + \sum_{\boldsymbol{\lambda} \succ \boldsymbol{\mu}} \Delta^e_{\boldsymbol{\lambda},\boldsymbol{\mu}}(v)\boldsymbol{\mu}.$$

(ii) *For each $\boldsymbol{\lambda} \in \Pi_{l,n}$, there exist polynomials $\Delta^\infty_{\boldsymbol{\lambda},\boldsymbol{\mu}}(v) \in \mathbb{Z}[v]$ for $\boldsymbol{\mu} \in \Pi_{l,n}$ such that we have the unitriangular expansion*

$$G_\infty(\boldsymbol{\lambda},\boldsymbol{s}) = \boldsymbol{\lambda} + \sum_{\boldsymbol{\lambda} \succ \boldsymbol{\mu}} \Delta^\infty_{\boldsymbol{\lambda},\boldsymbol{\mu}}(v)\boldsymbol{\mu}.$$

(iii) *For each pair $(\boldsymbol{\lambda}, \boldsymbol{\mu}) \in \Pi_{l,n} \times \Pi_{l,n}$, $\Delta^e_{\boldsymbol{\lambda},\boldsymbol{\mu}}(v)$ and $\Delta^\infty_{\boldsymbol{\lambda},\boldsymbol{\mu}}(v)$ are expressed by certain parabolic Kazhdan–Lusztig polynomials* (see § 5)*. In particular, they are polynomials with non-negative integer coefficients.*

Proof. We prove (i) and (ii) by arguments similar to those used in [**17**]. As in [**30**], it suffices to show that the matrix of the bar-involution is unitriangular with respect to $\succ$. Then the results immediately follow from the characterization of the canonical basis. We recall the bar-involution on the space $\wedge^{s+\infty/2}V_{e,l}$, which is defined in [**30**], where $s = s_1 + \cdots + s_l$. The space $\wedge^{s+\infty/2}V_{e,l}$ is the $\mathbb{Q}(v)$-vector space spanned by the semi-infinite monomials

$$u_{\boldsymbol{k}} = u_{k_1} \wedge u_{k_2} \wedge \cdots ,$$

where $k_i \in \mathbb{Z}$ for all $i \geqslant 1$, and $k_i = s - i + 1$ if $i \gg 0$. Its basis is given by the *ordered* monomials (i.e. the monomials with decreasing indices $k_1 > k_2 > \cdots$) because any monomial may be expressed as a linear combination of ordered monomials by 'straightening relations' in [**30**, Proposition 3.16]. Now, the procedure in § 3.1 yields a bijection τ_l from the set of ordered monomials to the set of pairs $(\boldsymbol{\lambda}, \boldsymbol{s})$ such that $\boldsymbol{\lambda} \in \Pi_{l,n}$ and $\boldsymbol{s} = (s_1, \dots, s_l)$ with $s = s_1 + \cdots + s_l$. This allows us to identify the space $\wedge^{s+\infty/2}V_{e,l}$ with $\bigoplus_{s_1+\cdots+s_l=s} \mathcal{F}_e^{\boldsymbol{s}}$. Let $u_{\boldsymbol{k}}$ be a semi-infinite (possibly non-ordered) monomial. Let $u_{\tilde{\boldsymbol{k}}}$ be the monomial obtained from $u_{\boldsymbol{k}}$ by reordering the k_i in strictly decreasing order. The bijection τ_l then allows us to associate a pair $(\boldsymbol{\lambda}, \boldsymbol{s})$ with $u_{\tilde{\boldsymbol{k}}}$ such that $\boldsymbol{\lambda} \in \Pi_{l,n}$ and $\boldsymbol{s} = (s_1, \dots, s_l)$. We define a map π on the set of semi-infinite monimials by

$$\pi(u_{\boldsymbol{k}}) = (\boldsymbol{\lambda}, \boldsymbol{s}).$$

In particular, τ_l and π coincide on the set of ordered monomials. Uglov defined a bar-involution on $\bigwedge^{s+\infty/2} V_{e,l}$ as follows. For all semi-infinite ordered monomials $u_{\boldsymbol{k}}$, we define

$$\bar{u}_{\boldsymbol{k}} := v^t u_{k_r} \wedge u_{k_{r-1}} \wedge \cdots \wedge u_{k_1} \wedge u_{k_{r+1}} \wedge u_{k_{r+2}} \wedge \cdots ,$$

where t is a certain integer (see [**30**, § 3.4] for its explicit definition) and r is a sufficiently large integer. Hence, to compute $\bar{\boldsymbol{\lambda}}$ in $\mathcal{F}_e^{\boldsymbol{s}}$, we set $u_{\boldsymbol{k}} = \tau_l^{-1}(\boldsymbol{\lambda}, \boldsymbol{s})$ and use the straightening relations to expand $\bar{u}_{\boldsymbol{k}}$ on the basis of the ordered monomials, and apply π to obtain the expression of $\bar{\boldsymbol{\lambda}}$ as a linear combination of l-partitions. We note that $\boldsymbol{\lambda}$ appears with coefficient 1 by [**30**, Remark 3.24]. Let $u_{\boldsymbol{p}}$ be an arbitrary semi-infinite monomial and assume that this is non-ordered. Then there exists $i \in \mathbb{N}$ such that $k_i < k_{i+1}$. The straightening relations then show how to express $u_{\boldsymbol{p}}$ in terms of semi-infinite monomials $u_{\boldsymbol{p}'}$ with $p'_i > p'_{i+1}$. Let us define $\pi(u_{\boldsymbol{p}}) = (\boldsymbol{\lambda}, \boldsymbol{s})$ and $\pi(u_{\boldsymbol{p}'}) = (\boldsymbol{\lambda}', \boldsymbol{s}')$. A study of the straightening relations shows that we have $\boldsymbol{s} = \boldsymbol{s}'$ and that $\boldsymbol{\lambda}$ and $\boldsymbol{\lambda}'$ are both obtained from the same l-partition $\boldsymbol{\nu}$ by adding a ribbon of fixed size m (see [**17**, § 4.2]). We consider the set

$$\{\beta_1, \dots, \beta_h\} := \{\nu_j^{(i)} - j + s_i - \alpha_i \text{ for } j = 1, \dots, n + s_i \text{ and } i = 1, \dots, l\}.$$

Then there exists a and b such that $\gamma(\boldsymbol{\lambda})$ is the sequence obtained by reordering the elements of $\{\beta_1, \dots, \beta_h\} \setminus \{\beta_a\} \cup \{\beta_a + m\}$ in decreasing order and $\gamma(\boldsymbol{\lambda}')$ is the sequence obtained by reordering the elements $\{\beta_1, \dots, \beta_h\} \setminus \{\beta_b\} \cup \{\beta_b + m\}$ in decreasing order. Then, mimicking the argument in [**17**, pp. 581–583], one can prove, by a careful study of the straightening rules, that

$$\beta_a > \beta_b.$$

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This implies that $\boldsymbol{\lambda} \succ \boldsymbol{\lambda}'$. In particular, all the ordered monomials $u_{\boldsymbol{k}'}$ which appear in the expansion of $\bar{u}_{\boldsymbol{k}}$ satisfy the following property. If $\pi(u'_{\boldsymbol{k}}) = (\boldsymbol{\lambda}', \boldsymbol{s})$, then $\boldsymbol{\lambda} \succ \boldsymbol{\lambda}'$. This proves (i) and (ii). The third part is a result of [**30**]. Uglov proved that the coefficients $\Delta^e_{\boldsymbol{\lambda},\boldsymbol{\mu}}(v)$ are expressed by parabolic Kazhdan–Lusztig polynomials, as we will see in § 5. By the results of [**22**], this implies that they have non-negative integer coefficients. □

Remark 3.10. The order $\succ$ does not coincide with the partial order in [**30**]. In that work, the partial order depends on e, so a common partial order could not be used in statements (i) and (ii) of Theorem 3.9. On the other hand, we have used the common partial order $\succ$ there.

As a direct consequence, we have the following theorem.

Theorem 3.11. *For each $\boldsymbol{\lambda} \in \Pi_l$, we may expand $G_e(\boldsymbol{\lambda}, \boldsymbol{s})$ as follows:*

$$G_e(\boldsymbol{\lambda}, \boldsymbol{s}) = \sum_{\boldsymbol{\nu} \in \Pi_l} d_{\boldsymbol{\lambda},\boldsymbol{\nu}}(v) G_\infty(\boldsymbol{\nu}, \boldsymbol{s}), \tag{3.2}$$

where

- $d_{\boldsymbol{\lambda},\boldsymbol{\lambda}}(v) = 1$,
- $d_{\boldsymbol{\lambda},\boldsymbol{\nu}}(v) \in v\mathbb{Z}[v]$ *if* $\boldsymbol{\lambda} \neq \boldsymbol{\nu}$,
- $d_{\boldsymbol{\lambda},\boldsymbol{\nu}}(v) \neq 0$ *only if* $\boldsymbol{\lambda} \succeq \boldsymbol{\nu}$.

Proof. This follows from parts (i) and (ii) of Theorem 3.9. □

Corollary 3.12. *For $\boldsymbol{\lambda} \in B_e(\boldsymbol{s})$, the formula (3.2) has the form*

$$G_e(\boldsymbol{\lambda}, \boldsymbol{s}) = \sum_{\boldsymbol{\nu} \in B_\infty(\boldsymbol{s})} d_{\boldsymbol{\lambda},\boldsymbol{\nu}}(v) G_\infty(\boldsymbol{\nu}, \boldsymbol{s}). \tag{3.3}$$

Proof. We have already observed that $V_e(\boldsymbol{s})$ may be regarded as a $\mathcal{U}'_v(\widehat{\mathfrak{sl}}_e)$-submodule of $V_\infty(\boldsymbol{s})$ that shares the common highest-weight vector $\boldsymbol{\emptyset}$. Thus, we may expand $G_e(\boldsymbol{\lambda}, \boldsymbol{s}) \in \mathcal{G}^\circ_e(\boldsymbol{s})$ on the basis $\mathcal{G}^\circ_\infty(\boldsymbol{s}) \subset \mathcal{G}_\infty(\boldsymbol{s})$, and Theorem 3.11 implies (3.3). □

Definition 3.13. We define

$$\begin{aligned}
\Delta_e(v) &= (\Delta^e_{\boldsymbol{\lambda},\boldsymbol{\mu}}(v))_{\boldsymbol{\lambda} \in \Pi_l,\, \boldsymbol{\mu} \in \Pi_l}, \\
\Delta_\infty(v) &= (\Delta^\infty_{\boldsymbol{\lambda},\boldsymbol{\mu}}(v))_{\boldsymbol{\lambda} \in \Pi_l,\, \boldsymbol{\mu} \in \Pi_l}, \\
\Delta^e_\infty(v) &= (d_{\boldsymbol{\lambda},\boldsymbol{\nu}}(v))_{\boldsymbol{\lambda} \in \Pi_l,\, \boldsymbol{\nu} \in \Pi_l}.
\end{aligned}$$

They depend on $\boldsymbol{s}$. Then we have

$$\Delta_e(v) = \Delta_\infty(v) \Delta^e_\infty(v).$$

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We also define the following submatrices:

$$D_e(v) = (\Delta^e_{\boldsymbol{\lambda},\boldsymbol{\mu}}(v))_{\boldsymbol{\lambda}\in B_e(\boldsymbol{s}),\, \boldsymbol{\mu}\in B_\infty(\boldsymbol{s})},$$
$$D_\infty(v) = (\Delta^\infty_{\boldsymbol{\lambda},\boldsymbol{\mu}}(v))_{\boldsymbol{\lambda}\in B_e(\boldsymbol{s}),\, \boldsymbol{\mu}\in B_\infty(\boldsymbol{s})},$$
$$D^e_\infty(v) = (d_{\boldsymbol{\lambda},\boldsymbol{\nu}}(v))_{\boldsymbol{\lambda}\in B_e(\boldsymbol{s}),\, \boldsymbol{\nu}\in B_\infty(\boldsymbol{s})}.$$

Then we have $D_e(v) = D_\infty(v)D^e_\infty(v)$.

Remark 3.14. If $l = 1$, then the matrix $D_\infty(v)$ is the identity and $D^e_\infty(v) = D_e(v)$.

4. Computation of $\boldsymbol{\Delta^e_\infty(v)}$ and $\boldsymbol{D^e_\infty(v)}$

Before proceeding further, we explain algorithmic aspects for computing $\Delta^e_\infty(v)$ and $D^e_\infty(v)$. As $\Delta^e_\infty(v) = \Delta^{-1}_\infty(v)\cdot\Delta_e(v)$, we start by computing $\Delta_\infty(v)$ and $\Delta_e(v)$. Two algorithms are already proposed: one by Uglov and the other by Yvonne. Both use a natural embedding of the Fock spaces $\mathcal{F}^{\boldsymbol{s}}_e$ into the space of semi-infinite wedge products and compute the canonical bases $\mathcal{G}_e(\boldsymbol{s})$ and $\mathcal{G}_\infty(\boldsymbol{s})$.

The algorithm described by Uglov [**30**] needs steps to compute the straightening laws of the wedge products. This soon starts to require enormous computational resources. It occurs especially in the case when the differences between two consecutive entries of $\boldsymbol{s}$ are large.

Yvonne's algorithm [**32**] is much more efficient but it requires subtle computation related to the commutation relations of

$$\mathcal{U}_v(\widehat{\mathfrak{sl}}_e)\otimes\mathcal{H}\otimes\mathcal{U}_{-v^{-1}}(\widehat{\mathfrak{sl}}_l)$$

on the space of semi-infinite wedge products, where $\mathcal{H}$ is the Heisenberg algebra. We do not pursue this direction and refer the reader to [**32**] for a complete description of this algorithm.

Once $\mathcal{G}_e(\boldsymbol{s})$ and $\mathcal{G}_\infty(\boldsymbol{s})$ are computed, we can efficiently compute $\Delta^e_\infty(v)$ from them (see § 4.1).

The computation of $D^e_\infty(v)$ is easier. We can compute it directly from the canonical bases $\mathcal{G}^\circ_e(\boldsymbol{s})$ and $\mathcal{G}^\circ_\infty(\boldsymbol{s})$ and we may compute the canonical bases by the algorithms proposed in [**16**, **23**]. The algorithm given in [**16**] was originally suited for multi-charges $\boldsymbol{s}$ such that $0 \leqslant s_1 \leqslant s_2 \leqslant \cdots \leqslant s_l < e$. However, we will see in § 4.2 that it also computes the canonical bases $\mathcal{G}^\circ_e(\boldsymbol{s})$ and $\mathcal{G}^\circ_\infty(\boldsymbol{s})$ (and thus the matrix $D^e_\infty(v)$) for arbitrary multi-charge $\boldsymbol{s}$. Observe that this only uses the $\mathcal{U}_v(\mathfrak{g})$-module structure of the Fock space.

4.1. A general procedure

Assume that we have computed the canonical bases $\mathcal{G}_e(\boldsymbol{s})$ and $\mathcal{G}_\infty(\boldsymbol{s})$. Using the unitriangularity of the decomposition matrices, one can obtain $\Delta^e_\infty(v)$ directly from the relation $\Delta_\infty^e(v) = \Delta_\infty^{-1}(v) \cdot \Delta_e(v)$. This can be done efficiently by applying the procedure below.

(1) Let $\boldsymbol{\lambda} \in \Pi_{l,n}$. We know by Theorem 3.11 that $G_e(\boldsymbol{\lambda},\boldsymbol{s})$ may be expanded on $\mathcal{G}_\infty(\boldsymbol{s})$. We define

$$\Lambda(\boldsymbol{\lambda}) := \{\boldsymbol{\nu} \in \Pi_{l,n} \mid d_{\boldsymbol{\lambda},\boldsymbol{\nu}}(v) \neq 0\}.$$

Our aim is to find the members of $\Lambda(\boldsymbol{\lambda})$, and determine $d_{\boldsymbol{\lambda},\boldsymbol{\nu}}(v)$ when $\boldsymbol{\nu}$ is a member. Set $\boldsymbol{\lambda}^0 := \boldsymbol{\lambda}$. Then $\boldsymbol{\lambda}^0$ is a member and $d_{\boldsymbol{\lambda},\boldsymbol{\lambda}^0}(v) = 1$.

(2) Let $k \in \mathbb{N}$. Suppose that we already know k members $\boldsymbol{\lambda}^0, \dots, \boldsymbol{\lambda}^{k-1}$ of $\Lambda(\boldsymbol{\lambda})$ and the polynomials $d_{\boldsymbol{\lambda},\boldsymbol{\lambda}^i}(v)$ for $i = 0, \dots, k-1$. Then we expand

$$G_e(\boldsymbol{\lambda},\boldsymbol{s}) - \sum_{i=0}^{k-1} d_{\boldsymbol{\lambda},\boldsymbol{\lambda}^i}(v) G_\infty(\boldsymbol{\lambda}^i,\boldsymbol{s})$$

into a linear combination of the standard basis of l-partitions and write

$$\sum_{\boldsymbol{\nu} \in \Lambda(\boldsymbol{\lambda}) \setminus \{\boldsymbol{\lambda}^0,\dots,\boldsymbol{\lambda}^{k-1}\}} d_{\boldsymbol{\lambda},\boldsymbol{\nu}}(v) G_\infty(\boldsymbol{\nu},\boldsymbol{s}) = \sum_{\boldsymbol{\mu} \in \Pi_{l,n}} b_{\boldsymbol{\mu}}(v)\boldsymbol{\mu}.$$

We have $b_{\boldsymbol{\mu}}(v) \in \mathbb{Z}[v]$ by Theorem 3.9. If the right-hand side is zero, we are done. Otherwise, let $\boldsymbol{\lambda}^k$ be a maximal l-partition in $\{\boldsymbol{\mu} \in \Pi_{l,n} \mid b_{\boldsymbol{\mu}}(v) \neq 0\}$, with respect to the partial order $\succ$.

(3) Consider $\boldsymbol{\nu} \in \Lambda(\boldsymbol{\lambda}) \setminus \{\boldsymbol{\lambda}^0, \dots, \boldsymbol{\lambda}^{k-1}\}$ that satisfies $\boldsymbol{\nu} \succ \boldsymbol{\lambda}^k$. If such $\boldsymbol{\nu}$ does not exist, then we have

$$\boldsymbol{\lambda}^k \in \Lambda(\boldsymbol{\lambda}) \setminus \{\boldsymbol{\lambda}^0, \dots, \boldsymbol{\lambda}^{k-1}\}.$$

Otherwise, let $\boldsymbol{\nu}^k$ be maximal among them. If $\boldsymbol{\nu}^k$ appears in $G_\infty(\boldsymbol{\nu},\boldsymbol{s})$ for $\boldsymbol{\nu} \in \Lambda(\boldsymbol{\lambda}) \setminus \{\boldsymbol{\lambda}^0, \dots, \boldsymbol{\lambda}^{k-1}\}$, then $\boldsymbol{\nu} \succeq \boldsymbol{\nu}^k \succ \boldsymbol{\lambda}^k$, so that the maximality implies $\boldsymbol{\nu} = \boldsymbol{\nu}^k$. Since $\boldsymbol{\nu}^k$ appears in $G_\infty(\boldsymbol{\nu}^k,\boldsymbol{s})$, it follows that $b_{\boldsymbol{\nu}^k}(v) \neq 0$, which is impossible by the maximality of $\boldsymbol{\lambda}^k$ and $\boldsymbol{\nu}^k \succ \boldsymbol{\lambda}^k$. Hence, $\boldsymbol{\lambda}^k$ is a maximal element of $\Lambda(\boldsymbol{\lambda}) \setminus \{\boldsymbol{\lambda}^0, \dots, \boldsymbol{\lambda}^{k-1}\}$. Therefore, $\boldsymbol{\lambda}^k$ does not appear in $G_\infty(\boldsymbol{\nu},\boldsymbol{s})$ for $\boldsymbol{\nu} \in \Lambda(\boldsymbol{\lambda}) \setminus \{\boldsymbol{\lambda}^0, \dots, \boldsymbol{\lambda}^k\}$, and it follows that $d_{\boldsymbol{\lambda},\boldsymbol{\lambda}^k}(v) = b_{\boldsymbol{\lambda}^k}(v)$.

(4) We increment k and go to step (2).

4.2. The computation of $\mathcal{G}_e^\circ(\boldsymbol{s})$ and $\mathcal{G}_\infty^\circ(\boldsymbol{s})$

Let $e \in \mathbb{Z}_{\geqslant 2} \cup \{\infty\}$. Assume first that

$$0 \leqslant s_1 \leqslant s_2 \leqslant \cdots \leqslant s_l < e.$$

It is proved in [**16**, **23**] that one can construct a sequence of elements in $\mathbb{Z}/e\mathbb{Z}$

$$\underbrace{k_1, \dots, k_1}_{u_1}, \underbrace{k_2, \dots, k_2}_{u_2}, \dots, \underbrace{k_s, \dots, k_s}_{u_s} \tag{4.1}$$

for each $\boldsymbol{\lambda} \in B_e(\boldsymbol{s})$ such that if we define

$$A_e(\boldsymbol{\lambda},\boldsymbol{s}) := f_{k_1}^{(u_1)} \cdots f_{k_s}^{(u_s)} \cdot \boldsymbol{\emptyset} \in V_e(\boldsymbol{s}),$$

then

$$\mathcal{A}_e(\boldsymbol{s}) = \{A_e(\boldsymbol{\lambda},\boldsymbol{s}) \mid \boldsymbol{\lambda} \in B_e(\boldsymbol{s})\}$$

is a basis of $V_e(\boldsymbol{s})$. It is easy to obtain the coefficients $\gamma_{\boldsymbol{\lambda},\boldsymbol{\mu}}(v) \in \mathbb{Z}[v, v^{-1}]$ in the expansion

$$G_e(\boldsymbol{\lambda}, \boldsymbol{s}) = \sum_{\boldsymbol{\mu} \in B_e(\boldsymbol{s})} \gamma_{\boldsymbol{\lambda},\boldsymbol{\mu}}(v) A_e(\boldsymbol{\mu}, \boldsymbol{s}). \tag{4.2}$$

When $e \in \mathbb{Z}_{\geqslant 2}$, we have seen in § 3.1 that there is an action of the (extended) affine symmetric group W on $\mathbb{Z}^l$ such that

$$\mathcal{B}^l := \{(s_1, \ldots, s_l) \in \mathbb{Z}^l \mid 0 \leqslant s_1 \leqslant \cdots \leqslant s_l < e\}$$

is a fundamental domain for this action. Hence, for any $\boldsymbol{v} := (v_1, \ldots, v_l) \in \mathbb{Z}^l$, there exist $\boldsymbol{s} := (s_1, \ldots, s_l) \in \mathcal{B}^l$ and $w \in W$ such that $\boldsymbol{v} = w \cdot \boldsymbol{s}$. Since $\boldsymbol{v}$ and $\boldsymbol{s}$ yield the same dominant weight, we have an isomorphism $\phi_{\boldsymbol{s},\boldsymbol{v}}$ from $V_e(\boldsymbol{s})$ to $V_e(\boldsymbol{v})$. We can assume that $\phi_{\boldsymbol{s},\boldsymbol{v}}(\boldsymbol{\emptyset}) = \boldsymbol{\emptyset}$. For each $\boldsymbol{\lambda} \in B_e(\boldsymbol{s})$, we set

$$A_e(\boldsymbol{\lambda}, \boldsymbol{v}) = f_{k_1}^{(r_1)} \cdots f_{k_s}^{(r_s)} \cdot \boldsymbol{\emptyset} \in V_e(\boldsymbol{v}),$$

where the pairs (k_a, r_a) are defined by (4.1). Then we have $\phi_{\boldsymbol{s},\boldsymbol{v}}(A_e(\boldsymbol{\lambda}, \boldsymbol{s})) = A_e(\boldsymbol{\lambda}, \boldsymbol{v})$. By the uniqueness of the crystal basis on $V_e(\boldsymbol{v})$ proved by Kashiwara, we also have $\phi_{\boldsymbol{s},\boldsymbol{v}}(G_e(\boldsymbol{\lambda}, \boldsymbol{s})) = G_e(\varphi_{\boldsymbol{s},\boldsymbol{v}}(\boldsymbol{\lambda}), \boldsymbol{v})$, where $\varphi_{\boldsymbol{s},\boldsymbol{v}}$ is the crystal isomorphism from $B_e(\boldsymbol{s})$ to $B_e(\boldsymbol{v})$ (see [**19**] for a combinatorial description of $\varphi_{\boldsymbol{s},\boldsymbol{v}}$). By applying $\phi_{\boldsymbol{s},\boldsymbol{v}}$ to (4.2), we obtain

$$G_e(\boldsymbol{\nu}, \boldsymbol{v}) = \sum_{\boldsymbol{\mu} \in B_e(\boldsymbol{s})} \gamma_{\varphi^{-1}_{\boldsymbol{s},\boldsymbol{v}}(\boldsymbol{\nu}),\boldsymbol{\mu}}(v) A_e(\boldsymbol{\mu}, \boldsymbol{v}),$$

for $\boldsymbol{\nu} \in B_e(\boldsymbol{v})$, and it follows that

$$\mathcal{G}_e(\boldsymbol{v}) = \left\{ \sum_{\boldsymbol{\mu} \in B_e(\boldsymbol{s})} \gamma_{\boldsymbol{\lambda},\boldsymbol{\mu}}(v) A_e(\boldsymbol{\mu}, \boldsymbol{v}) \;\middle|\; \boldsymbol{\lambda} \in B_e(\boldsymbol{s}) \right\}.$$

Hence, the algorithms in [**17**, **23**] compute the canonical basis $\mathcal{G}_e(\boldsymbol{v})$ for any multi-charge $\boldsymbol{v} = (v_1, \ldots, v_l) \in \mathbb{Z}^l$. Applying the general procedure in § 4.2 restricted to the canonical bases $\mathcal{G}_e^\circ(\boldsymbol{v})$ and $\mathcal{G}_\infty^\circ(\boldsymbol{v})$, we may compute $D_\infty^e(v)$.

Remark 4.1. Another algorithm was recently proposed [**8**] for computing the canonical basis of the highest-weight $\mathcal{U}_v'(\widehat{\mathfrak{sl}}_e)$-modules that is realized in the tensor product of level-1 Fock spaces.

4.3. Example

We set $e = 2$. Then the matrix $D_e(v)$ of the canonical basis of the $\mathcal{U}_v(\widehat{\mathfrak{sl}}_e)$-module $V_e(0,0)$ is

$$\begin{array}{c} (\emptyset,(3)) \\ ((3),\emptyset) \\ ((1),(2)) \\ ((2),(1) \\ (\emptyset,(2\cdot 1)) \\ ((2\cdot 1),\emptyset) \\ ((1),(1\cdot 1)) \\ ((1\cdot 1),(1)) \\ (\emptyset,(1\cdot 1\cdot 1)) \\ ((1\cdot 1\cdot 1),\emptyset) \end{array} \quad \begin{pmatrix} 1 & \cdot & \cdot \\ v & \cdot & \cdot \\ v & 1 & \cdot \\ v^2 & v & \cdot \\ \cdot & \cdot & 1 \\ \cdot & \cdot & v \\ v & v^2 & \cdot \\ v^2 & v^3 & \cdot \\ v^2 & \cdot & \cdot \\ v^3 & \cdot & \cdot \end{pmatrix},$$

where dots mean 0 and each row is labelled by a 2-partition of rank 3. The matrix $D_\infty(v)$ of the canonical basis of the $\mathcal{U}_v(\mathfrak{sl}_\infty)$-module $V_\infty(0,0)$ is

$$\begin{array}{c} (\emptyset,(3)) \\ ((3),\emptyset) \\ ((1),(2)) \\ ((2),(1) \\ (\emptyset,(2.1)) \\ ((2\cdot 1),\emptyset) \\ ((1),(1\cdot 1)) \\ ((1\cdot 1),(1)) \\ (\emptyset,(1\cdot 1\cdot 1)) \\ ((1\cdot 1\cdot 1),\emptyset) \end{array} \quad \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot \\ v & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & v & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & v & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & v & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & v \end{pmatrix}$$

The matrix $D^e_\infty(v)$ obtained from our algorithm is

$$\begin{pmatrix} 1 & \cdot & \cdot \\ v & 1 & \cdot \\ \cdot & \cdot & 1 \\ v & v^2 & \cdot \\ v^2 & \cdot & \cdot \end{pmatrix}$$

and one can check that we have

$$D_e(v) = D_\infty(v) \cdot D^e_\infty(v).$$

5. Positivity of the coefficients in $d_{\boldsymbol{\lambda},\boldsymbol{\nu}(v)}$

The aim of this section is to study the entries of the matrix $D^e_\infty(v)$. The main result asserts that they are polynomials with non-negative integer coefficients.

5.1. Some notation on Kazhdan–Lusztig polynomials

Let H be the extended affine Hecke algebra of the symmetric group S_r. Namely, it is generated by $T_1, \dots, T_{r-1}$ and X^λ for $\lambda \in \bigoplus_{i=1}^r \mathbb{Z}\varepsilon_i$, such that the defining relations are

$$\begin{gathered}(T_i - v^{-1})(T_i + v) = 0,\\ X^\lambda T_i = T_i X^{s_i\lambda} + (v - v^{-1})\frac{X^{s_i\lambda} - X^\lambda}{1 - X^{\alpha_i}},\\ X^\lambda X^\mu = X^\mu X^\lambda,\\ X^\lambda X^{-\lambda} = 1\end{gathered}$$

and the Artin braid relations for $T_1, \dots, T_{r-1}$. The affine Hecke algebra admits a canonical basis $\{C'_w \mid w \in W\}$ such that

$$C'_w = v^{\ell(w)} \sum_{y \in W, y \leqslant w} P_{y,w}(v^{-2}) v^{-\ell(y)} T_y,$$

where $\leqslant$ is the Bruhat order on W. We refer the reader to [**22**, **27**] for a detailed review on affine Hecke algebras, the definition of the relevant length function and the Kazhdan–Lusztig basis. The polynomials $P_{y,w}(v^{-2})$ are the affine Kazhdan–Lusztig polynomials. They admit non-negative integer coefficients. We also recall the following property:

$$P_{y,w} = P_{s_i y,w} \tag{5.1}$$

for any $y < w$ in W and $i = 1, \dots, r$ such that $s_i w < w$.

5.2. Expression of the coefficients $\Delta^e_{\boldsymbol{\lambda},\boldsymbol{\mu}}(v)$ in terms of Kazhdan–Lusztig polynomials

The aim of this section is to recall Uglov's construction of a finite wedge product [**30**] and to show in a simpler manner than [**30**] that the entries $\Delta^e_{\boldsymbol{\lambda},\boldsymbol{\mu}}(v)$ are expressed in terms of parabolic Kazhdan–Lusztig polynomials.

We want to introduce the space of finite wedge products. Consider $a \in A^r$ and $b \in B^r$. We define ${}_aW$, W_b, w_a and w_b as in § 3.1. The subgroups ${}_aW$ and W_b define parabolic subalgebras H_a and H_b of the affine Hecke algebra H. If we define

$$J = \{i \mid 1 \leqslant i \leqslant r-1,\ b_i = b_{i+1}\},$$

then the parabolic subgroup W_J is merely the subgroup W_b. Let 1_a^+ (respectively, 1_b^-) be the right H_a-module (respectively, left H_b-module) such that

$$\left.\begin{aligned} 1_a^+ T_i &= v^{-1} 1_a^+, \quad s_i \in {}_aW,\\ T_i 1_b^- &= -v 1_b^-, \quad s_i \in W_b.\end{aligned}\right\} \tag{5.2}$$

We define $\Lambda^r(a,b) = 1_a^+ \otimes_{H_a} H \otimes_{H_b} 1_b^-$. Then, the *space of finite wedges* Λ^r is the direct sum of the $\Lambda^r(a,b)$ for $a \in A^r$ and $b \in B^r$. We define the bar-involution on Λ^r by

$$\overline{1_a^+ \otimes h \otimes 1_b^-} = 1_a^+ \otimes \bar{h} \otimes 1_b^-.$$

Definition 5.1. Let $\xi \in P$. Then there are unique $a \in A^r$ and $x \in {}^aW$ such that $\xi = ax$. We denote this x by $x(\xi)$.

We say that ξ is *J-dominant* and write $\xi \in P_b^{++}$ if $\xi_i > \xi_{i+1}$ whenever $b_i = b_{i+1}$. Similarly, we say that $\xi \in P_b^{+}$ if $\xi_i \geqslant \xi_{i+1}$ whenever $b_i = b_{i+1}$. Note that $\zeta(\boldsymbol{\lambda}) \in P_b^{++}$ for $\boldsymbol{\lambda} \in \Pi_l$. If $\xi \in P_b^{++}$, it follows by [**30**, Proposition 3.8] that $x(\xi)s < x(\xi)$ in the Bruhat order for any $s \in W_b$. So $x(\xi)w_b$ is the minimal length coset representative of ${}_aWx(\xi)W_b$.

By [**30**, Lemma 3.19, Proposition 3.20], the *wedge basis* of $\Lambda^r(a,b)$ is given by

$$\{|\boldsymbol{\lambda}\rangle = 1_a^+ \otimes T_{x(\zeta(\boldsymbol{\lambda}))w_b} \otimes 1_b^- = (-v)^{-\ell(w_b)} 1_a^+ \otimes T_{x(\zeta(\boldsymbol{\lambda}))} \otimes 1_b^- \mid \zeta(\boldsymbol{\lambda}) \in aW\}.$$

Here, for brevity, we have written $a = a(\boldsymbol{k})$ and $b = b(\boldsymbol{k}) = b(\boldsymbol{\lambda})$, where $\boldsymbol{k} = \tau_l^{-1}(\boldsymbol{\lambda}, \boldsymbol{s})$. We put $x = x(\zeta(\boldsymbol{\lambda}))w_b$. Then, by Kazhdan–Lusztig theory,

$$C'_{w_a x} = v^{\ell(w_a x)} \sum_{y \in W} P_{y, w_a x}(v^{-2}) v^{-\ell(y)} T_y$$

is bar invariant. As

$$W \simeq {}_aW \times \{x(\eta) \mid \eta \in aW\}, \tag{5.3}$$

we have

$$\begin{aligned} C'_{w_a x} &= v^{\ell(w_a x)} \sum_{\eta \in aW} \sum_{u \in {}_aW} P_{ux(\eta), w_a x}(v^{-2}) v^{-\ell(u) - \ell(x(\eta))} T_u T_{x(\eta)} \\ &= v^{\ell(w_a x)} \sum_{\eta \in aW} \sum_{u \in {}_aW} P_{w_a x(\eta), w_a x}(v^{-2}) v^{-\ell(u) - \ell(x(\eta))} T_u T_{x(\eta)} \end{aligned}$$

where the last equality is a consequence of (5.1). Set

$$C_e^+(\boldsymbol{\lambda}) = \frac{v^{-\ell(w_a)}}{\sum_{u \in {}_aW} v^{-2\ell(u)}} 1_a^+ \otimes C'_{w_a x} \otimes 1_b^-,$$

where $x = x(\xi)w_b$ and $\xi = \zeta(\boldsymbol{\lambda})$. Then, using (5.2), we have that

$$C_e^+(\boldsymbol{\lambda}) = \sum_{\eta \in aW} v^{\ell(x) - \ell(x(\eta))} P_{w_a x(\eta), w_a x}(v^{-2}) 1_a^+ \otimes T_{x(\eta)} \otimes 1_b^-$$

is bar invariant. When η admits repeated entries, one can verify that $1_a^+ \otimes T_{x(\eta)} \otimes 1_b^-$ is equal to 0. Here we refer the reader to [**30**, § 3.3] for a detailed proof (which justifies the terminology of Fock space used). Now, we rewrite $C_e^+(\boldsymbol{\lambda})$ as the expression

$$C_e^+(\boldsymbol{\lambda}) = \sum_{\eta \in aW \cap P_b^{++}} \sum_{u \in W_b} v^{\ell(x) - \ell(x(\eta)w_b u)} P_{w_a x(\eta) w_b u, w_a x}(v^{-2}) (-v)^{\ell(u)} 1_a^+ \otimes T_{x(\eta)w_b} \otimes 1_b^-.$$

Recall that

$$P^{J,-1}_{w_a x(\eta) w_b, w_a x(\xi) w_b}(v^{-2}) = \sum_{u \in W_b} (-1)^{\ell(u)} P_{w_a x(\eta) w_b u, w_a x(\xi) w_b}(v^{-2})$$

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is a parabolic Kazhdan–Lusztig polynomial. These polynomials were introduced by Deodhar [**6**]. As

$$x = x(\xi)w_b \quad \text{and} \quad v^{\ell(x)-\ell(x(\eta)w_b u)} = v^{\ell(x(\xi))-\ell(x(\eta))-\ell(u)},$$

we have

$$C_e^+(\boldsymbol{\lambda}) = \sum_{\eta \in aW \cap P_b^{++}} v^{\ell(x(\xi))-\ell(x(\eta))} P^{J,-1}_{w_a x(\eta) w_b, w_a x(\xi) w_b}(v^{-2}) 1_a^+ \otimes T_{x(\eta)w_b} \otimes 1_b^-. \tag{5.4}$$

It satisfies the defining properties of the plus canonical basis introduced by Uglov [**30**]. Thus, we have recovered Uglov's result, Theorem 3.9 (iii). To be more precise, let $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \Pi_{l,n}$. Choose $r \in \mathbb{N}$ as in § 3.1, and define $\boldsymbol{k}, \boldsymbol{l} \in \mathbb{Z}^r$ as

$$\boldsymbol{k} = \tau_l^{-1}(\boldsymbol{\lambda}, \boldsymbol{s}) \quad \text{and} \quad \boldsymbol{l} = \tau_l^{-1}(\boldsymbol{\mu}, \boldsymbol{s}).$$

Define $a(\boldsymbol{k}), a(\boldsymbol{l})$ and $b(\boldsymbol{\lambda}), b(\boldsymbol{\mu})$ as in § 3.1, and set $\xi = \zeta(\boldsymbol{\lambda})$ and $\eta = \zeta(\boldsymbol{\mu})$.

Theorem 5.2. *With the above notation, we have the following.*

(i) *If $a(\boldsymbol{k}) \neq a(\boldsymbol{l})$, or $b(\boldsymbol{\lambda}) \neq b(\boldsymbol{\mu})$, then $\Delta^e_{\boldsymbol{\lambda},\boldsymbol{\mu}}(v) = 0$.*

(ii) *If $a(\boldsymbol{k}) = a(\boldsymbol{l}) = a \in A^r$ and $b(\boldsymbol{\lambda}) = b(\boldsymbol{\mu}) = b \in B^r$, then*

$$\Delta^e_{\boldsymbol{\lambda},\boldsymbol{\mu}}(v) = v^{\ell(x(\xi))-\ell(x(\eta))} P^{J,-1}_{w_a x(\eta) w_b, w_a x(\xi) w_b}(v^{-2}). \tag{5.5}$$

5.3. Stabilization for $e = \infty$

Now we assume that $\boldsymbol{s} \in \mathbb{Z}^l$ and $\boldsymbol{\lambda} \in \Pi_l$ are fixed and we increase e. By Remark 3.6, we have seen that, for any $e' > e$, one can choose r such that $\xi = \zeta(\boldsymbol{\lambda})$ coincide for e and e'. Since $\boldsymbol{s}$ and $\boldsymbol{\lambda}$ are fixed, when e' is sufficiently large, there exist $\tilde{x}(\xi) \in S_r$ and $\tilde{a} = (\tilde{a}_1, \dots, \tilde{a}_r)$ such that

$$\tilde{a}_1 \leqslant \cdots \leqslant \tilde{a}_r, \quad \tilde{x}(\xi) \in {}^{\tilde{a}}S_r \quad \text{and} \quad \xi = \tilde{a}\tilde{x}(\xi). \tag{5.6}$$

This only means that we do not need translations by $e'\mu$, for $\mu \in P$, to reach the fundamental domain when e' is sufficiently large. In what follows, we refer to this stabilization phenomenon as the $e = \infty$ case. By Remark 3.6 we have the following expression for the $e = \infty$ case:

$$\Delta^\infty_{\boldsymbol{\lambda},\boldsymbol{\mu}}(v) = v^{\ell(\tilde{x}(\zeta(\boldsymbol{\lambda})))-\ell(\tilde{x}(\zeta(\boldsymbol{\mu})))} P^{J,-1}_{w_{\tilde{a}}\tilde{x}(\zeta(\boldsymbol{\mu}))w_b, w_{\tilde{a}}\tilde{x}(\zeta(\boldsymbol{\lambda}))w_b}(v^{-2}) \quad \text{for } \boldsymbol{\lambda}, \boldsymbol{\mu} \in \Pi_l.$$

Moreover, one can assume that r is adjusted such that b and $\xi = \zeta(\boldsymbol{\lambda})$ are the same for e finite (fixed) and $e = \infty$. In particular, we have $\xi \in \tilde{a}S_r$ for $\boldsymbol{\lambda} \in \Pi_l$, as before. Then Theorem 5.2 (ii) implies that we may assume $\eta S_r = \tilde{a}S_r$ for $\eta = \zeta(\boldsymbol{\mu})$.

Recall that $\xi = \zeta(\boldsymbol{\lambda})$ and $\eta = \zeta(\boldsymbol{\mu})$ belong to P_b^{++}. Then Theorems 3.11 and 5.2 imply that there exist polynomials

$$d_{\gamma\xi}(v) \in \mathbb{Z}[v] \quad \text{for } \gamma \in P_b^{++},$$

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such that

$$v^{\ell(x(\xi))-\ell(x(\eta))} P^{J,-1}_{w_a x(\eta) w_b, w_a x(\xi) w_b}(v^{-2}) = \sum_{\gamma \in \tilde{a} S_r \cap P_b^{++}} v^{\ell(\tilde{x}(\gamma))-\ell(\tilde{x}(\eta))} P^{J,-1}_{w_{\tilde{a}} \tilde{x}(\eta) w_b, w_{\tilde{a}} \tilde{x}(\gamma) w_b}(v^{-2}) d_{\gamma\xi}(v).$$

Define a linear map $\psi\colon \Lambda(\tilde{a}, b) \hookrightarrow \Lambda(a, b)$ as

$$1^+_{\tilde{a}} \otimes T_{\tilde{x}(\xi) w_b} \otimes 1^-_b \mapsto 1^+_a \otimes T_{x(\xi) w_b} \otimes 1^-_b = 1^+_a \otimes T_{x(\tilde{a})} T_{\tilde{x}(\xi) w_b} \otimes 1^-_b.$$

Then, in view of (5.4), the above equality is equivalent to

$$C^+_e(\boldsymbol{\lambda}) = \sum_{\tilde{a} \in aW \cap P^-} \sum_{\gamma := \zeta(\boldsymbol{\nu}) \in \tilde{a} S_r \cap P_b^{++}} d_{\gamma\xi}(v) \psi(C^+_\infty(\boldsymbol{\nu})), \tag{5.7}$$

where

$$C^+_\infty(\boldsymbol{\nu}) = \frac{v^{-\ell(w_{\tilde{a}})}}{\sum_{u \in \tilde{a} W} v^{-2\ell(u)}} 1^+_{\tilde{a}} \otimes C'_{w_{\tilde{a}} \tilde{x}(\zeta(\boldsymbol{\nu})) w_b} \otimes 1^-_b. \tag{5.8}$$

5.4. Proof of the positivity

The idea of the proof is to expand $C^+_e(\boldsymbol{\lambda})$ into a linear combination of $\psi(C^+_\infty(\boldsymbol{\mu}))$ and compare it with (5.7). The famous positivity result of the multiplicative structure constants with respect to the Kazhdan–Lusztig basis and its generalization in [**14**] then yields the desired positivity.* Recall the basis

$$C'_w = v^{\ell(w)} \sum_{y \in W} P_{y,w}(v^{-2}) v^{-\ell(y)} T_y.$$

For $y \in W$, we write $y = y'y''$, where $y'' \in S_r$ and y' is the minimal length coset representative of yS_r. Then we define

$$U_y = T_{y'} C'_{y''}. \tag{5.9}$$

It is clear that we may write

$$C'_w = \sum_{y \in W} A_{y,w}(v) U_y, \tag{5.10}$$

where $A_{y,w}(v) \in \mathbb{Z}[v, v^{-1}]$. By [**14**, Corollary 3.9], we have in fact $A_{y,w}(v) \in \mathbb{N}[v, v^{-1}]$.

We write $y = ux(\gamma)$, for $u \in {}_aW$ and $\gamma \in aW$, by (5.3). Then we have

$$U_y = U_{ux(\gamma)} = T_u U_{x(\gamma)}$$

and this implies that

$$T_i U_y = \begin{cases} U_{s_i y}, & s_i y > y, \\ (v^{-1} - v) U_y + U_{s_i y}, & s_i y < y. \end{cases}$$

* One purpose of [**14**] is to introduce LLT polynomials for general root systems. Note that LLT polynomials for finite root systems other than type A had been introduced independently in [**25**]. It is interesting to compare the two definitions.

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Let $w = w_a x(\xi) w_b$ and $\xi = \zeta(\boldsymbol{\lambda})$. As $s_i w < w$, for $s_i \in {}_aW$, we deduce

$$\begin{aligned} v^{-1} C'_w = T_i C'_w &= \sum_{s_i y > y} A_{y,w}(v) U_{s_i y} + \sum_{s_i y < y} A_{y,w}(v)((v^{-1} - v) U_y + U_{s_i y}) \\ &= \sum_{s_i y < y} (A_{s_i y,w}(v) + (v^{-1} - v) A_{y,w}(v)) U_y + \sum_{s_i y > y} A_{s_i y,w}(v) U_y. \end{aligned}$$

Thus, $A_{s_i y,w}(v) = v^{-1} A_{y,w}(v)$ if $s_i y > y$, and it follows that

$$A_{y,w}(v) = v^{-\ell(u)} A_{x(\gamma),w}(v) \quad \text{for } y = u x(\gamma).$$

Therefore, we have

$$\left(\sum_{u \in {}_aW} v^{-\ell(u)} T_u \right) \left(\sum_{\gamma \in aW} A_{x(\gamma),w}(v) U_{x(\gamma)} \right) = C'_w. \tag{5.11}$$

Hence, for any $\boldsymbol{\lambda} \in \Pi_l$, the plus canonical basis is given by

$$\begin{aligned} C_e^+(\boldsymbol{\lambda}) &= \frac{v^{-\ell(w_a)}}{\sum_{u \in {}_aW} v^{-2\ell(u)}} 1_a^+ \otimes C'_w \otimes 1_b^- \\ &= \sum_{\gamma \in aW} v^{-\ell(w_a)} A_{x(\gamma), w_a x(\xi) w_b}(v) 1_a^+ \otimes U_{x(\gamma)} \otimes 1_b^- \\ &= \sum_{\tilde{a} \in aW \cap P^-} \sum_{z \in S_r} v^{-\ell(w_a)} A_{x(\tilde{a})z, w_a x(\xi) w_b}(v) 1_a^+ \otimes T_{x(\tilde{a})} C'_z \otimes 1_b^- \end{aligned} \tag{5.12}$$

where the second equality follows from $w = w_a x(\xi) w_b$, (5.2) and (5.11), the third follows from (5.9). Note that ${}_{\tilde{a}}W = S_r \cap x(\tilde{a})^{-1} {}_aW x(\tilde{a})$ by $\tilde{a} = a x(\tilde{a})$. Then (5.2) allows us to write

$$1_a^+ \otimes T_{x(\tilde{a})} C'_z \otimes 1_b^- = \frac{1}{\sum_{u \in {}_{\tilde{a}}W} v^{-2\ell(u)}} 1_a^+ \otimes T_{x(\tilde{a})} \left(\sum_{u \in {}_{\tilde{a}}W} v^{-\ell(u)} T_u \right) C'_z \otimes 1_b^-.$$

As the left multiplication by

$$\sum_{u \in {}_{\tilde{a}}W} v^{-\ell(u)} T_u$$

gives the subspace of dimension $|S_r|/|{}_{\tilde{a}}W|$ in the Hecke algebra $H(S_r)$, it has the basis $\{C'_{w_{\tilde{a}} y} \mid y \in_{\tilde{a}} W \setminus S_r\}$. By the positivity of the structure constants, we may write

$$\left(\sum_{u \in {}_{\tilde{a}}W} v^{-\ell(u)} T_u \right) C'_z = \sum_{y \in_{\tilde{a}} W \setminus S_r} B_{y,z}(v) C'_{w_{\tilde{a}} y},$$

where $B_{y,z}(v) \in \mathbb{N}[v, v^{-1}]$. Thus,

$$1_a^+ \otimes T_{x(\tilde{a})} C'_z \otimes 1_b^- = \sum_{\gamma \in \tilde{a} S_r} B_{\tilde{x}(\gamma),z}(v) \frac{1}{\sum_{u \in {}_{\tilde{a}}W} v^{-2\ell(u)}} 1_a^+ \otimes T_{x(\tilde{a})} C'_{w_{\tilde{a}} \tilde{x}(\gamma)} \otimes 1_b^-.$$

For each $\gamma \in \tilde{a}S_r$, define

$$d'_{\gamma,\xi}(v) = v^{-\ell(w_a)} \sum_{z \in S_r} v^{\ell(w_{\tilde{a}})} A_{x(\tilde{a})z, w_a x(\xi) w_b}(v) B_{\tilde{x}(\gamma),z}(v). \tag{5.13}$$

Then, $d'_{\gamma,\xi}(v) \in \mathbb{N}[v, v^{-1}]$ and we have

$$\begin{aligned}
&\sum_{z \in S_r} v^{-\ell(w_a)} A_{x(\tilde{a})z, w_a x(\xi) w_b}(v) 1_a^+ \otimes T_{x(\tilde{a})} C'_z \otimes 1_b^- \\
&\qquad = \sum_{\gamma \in \tilde{a}S_r} d'_{\gamma,\xi}(v) \frac{v^{-\ell(w_{\tilde{a}})}}{\sum_{u \in \tilde{a}W} v^{-2\ell(u)}} 1_a^+ \otimes T_{x(\tilde{a})} C'_{w_{\tilde{a}}\tilde{x}(\gamma)} \otimes 1_b^- \\
&\qquad = \sum_{\gamma \in \tilde{a}S_r \cap P_b^+} \sum_{t \in W_b} d'_{\gamma w_b t,\xi}(v) \frac{v^{-\ell(w_{\tilde{a}})}}{\sum_{u \in \tilde{a}W} v^{-2\ell(u)}} 1_a^+ \otimes T_{x(\tilde{a})} C'_{w_{\tilde{a}}\tilde{x}(\gamma) w_b t} \otimes 1_b^-,
\end{aligned}$$

where we slightly abuse the notation by using the same index γ in the last two expressions. If $xs_i < x$ for some $s_i \in W_b$, then

$$v^{-1} C'_x \otimes 1_b^- = C'_x T_i \otimes 1_b^- = -v C'_x \otimes 1_b^-$$

and $C'_x \otimes 1_b^- = 0$. Thus, we have, in fact,

$$\begin{aligned}
&\sum_{z \in S_r} v^{-\ell(w_a)} A_{x(\tilde{a})z, w_a x(\xi) w_b}(v) 1_a^+ \otimes T_{x(\tilde{a})} C'_z \otimes 1_b^- \\
&\qquad = \sum_{\gamma \in \tilde{a}S_r \cap P_b^+} d'_{\gamma w_b,\xi}(v) \frac{v^{-\ell(w_{\tilde{a}})}}{\sum_{u \in \tilde{a}W} v^{-2\ell(u)}} 1_a^+ \otimes T_{x(\tilde{a})} C'_{w_{\tilde{a}}\tilde{x}(\gamma) w_b} \otimes 1_b^-.
\end{aligned}$$

By using the last expression in (5.12), we derive

$$C_e^+(\boldsymbol{\lambda}) = \sum_{\tilde{a} \in aW \cap P^-} \sum_{\gamma \in \tilde{a}S_r \cap P_b^{++}} d'_{\gamma w_b,\xi}(v) \frac{v^{-\ell(w_{\tilde{a}})}}{\sum_{u \in \tilde{a}W} v^{-2\ell(u)}} 1_a^+ \otimes T_{x(\tilde{a})} C'_{w_{\tilde{a}}\tilde{x}(\gamma) w_b} \otimes 1_b^-.$$

By using (5.8), this can also be rewritten

$$C_e^+(\boldsymbol{\lambda}) = \sum_{\tilde{a} \in aW \cap P^-} \sum_{\gamma = \zeta(\boldsymbol{\nu}) \in \tilde{a}S_r \cap P_b^{++}} d'_{\gamma w_b,\xi}(v) \psi(C_\infty^+(\boldsymbol{\nu})).$$

Hence, comparing this with (5.7), we obtain $d_{\gamma\xi}(v) = d'_{\gamma w_b,\xi}(v) \in \mathbb{N}[v, v^{-1}]$. We have established the following desired positivity result.

Theorem 5.3. *The polynomials $d_{\boldsymbol{\lambda},\boldsymbol{\nu}}(v)$ that appear in (3.2) have non-negative integer coefficients.*

5.5. The case $v = 1$

The proof of the positivity we have obtained does not properly yield a geometric interpretation of the coefficients $d_{\gamma\xi}(v)$. The purpose of this section is to show that their specializations $d_{\gamma\xi}(1)$ may be interpreted as composition multiplicities. Let us rewrite the right action in a more coordinate-free manner. For this, we consider

$$\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}] = \mathfrak{sl}_r(\mathbb{C}) \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c,$$

where $\mathfrak{g} = \mathfrak{sl}_r(\mathbb{C}) \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d$ is the Kac–Moody Lie algebra of type $A^{(1)}_{r-1}$. Then the fundamental weights $\Lambda_0, \dots, \Lambda_{r-1}$ remain linearly independent on

$$\mathfrak{h}' = \bigoplus_{i=0}^{r-1} \mathbb{C}\alpha_i^\vee$$

and we may write its dual space as follows:

$$\mathfrak{h}'^* = \mathfrak{h}^*/\mathbb{C}\delta = \bigoplus_{i=0}^{r-1} \mathbb{C}\Lambda_i.$$

We identify the weight lattice P of $\mathfrak{sl}_r(\mathbb{C})$ with the set of level zero integral weights in $\mathfrak{h}'^*$ by

$$P = \frac{\bigoplus_{i=1}^{r} \mathbb{Z}\varepsilon_i}{\mathbb{Z}(\varepsilon_1 + \cdots + \varepsilon_r)} = \bigoplus_{i=1}^{r-1} \mathbb{Z}(\Lambda_i - \Lambda_0) \subseteq \mathfrak{h}'^*,$$

where

$$\xi = \sum_{i=1}^{r} \xi_i \varepsilon_i \mapsto \sum_{i=1}^{r-1} (\xi_i - \xi_{i+1})(\Lambda_i - \Lambda_0)$$

(we drop 'modulo $\mathbb{Z}(\varepsilon_1 + \cdots + \varepsilon_r)$' by abuse of notation). For $\xi \in P$, we define

$$\hat{\xi} = -\xi + e\Lambda_0 \in \mathfrak{h}'^*.$$

The Weyl group action on $\mathfrak{h}'^*$ preserves $P + e\Lambda_0$. Moreover, if we define $w\xi$ for $w \in W$ and $\xi \in P$ by $w\hat{\xi} = -w\xi + e\Lambda_0$, where $\hat{\xi} \mapsto w\hat{\xi}$ is the Weyl group action on $\mathfrak{h}'^*$, then

$$s_i\xi = \xi_{i+1}\varepsilon_i + \xi_i\varepsilon_{i+1} + \sum_{j \neq i, i+1} \xi_j \varepsilon_j,$$

for $1 \leqslant i \leqslant r-1$, and

$$s_0\xi = (\xi_r - e)\varepsilon_1 + (\xi_1 + e)\varepsilon_r + \sum_{j \neq 1, r} \xi_j \varepsilon_j.$$

Thus, $\xi \cdot w := w^{-1}\xi$, for $\xi \in P$ and $w \in W$, is nothing but the right action of W.

Let $J \subset \{1, \dots, r-1\}$ and let μ be the composition of r defined by J. Write $\mathfrak{p}_\mu(\mathbb{C})$ for the parabolic subalgebra of $\mathfrak{g}$ defined by μ and $\mathfrak{l}_\mu(\mathbb{C})$ for the standard Levi subalgebra of $\mathfrak{p}_\mu(\mathbb{C})$. For $\eta \in P_b^{++}$, we denote by $V(w_b\hat{\eta})$ the finite-dimensional irreducible

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$\mathfrak{l}_\mu(\mathbb{C}) \oplus \mathbb{C}c$-module with highest weight $w_b\hat{\eta} - \rho$, where ρ is such that $\langle \rho, \alpha_i^\vee \rangle = 1$ for $0 \leqslant i \leqslant r-1$. Thus, the canonical central element c acts as the scalar $e - r$. We view $V(w_b\hat{\eta})$ as a $\mathfrak{p}_\mu(\mathbb{C}) \oplus \mathbb{C}c$-module. Then, through the evaluation homomorphism

$$\mathfrak{p}_\mu = \{X \in \mathfrak{sl}_r(\mathbb{C}[t]) \mid X|_{t=0} \in \mathfrak{p}_\mu(\mathbb{C})\} \oplus \mathbb{C}c \to \mathfrak{p}_\mu(\mathbb{C}) \oplus \mathbb{C}c,$$

we may view it as a $\mathfrak{p}_\mu$-module as well. We define the following $\mathfrak{g}'$-module:

$$M_\mu(w_b\hat{\eta}) = U(\mathfrak{g}') \otimes_{U(\mathfrak{p}_\mu)} V(w_b\hat{\eta}).$$

If $X \in \mathfrak{p}_\mu$, then

$$Xu \otimes v = [X, u] \otimes v + u \otimes Xv, \quad u \in U(\mathfrak{g}'),\ v \in V(w_b\hat{\eta}).$$

Hence, $M_\mu(w_b\hat{\eta})$ is isomorphic to the tensor product representation of the adjoint representation on $U(\mathfrak{g}')$ and $V(w_b\hat{\eta})$ as a $\mathfrak{p}_\mu$-module. Thus $M_\mu(w_b\hat{\eta})$ is an integrable $\mathfrak{p}_\mu$-module.

For any ζ in $\mathfrak{h}'^*$, we denote by $M(\zeta)$ the Verma $\mathfrak{g}'$-module with highest weight $\zeta - \rho$. Then, by the Weyl character formula we have, for $\eta \in P_b^{++}$,

$$M_\mu(w_b\hat{\eta}) = \sum_{u \in W_b} (-1)^{\ell(u)} M(uw_b\hat{\eta}).$$

We consider the following maximal parabolic subalgebra of $\mathfrak{g}'$.

$$\mathfrak{g}'_0 = \mathfrak{sl}_r(\mathbb{C}[t]) \oplus \mathbb{C}c \subseteq \mathfrak{g}'.$$

We define

$$M_0(w_b\hat{\eta}) = U(\mathfrak{g}') \otimes_{U(\mathfrak{g}'_0)} L(w_b\hat{\eta}),$$

where $L(w_b\hat{\eta})$ is the irreducible highest-weight $\mathfrak{g}'_0$-module whose highest weight is $w_b\hat{\eta} - \rho$.

Now, with the notation of § 5.3, observe that $\langle \tilde{a}, \alpha_i^\vee \rangle \leqslant 0$ for $1 \leqslant i \leqslant r-1$. Moreover, we have

$$-uw_b\eta = -uw_b\tilde{x}(\eta)^{-1}\tilde{a}$$

such that $w_{\tilde{a}}\tilde{x}(\eta)w_bu^{-1}$ is the maximal-length coset representative of $W_{\tilde{a}}\tilde{x}(\eta)w_bu^{-1}$. Now we apply the classical Kazhdan–Lusztig conjecture for semi-simple Lie algebras, which is the theorem by Beilinson and Bernstein, and Brylinski and Kashiwara. Here, the Lie algebra is $\mathfrak{sl}_r(\mathbb{C})$ and it gives

$$M(uw_b\hat{\eta}) = \sum_{\gamma \in \tilde{a}S_r} P_{w_{\tilde{a}}\tilde{x}(\eta)w_bu^{-1}, w_{\tilde{a}}\tilde{x}(\gamma)w_b}(1) M_0(w_b\hat{\gamma})$$

for $u \in W_b$. This implies that

$$\begin{aligned} M_\mu(w_b\hat{\eta}) &= \sum_{u \in W_b} (-1)^{\ell(u)} M(uw_b\hat{\eta}) \\ &= \sum_{\gamma \in \tilde{a}S_r} P^{J,-1}_{w_{\tilde{a}}\tilde{x}(\eta)w_b, w_{\tilde{a}}\tilde{x}(\gamma)w_b}(1) M_0(w_b\hat{\gamma}). \end{aligned}$$

By the integrality as a $\mathfrak{p}_\mu$-module, we have

$$M_\mu(w_b\hat{\eta}) = \sum_{\gamma\in\tilde{a}S_r\cap P_b^{++}} P^{J,-1}_{w_{\tilde{a}}\tilde{x}(\eta)w_b,w_{\tilde{a}}\tilde{x}(\gamma)w_b}(1)M_0(w_b\hat{\gamma}).$$

Note also that

$$\hat{a} = -\sum_{i=1}^{r-1}(a_i - a_{i+1})(\Lambda_i - \Lambda_0) + e\Lambda_0$$

satisfies

$$\langle\hat{a}, \alpha_i^\vee\rangle = \begin{cases} a_{i+1} - a_i \geqslant 0, & 1 \leqslant i \leqslant r-1, \\ e + a_1 - a_r \geqslant 1 > 0, & i = 0, \end{cases}$$

and we have

$$uw_b\hat{\eta} = uw_bx(\eta)^{-1}\hat{a}$$

such that $w_ax(\eta)w_bu^{-1}$ is the maximal-length coset representative of ${}_aWx(\eta)w_bu^{-1}$ for $u \in W_b$. Thus, by the Kazhdan–Lusztig conjecture again, this time for $\mathfrak{g}$ [**31**],

$$M(uw_b\hat{\eta}) = \sum_{\xi\in aW} P_{w_ax(\eta)w_bu^{-1},w_ax(\xi)w_b}(1)L(w_b\hat{\xi})$$

for $u \in W_b$. This implies that

$$\begin{aligned} M_\mu(w_b\hat{\eta}) &= \sum_{u\in W_b} (-1)^{\ell(u)}M(uw_b\hat{\eta}) \\ &= \sum_{\xi\in aW} P^{J,-1}_{w_ax(\eta)w_b,w_ax(\xi)w_b}(1)L(w_b\hat{\xi}). \end{aligned}$$

By the integrality as a $\mathfrak{p}_\mu$-module again, we obtain

$$M_\mu(w_b\hat{\eta}) = \sum_{\xi\in aW\cap P_b^{++}} P^{J,-1}_{w_ax(\eta)w_b,w_ax(\xi)w_b}(1)L(w_b\hat{\xi}).$$

Therefore, if we write

$$M_0(w_b\hat{\gamma}) = \sum_{\xi\in aW\cap P_b^{++}} d_{\gamma\xi}L(w_b\hat{\xi})$$

for $d_{\gamma\xi} \in \mathbb{N}$, in other words $[M_0(w_b\hat{\gamma}) : L(w_b\hat{\xi})] = d_{\gamma\xi}$, we have

$$P^{J,-1}_{w_ax(\eta)w_b,w_ax(\xi)w_b}(1) = \sum_{\gamma\in\tilde{a}S_r\cap P_b^{++}} P^{J,-1}_{w_{\tilde{a}}\tilde{x}(\eta)w_b,w_{\tilde{a}}\tilde{x}(\gamma)w_b}(1)d_{\gamma\xi}.$$

Hence, we have the following interpretation of $d_{\boldsymbol{\lambda},\boldsymbol{\nu}}(1)$.

Proposition 5.4. *For the relative decomposition numbers evaluated at $v = 1$, we have the equalities*

$$d_{\boldsymbol{\lambda},\boldsymbol{\nu}}(1) = [M_0(w_b\hat{\gamma}) : L(w_b\hat{\xi})],$$

where $\xi = \zeta(\boldsymbol{\lambda})$ and $\gamma = \zeta(\boldsymbol{\nu})$.

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It would be desirable to understand $d_{\boldsymbol{\lambda},\boldsymbol{\nu}}(v)$ in terms of Jantzen filtration. In the case when W_b is trivial, we expect the Verma module to be rigid and the Jantzen conjecture to hold.

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