University, USA, and Arpita Chakraborty for their continuous support and encouragement

References

1. N. Haque, A Visual Proof that $e \leqslant A \leqslant B \Rightarrow A^B > B^A$, *Resonance*, **26**(1) (2021) pp. (127-128).
2. A. Mukherjee and B. Chakraborty, Yet Another Visual Proof that $\pi^e < e^\pi$, *Math. Intelligencer*, **41**(2) (2019) p. 60.
3. C. Gallant, $A^B > B^A$ for $e \leqslant A \leqslant B$, *Math. Mag*. **64**(1) (1991) p. 31.
4. N. Yuefeng, Proof Without Words: Jordan's inequality, *Math. Mag*., **69**(2) (1996) p. 126.
5. R. Nelsen, *Proof without words II* (2000), MAA.
6. A. Plaza, Harmonic, Logarithmic, and Arithmetic Means and corollaries, *Amer. Math. Monthly*, **127**(5) (2020), p. 427.
7. A. Plaza, HM-LM-AM Inequalities, *Math. Mag*., **94**(2) (2021), p. 148.

10.1017/mag.2023.20

NAZRUL HAQUE
Ramakrishna Mission Vivekananda Centenary College, West Bengal, India
e-mail: *nazrul@rkmvccrahara.org*

ÁNGEL PLAZA
University of Las Palmas de Gran Canaria, Spain
e-mail: *angel.plaza@ulpgc.es*

107.07 A geometric illustration for infinite sequences and series

We can use a positive vanishing sequence to construct a sequence of squares. This so-called *square set* can then be used to visualise sums involving the sequence. As a first example, we use the sequence $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$ to construct the arrangement of squares seen in Figure 1, which we call the *square set for* $\frac{1}{2^n}$. We define l to be the length of this set, h to be the height to which the squares converge, and A to be the total area of the set.

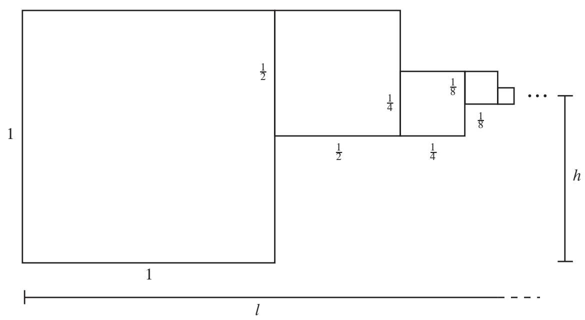


FIGURE 1: The square set for $\frac{1}{2^n}$

Such objects provide a useful visual representation of positive or alternating series. For example, notice that the boundedness of the alternating series $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \dots$ is immediately clear in Figure 1. To calculate both l and h, we can use the geometric series sum formula [1]:

$$l = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots = \frac{1}{1 - \frac{1}{2}} = 2,$$

$$h = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \dots = \frac{1}{1 + \frac{1}{2}} = \frac{2}{3}.$$

There are other approaches for calculating h. A series totalling the lengths of the vertical segments only approaching h from below arrives at the same result as the first alternating sum:

$$h = \frac{1}{2} + \frac{1}{8} + \frac{1}{32} + \dots = \frac{\frac{1}{2}}{1 - \frac{1}{4}} = \frac{2}{3}.$$

A more entertaining approach makes use of the self-similarity of the square set; observe that in Figure 2 the shaded section is a quarter-scale copy of the entire object. This property gives rise to the equation $h = \frac{h}{4} + \frac{1}{2}$, whose solution is $h = \frac{2}{3}$.

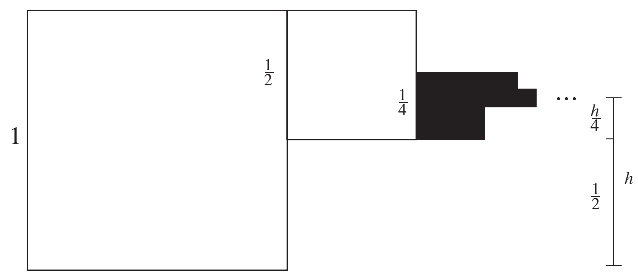


FIGURE 2: Square sets display self-similarity

Moving on from h, we can also calculate the area A of the square set:

$$A = 1 + \frac{1}{4} + \frac{1}{16} + \dots = \frac{1}{1 - \frac{1}{4}} = \frac{4}{3}.$$

We then see that $l \cdot h = A$, as $2 \cdot \frac{2}{3} = \frac{4}{3}$. Surprisingly, this identity holds for all square sets whose side-lengths are given by a vanishing positive geometric sequence!

Theorem: If S is the square set with length l, height h and area A whose sides are given by the geometric sequence $1, r, r^2, r^3, \dots$ with $0 < r < 1$, then $l \cdot h = A$.

https://doi.org/10.1017/mag.2023.21 Published online by Cambridge University Press

Proof

We use the geometric series sum formula to calculate as follows:

$$l = 1 + r + r^2 + r^3 + \ldots = \frac{1}{1 - r},$$

$$h = 1 - r + r^2 - r^3 + \ldots = \frac{1}{1 + r},$$

$$A = 1 + r^2 + r^4 + r^6 + \ldots = \frac{1}{1 - r^2}.$$

Then $l \cdot h = \frac{1}{1 - r} \cdot \frac{1}{1 + r} = \frac{1}{1 - r^2} = A$, as claimed.

While simple and sharp, this algebraic proof is not visually illuminating. As such, we present a geometric proof of the previous result.

Proof

Let T be a right-angled triangle with legs of $l = \frac{1}{1 - r}$ and $2h = \frac{2}{1 + r}$. Then the area of T is $\frac{1}{2} \cdot l \cdot 2h = l \cdot h$. To see that the square set S and triangle T have equal areas, rearrange S into the set S', then coordinatise T and S' as shown in Figure 3.

FIGURE 3: A coordinatisation of T and S'

Observe that the dark-shaded triangles inside T and above S' correspond to the portion of S' that lies outside of T, (see Figure 4). As such, the area of T is indeed equal to the area of S' and S.

https://doi.org/10.1017/mag.2023.21 Published online by Cambridge University Press

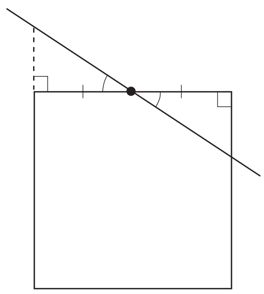


FIGURE 4: Congruent triangles inside and outside T

To confirm that the hypotenuse of T does indeed intersect the midpoints of the top edges of the squares of S', we first write the equation of the hypotenuse line, which is $(1 - r)x + \frac{1}{2}(1 + r)\, y = 1$.

Notice that $\left(\frac{1}{2},\ 1\right)$ is both a solution to this linear equation and the midpoint of the top edge of the first square in S'. A dilation with centre $\left(\frac{1}{1 - r},\ 0\right)$ and scale factor r maps each square to its right-hand neighbour, allowing us to conclude that the hypotenuse maintains this behaviour for every square in S'.

The relation $l \cdot h = A$ does not hold for square sets constructed using general sequences. A quick counterexample can be seen with the harmonic series:

$$l = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ \dots \text{ is divergent,}$$

$$h = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ \dots = \ln 2,$$

$$A = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \ \dots = \frac{\pi^2}{6}.$$

As l is unbounded, we immediately know that $l \cdot h \neq A$. It is also worth noting that h is calculated using the Maclaurin series for $\ln(1 + x)$ [1], A is the result of the Basel problem [2], and this shrinking square set is an object with infinite length and finite area, similar to the integral $\int_1^\infty \frac{1}{x^2}dx$. If one wishes to see a counterexample in which l is bounded, the sequence $\frac{1}{n^2}$ suffices. It is not yet known if the sequence describing the square set must be geometric to ensure that $l \cdot h = A$.

References

1. J. Stewart, *Essential calculus* (2nd edn), Brooks/Cole (2013).
2. W. Dunham, *Euler: the master of us all*, Mathematical Association of America (1999).

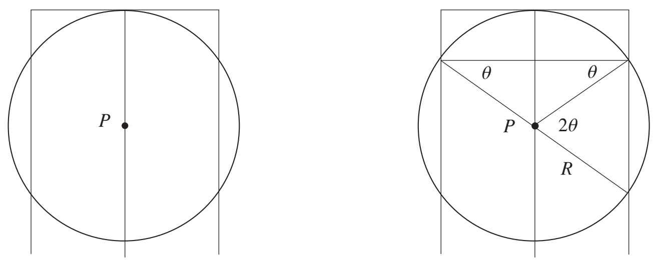
10.1017/mag.2023.21

RICHARD G. LIGO
109 University Square,
Erie, PA 16541 USA
e-mail: *ligo001@gannon.edu*

107.08 An interesting equivalent of squaring the circle

Consider the following task.

> Given the rectangular strip shown in Figure 1(a), construct the point P on the mid-line such that, when the circle with centre P which touches the horizontal edge of the strip is drawn, the area outside the circle at the top of the strip is equal to the area of the segment on the side.



(a) (b)

FIGURE 1

With notation added as in Figure 1(b), the area, A_s, of the segment on the side is given by

$$A_s = \tfrac{1}{2}R^2\left(2\theta - \sin 2\theta\right) = R^2\theta - \tfrac{1}{2}R^2 \sin 2\theta.$$

The area, A_t, of the area outside the circle at the top of the strip is given by

$$\begin{aligned} A_t &= \tfrac{1}{2}R^2 \sin(\pi - 2\theta) + 2R\cos\theta \cdot R\left(1 - \sin\theta\right) - \tfrac{1}{2}R^2(\pi - 2\theta) \\ &= \tfrac{1}{2}R^2 \sin 2\theta + 2R^2\cos\theta - R^2 \sin 2\theta - \tfrac{1}{2}R^2\pi + R^2\theta \\ &= 2R^2\cos\theta - \tfrac{1}{2}R^2\pi + R^2\theta - \tfrac{1}{2}R^2\sin 2\theta. \end{aligned}$$

Equality of areas, $A_s = A_t$, thus necessitates $\cos\theta = \dfrac{\pi}{4}$.

If point P were constructible using ruler and compasses, we could continue the construction as shown in Figure 2 where chord ABC produced is parallel to the horizontal chord in Figure 1(b). Point D is on the line ABC such that $BD = R$. Point E is on the extension of the left-hand edge of the strip with $DE = DA$ and $CEFG$ is a square.