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BOUNDEDNESS CRITERIA FOR GENERALIZED HANKEL CONJUGATE TRANSFORMATIONS

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1. Introduction. This paper completes a study, begun in [**7**], of conditions under which a generalized Hankel conjugate transformation H_λ is bounded between a pair of μ_α-rearrangement invariant function spaces, the measure μ_α being defined by $d\mu_\alpha(t) = t^{\alpha-1}dt$. Examples of such spaces are the $L^p(\mu_\alpha)$ of Lebesgue and generalizations of them due respectively to Orlicz and Lorentz.

A generalized Hankel conjugate transformation H_λ, $\lambda > -1$, is defined by

$$(1.1) \qquad (H_\lambda f)(y) = \lim_{x\to 0+} \int_0^\infty Q_\lambda(x, y, z) f(z) z^{2\lambda} dz,$$

the kernel $Q_\lambda(x, y, z)$ having the following expression in terms of Bessel functions

$$(1.2) \qquad Q_\lambda(x, y, z) = -(yz)^{-\lambda+1/2} \int_0^\infty e^{-xt} J_{\lambda+1/2}(yt) J_{\lambda-1/2}(zt) t dt.$$

The f in (1.1) is assumed to belong to the class $M(0, \infty)$ of functions which are Lebesgue-measurable on $(0, \infty)$. It is understood there is a set E of Lebesgue measure zero (depending on f) so that for fixed $y \notin E$ the integral in (1.1) is defined for all $x > 0$; furthermore, the resulting function of x has the indicated limit.

Our aim is to prove the continuity of the H_λ is equivalent to that of simpler operators T of the general form

$$(1.3) \qquad (Tf)(t) = \int_0^\infty a(s) f(st) dt, \quad t > 0,$$

whose domain consists of all functions in $M(0, \infty)$ for which the required integral exists a.e. More specifically, denote by $[X, Y]$ the space of linear operators bounded between the Banach spaces X and Y with $[X]$ an abbreviation for $[X, X]$; let P_p, Q_q be the operators of form (1.3) with kernels $a(t)$ equal to

$$(1.4) \qquad t^{1/p-1}\chi_{(0,1)}(t),\ 1 \leqq p < \infty \quad \text{and} \quad t^{1/q-1}\chi_{(1,\infty)}(t),\ 1 < q \leqq \infty,$$

respectively; then the main result, Theorem 2.1, may be stated as follows:

Let ρ_1 and ρ_2 be μ_α-rearrangement invariant norms on $M(0, \infty)$, generated by σ_1 and σ_2 respectively. Then $H_\lambda \in [L^{\rho_1}, L^{\rho_2}]$ if and only if $P_p + Q_q \in [L^{\sigma_1}, L^{\sigma_2}]$.

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The values of p and q depend on the relationship between α and λ. For example, in [7], it was shown that $p = 1$ and $q = \infty$ when $\lambda > -\frac{1}{2}$ and $-1 \leqq \alpha \leqq 2\lambda + 1$. We remark that the notations P and P' were used there in place of P_1 and Q_∞.

In Theorem 2.2 the results are specialized to the case $\sigma_1 = \sigma_2 = \sigma$. Here the boundedness criteria may be expressed in terms of the Boyd indices of the space L^σ. These indices have been calculated for the Lorentz and Orlicz spaces. See [**2**] and [**6**].

The operators $P_p + Q_q$ can be used to determine the possible range and domain spaces of a fixed H_λ and indeed to construct the optimal continuous partner of a suitable range or domain space. We will report on this in another paper.

For background on rearrangement invariant spaces and operators of the form (1.3)—in particular the P_p and Q_q—the reader is referred to the papers [**2**; **3**; **4** and **5**] of Boyd. Observe that we use the definitions of P_p and Q_q given in [**3**] rather than those of [**4**]. As in [**7**], Lebesgue measure on $(0, \infty)$ is denoted by m rather than μ_1. Lastly, the operators T and T' will be called *α-associates* if for all Lebesgue-measurable subsets E and F of $(0, \infty)$ with $\mu_\alpha(E)$, $\mu_\alpha(F) < \infty$

$$(1.5) \qquad \int_0^\infty \chi_F T\chi_E d\mu_\alpha = \int_0^\infty \chi_E T'\chi_F d\mu_\alpha.$$

If T is of form (1.3) and the measure is Lebesgue's—in which case T' will also be of this type, the kernel being $1/t\, a(1/t)$—the equation in (1.5) holds for all non-negative f, g in $M(0, \infty)$.

2. The boundedness criteria. As our starting point we give certain estimates for the kernel $Q_\lambda(x, y, z)$ which were established (for $Q_{\lambda,\alpha}(x, y, z) = z^{2\lambda+1-\alpha}Q_\lambda(x, y, z)$ rather than $Q_\lambda(x, y, z)$) in Lemma 2.1 of [**7**] in the special case $k = 2$. Those estimates were a somewhat sharpened version of ones given in [**8**] for $\lambda > 0$.

LEMMA 2.1. *If* $\lambda > -1$, $k > 1$, *then*

(i) $Q_\lambda(x,y,z) = O(y^{-2\lambda-1})$, *if* $0 < z < k^{-1}y$,
$\qquad\qquad\quad = O(yz^{-2\lambda-2})$, *if* $z \geqq ky$;

(ii) $$Q_\lambda(x, y, z) = c_\lambda y^{-\lambda} z^{-\lambda} \frac{y - z}{x^2 + (y - z)^2} + O\left[(yz)^{-\lambda-1/2}\left(1 + \log^+ \frac{yz}{(y - z)^2}\right)\right],$$

if $k^{-1}y \leqq z \leqq ky$.

Proof. Essentially the same as for Lemma 2.1 of [**7**].

Remarks. 1. The results of Lemma 2.1 yield their counterparts for the αth associate kernels $Q_{\lambda,\alpha}'(x, y, z) = y^{2\lambda+1-\alpha}Q_\lambda(x, z, y)$. Indeed, one can give an analysis of $H_{\lambda,\alpha}'$ entirely similar to that given below for H_λ. However, as this operator will not play as great a role as in [7] we do not deal with it here.

2. As in [7] (see the remark following Lemma 2.1), the first inequality in (i) can be improved when $\lambda = -\frac{1}{2}$. The consequences of this will be discussed at the end of the section.

On the basis of Lemma 2.1 we write $H_\lambda = \sum_{m=1}^{4} H_{\lambda,k}{}^{(m)}$, with $H_{\lambda,k}{}^{(m)}$ being defined as in (1.1), $Q_{\lambda,k}{}^{(m)}$ replacing $Q_\lambda(x, y, z)$, where

$$\begin{aligned} Q_{\lambda,k}{}^{(1)}(x, y, z) &= Q_\lambda(x, y, z)\chi_{(0,y/k)}(z), \\ Q_{\lambda,k}{}^{(2)}(x, y, z) &= Q_\lambda(x, y, z)\chi_{(ky,\infty)}(z), \\ Q_{\lambda,k}{}^{(4)}(x, y, z) &= c_\lambda y^{-\lambda} z^{-\lambda} \frac{y - z}{x^2 + (y - z)^2} \chi_{(y/k,ky)}(z), \\ Q_{\lambda,k}{}^{(3)}(x, y, z) &= Q_\lambda(x, y, z)\chi_{(y/k,ky)}(z) - Q_{\lambda,k}{}^{(4)}(x, y, z). \end{aligned} \tag{2.1}$$

Boundedness properties of the first three $H_{\lambda,k}{}^{(m)}$ will be stated, in Lemma 2.3 below, in terms of certain operators $T_{\lambda,k}{}^{(m)}$ $(m = 1, 2, 3)$, whose definition is motivated by the estimates of Lemma 2.1. In fact, $T_{\lambda,k}{}^{(m)}$ is an operator of form (1.3) having kernel $a_{\lambda,k}{}^{(m)}(z)$,

$$\begin{aligned} a_{\lambda,k}{}^{(1)}(z) &= z^{2\lambda}\chi_{(0,1/k)}(z), \\ a_{\lambda,k}{}^{(2)}(z) &= z^{-2}\chi_{(k,\infty)}(z), \\ a_{\lambda,k}{}^{(3)}(z) &= z^{\lambda-1/2}\left(1 + \log^+ \frac{z}{(1 - z)^2}\right)\chi_{(1/k,k)}(z). \end{aligned} \tag{2.2}$$

Criteria for the continuity of the $T_{\lambda,k}{}^{(m)}$ are given in

LEMMA 2.2. *Let ρ_1 and ρ_2 be μ_α-rearrangement invariant norms on $M(0, \infty)$ $(\alpha \neq 0)$, generated by σ_1 and σ_2 respectively. Then, for $\lambda > -1$, $k > 1$, $\beta = \alpha/(2\lambda + 1)$,*

(i) *a necessary and sufficient condition that $T_{\lambda,k}{}^{(1)} \in [L^{\rho_1}, L^{\rho_2}]$ is*

$$Q_\beta \in [L^{\sigma_1}, L^{\sigma_2}], \quad \alpha < 2\lambda + 1 < 0, \quad \textit{and}$$
$$P_\beta \in [L^{\sigma_1}, L^{\sigma_2}] \quad \alpha > 2\lambda + 1 > 0;$$

(ii) *a necessary and sufficient condition that $T_{\lambda,k}{}^{(2)} \in [L^{\rho_1}, L^{\rho_2}]$ is*

$$P_{-\alpha} \in [L^{\sigma_1}, L^{\sigma_2}] \quad \textit{when } \alpha < -1;$$

(iii) $T_{\lambda,k}{}^{(3)} \in [L^{\rho_1}, L^{\rho_2}]$ *if and only if* $L^{\rho_1} \subset L^{\rho_2}$.

Proof. In Andersen [**1**], it is shown that, for $\alpha > 0$, the membership in $[L^{\rho_1}, L^{\rho_2}]$ of a positive operator of form (1.3) with kernel $a(t)$ is equivalent to that in $[L^{\sigma_1}, L^{\sigma_2}]$ of another such operator whose kernel is

$$|\alpha|^{-1}t^{1/\alpha-1}a(t^{1/\alpha}). \tag{2.3}$$

The proof easily extends to the case $\alpha < 0$. Up to a constant multiple, the

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function (2.3) corresponding to $T_{\lambda,k}^{(1)}$ is

(2.4) $t^{1/\beta-1}\chi_{(k^{-\alpha},\infty)}(t)$, if $\alpha < 0$, and $t^{1/\beta-1}\chi_{(0,k^{-\alpha})}(t)$, if $\alpha > 0$.

The proof of (i) will be complete for the case $\alpha < 2\lambda + 1 < 0$ once it is shown the operator T with kernel $a(t) = t^{1/\beta-1}\chi_{(1,k^{-\alpha})}(t)$ belongs to $[L^{\sigma_1}, L^{\sigma_2}]$ whenever one of Q_β or the operator with kernel (2.4) does. But, if either of these operators is in $[L^{\sigma_1}, L^{\sigma_2}]$, it then follows from [**2**, Lemma 3.3(b)] that $L^{\sigma_1} \subset L^{\sigma_2}$. In view of [**2**, Lemma 3.2(c) and Theorem 3.1], this means $T \in [L^{\sigma_1}, L^{\sigma_2}]$. A like approach disposes of the case $\alpha > 2\lambda + 1 > 0$.

The proof of (ii) is similar to that of (i).

Another appeal to Lemma 3.2(c) and Theorem 3.1 of [2] suffices to establish (iii).

LEMMA 2.3. *Let ρ_1 and ρ_2 be μ_α-rearrangement invariant norms on $M(0, \infty)$, generated by σ_1 and σ_2 respectively. Then, for $\lambda > -1$, $k > 1$,*

(i) *$H_{\lambda,k}^{(m)}$ $(m = 1, 2, 3)$ belongs to $[L^{\rho_1}, L^{\rho_2}]$ whenever $T_{\lambda,k}^{(m)}$ does. In particular, $H_{\lambda,k}^{(3)} \in [L^{\rho_1}, L^{\rho_2}]$ whenever $L^{\rho_1} \subset L^{\rho_2}$.*

(ii) *$H_{\lambda,k}^{(4)}$ is defined for all locally μ_α-integrable functions (α any real number) and it belongs to $[L^{\rho_1}, L^{\rho_2}]$ whenever $P_1 + Q_\infty \in [L^{\sigma_1}, L^{\sigma_2}]$.*

Proof. The first assertion in (i) follows from the estimates of Lemma 2.1 and the use of Lebesgue's theorem on dominated convergence. See the arguments around (2.26) and (2.32) in Lemma 2.4 and Theorem 2.1 of [**7**].

$H_{\lambda,k}^{(4)}$ has essentially been dealt with in [**7**]—more precisely, it was the operator defined as in (1.1) with $Q_\lambda(x, y, z)$ replaced by $y^{2\lambda+1-\alpha}Q_{\lambda,2}^{(4)}(x, y, z)z^{\alpha-2\lambda-1}$. (All references, but one, in the remainder of the proof will be to [**7**].) Thus, the proof that $H_{\lambda,k}^{(4)}$ and its αth associate (see corollaries 2.1.1 and 2.1.2) are defined a.e. $[m]$ is readily extracted from the discussion surrounding (2.27). Again, considerations like those of Theorem 2.1 from (2.33) on, as well as their counterparts in Theorem 2.2, yield, as in Lemma 3.1, that if $f \in M(0, \infty)$ satisfies

$$(2.5) \qquad \int_0^\infty f^*(t) \sinh^{-1} (1/t)dt < \infty,$$

then $H_{\lambda,k}^{(4)}f$ is defined and

$$(2.6) \qquad (H_{\lambda,k}^{(4)}f)^{**}(t) \leqq c \int_0^\infty \frac{f^{**}(u)}{\sqrt{t^2 + u^2}}\, du,$$

where c is a positive constant independent of f. One may now complete the proof of (ii) using the argument of [**2**, Theorem 2.1].

THEOREM 2.1. *Let ρ_1 and ρ_2 be μ_α-rearrangement invariant norms on $M(0, \infty)$, generated by σ_1 and σ_2 respectively. Then $H_\lambda \in [L^{\rho_1}, L^{\rho_2}]$ if and only if $P_p + Q_q \in [L^{\sigma_1}, L^{\sigma_2}]$, where*

(i) $p = -\alpha, q = \alpha/(2\lambda + 1)$ *in case* $-1 < \lambda < -\frac{1}{2}, \alpha < -1$;
(ii) $p = 1, q = \alpha/(2\lambda + 1)$ *in case* $-1 < \lambda < -\frac{1}{2}, -1 \leqq \alpha < 2\lambda + 1$;
(iii) $p = -\alpha, q = \infty$ *in case* $\lambda > -\frac{1}{2}, \alpha < -1$;
(iv) $p = 1, q = \infty$ *in case* $\lambda > -\frac{1}{2}, -1 \leqq \alpha \leqq 2\lambda + 1$;
(v) $p = \alpha/(2\lambda + 1), q = \infty$ *in case* $\lambda > -\frac{1}{2}, \alpha > 2\lambda + 1$.

Proof. We first prove the if parts. To start, observe that in each case the proposed condition implies (keeping Lemma 2.2 in mind) that $P_1 + Q_\infty \in [L^{\sigma_1}, L^{\sigma_2}]$ and $T_{\lambda,2}^{(1)} + T_{\lambda,2}^{(2)} \in [L^{\rho_1}, L^{\rho_2}]$. Thus, for example, in case (i) elementary inequalities involving their kernels ensure that $P_1 + Q_\infty \in [L^{\sigma_1}, L^{\sigma_2}]$ along with $P_p + Q_q$. Now, $P_1 + Q_\infty \in [L^{\sigma_1}, L^{\sigma_2}]$ implies $L^{\rho_1} \subset L^{\rho_2}$. (See [**7**, Theorem 4.1].) Therefore, by Lemma 2.3, $H_{\lambda,2}^{(3)}, H_{\lambda,2}^{(4)} \in [L^{\rho_1}, L^{\rho_2}]$. Finally $T_{\lambda,2}^{(1)}, T_{\lambda,2}^{(2)} \in [L^{\rho_1}, L^{\rho_2}]$ yields $H_\lambda \in [L^{\rho_1}, L^{\rho_2}]$.

As for the only if parts, observe firstly that from (2.3) and (2.8) of [**7**] together with (16.5′) on page 84 of [**8**]

$$
\begin{aligned}
Q_\lambda(0, y, z) = &-\frac{(2\lambda + 1)}{\pi} 2^{-(\lambda+1/2)} y \int_0^\pi \sin^{2\lambda+1}\phi D^{-\lambda-1} d\phi \\
&+ \frac{2(\lambda + 1)}{\pi} yz^2 \int_0^\pi \sin^{2\lambda+1}\phi D^{-\lambda-2} d\phi \\
&- \frac{2(\lambda + 1)}{\pi} y^2 z \int_0^\pi \cos\phi \sin^{2\lambda+1}\phi D^{-\lambda-2} d\phi,
\end{aligned}
\tag{2.7}
$$

where $D = y^2 + z^2 - 2yz\cos\phi$. Thus, if $t = z/y$ and $E = 1 + t^2 - 2t\cos\phi$, $Q_\lambda(0, y, z)$ may be expressed in the form $K_\lambda(t) y^{-2\lambda-1}$,

$$
\begin{aligned}
K_\lambda(t) = -c_\lambda \int_0^\pi \sin^{2\lambda+1}\phi E^{-\lambda-1} d\phi - \left[c_\lambda' \int_0^\pi \cos\phi \sin^{2\lambda+1}\phi E^{-\lambda-2} d\phi\right] t \\
+ \left[c_\lambda' \int_0^\pi \sin^{2\lambda+1}\phi E^{-\lambda-2} d\phi\right] t^2,
\end{aligned}
\tag{2.8}
$$

with $c_\lambda = (2\lambda + 1) 2^{-(\lambda+1/2)}/\pi$ (not the same as in Lemma 2.1) and $c_\lambda' = 2(\lambda + 1)/\pi$. Again, if $\tau = y/z$ and $F = 1 + \tau^2 - 2\tau\cos\phi$, then $Q_\lambda(0, y, z)$ may be expressed in the form $L_\lambda(t) yz^{-2\lambda-2}$, where

$$
\begin{aligned}
L_\lambda(t) = -c_\lambda \int_0^\pi \sin^{2\lambda+1}\phi F^{-\lambda-1} d\phi + c_\lambda' \int_0^\pi \sin^{2\lambda+1}\phi F^{-\lambda-2} d\phi \\
- c_\lambda' \left[\int_0^\pi \cos\phi \sin^{2\lambda+1}\phi F^{-\lambda-2} d\phi\right] \tau.
\end{aligned}
\tag{2.9}
$$

Hence, for each $\lambda > -1, \lambda \neq -\frac{1}{2}$, there exists $k(\lambda) > 1$ so that

$$
\begin{aligned}
Q_\lambda(0, y, z) &\geqq (d_\lambda/2) y^{-2\lambda-1}, \quad -1 < \lambda < -\tfrac{1}{2}, \\
-Q_\lambda(0, y, z) &\geqq (d_\lambda/2) y^{-2\lambda-1}, \quad \lambda > -\tfrac{1}{2},
\end{aligned}
\tag{2.10}
$$

for $0 < z/y < [k(\lambda)]^{-1}$, while

$$
Q_\lambda(0, y, z) \geqq (e_\lambda/2) yz^{-2\lambda-2} \tag{2.11}
$$

for $0 < y/z < [k(\lambda)]^{-1}$. Here

(2.12) $$d_\lambda = |c_\lambda| \int_0^\pi \sin^{2\lambda+1} \phi d\phi \quad \text{and} \quad 0 < e_\lambda = [c_\lambda' - c_\lambda] \int_0^\pi \sin^{2\lambda+1} \phi d\phi.$$

Next, the first remark following Corollary 3.1.1 of [7] yields $P_1 + Q_\infty \in [L^{\sigma_1}, L^{\sigma_2}]$ whenever $H_\lambda \in [L^{\rho_1}, L^{\rho_2}]$. This completes the proof of (iv). In addition, it means that $H_\lambda \in [L^{\rho_1}, L^{\rho_2}]$ necessitates $H_{\lambda,k}{}^{(3)}$, $H_{\lambda,k}{}^{(4)} \in [L^{\rho_1}, L^{\rho_2}]$ for all $k > 1$, as follows from Lemma 2.3.

We now treat case (iii) which is similar to both (ii) and (v). For these values of the parameters $Q_\infty \in [L^{\sigma_1}, L^{\sigma_2}]$ implies the operator with kernel $a(t)$ given by (2.4) also belongs to that class. Hence $T_{\lambda,k}{}^{(1)}$ (and so, by Lemma 2.3, $H_{\lambda,k}{}^{(1)}$) is in $[L^{\rho_1}, L^{\rho_2}]$ for all $k > 1$. It follows that $H_{\lambda,k}{}^{(2)}$ must be in $[L^{\rho_1}, L^{\rho_2}]$ for all $k > 1$. Given non-negative $f \in M(0, \infty)$ vanishing outside a compact interval, it is seen from (2.11) that for $y > 0$

(2.13) $$0 \leqq (T_{\lambda,k(\lambda)}{}^{(2)} f)(y) \leqq (2/e_\lambda) \int_{k(\lambda)y}^\infty Q_\lambda(0, y, z) f(z) z^{2\lambda} dz = (2/e_\lambda)(H_{\lambda,k(\lambda)}{}^{(2)} f)(y).$$

Its kernel being positive, we get $T_{\lambda,k(\lambda)}{}^{(2)} \in [L^{\rho_1}, L^{\rho_2}]$ and so, by Lemma 2.2, $P_{-\alpha} \in [L^{\sigma_1}, L^{\sigma_2}]$.

Finally, consider (i). Given such restrictions on the parameters, we have, for all non-negative $f \in M(0, \infty)$ which vanish outside a compact interval,

(2.14) $$0 \leqq ([T_{\lambda,k(\lambda)}{}^{(1)} + T_{\lambda,k(\lambda)}{}^{(2)}] f)(y) \leqq (2/d_\lambda) \int_0^{y[k(\lambda)]^{-1}} Q_\lambda(0, y, z) f(z) z^{2\lambda} dz + (2/e_\lambda) \int_{k(\lambda)y}^\infty Q_\lambda(0, y, z) f(z) z^{2\lambda} dz.$$

The righthand side of (2.14) is bounded by a constant times $(H_\lambda - H_{\lambda,k(\lambda)}{}^{(3)} - H_{\lambda,k(\lambda)}{}^{(4)})$ which is in $[L^{\rho_1}, L^{\rho_2}]$. Hence $T_{\lambda,k(\lambda)}{}^{(1)} + T_{\lambda,k(\lambda)}{}^{(2)}$ is in $[L^{\rho_1}, L^{\rho_2}]$— that is, $P_p + Q_q \in [L^{\sigma_1}, L^{\sigma_2}]$ for $p = -\alpha$, $q = \alpha/(2\lambda + 1)$.

Remark. For the values of the parameters not considered in Theorem 2.1, namely $-1 < \lambda < -\frac{1}{2}$ and $\alpha \geqq 2\lambda + 1$, $H_\lambda \notin [L^{\rho_1}, L^{\rho_2}]$ for any μ_α-rearrangement invariant norms ρ_1 and ρ_2. This follows once it is seen that $H_\lambda \notin [L'_{\text{loc}}, L_{\text{loc}}]$. For, as shown in [7], the space L_{loc} of locally μ_α-integrable functions is the largest μ_α-rearrangement invariant space while its associate space L'_{loc} is the smallest.

Suppose, if possible, that $H_\lambda \in [L'_{\text{loc}}, L_{\text{loc}}]$. An argument using (2.10) and (2.11) as in the proof of Theorem 2.1 yields $T_{\lambda,k(\lambda)}{}^{(1)}$, $T_{\lambda,k(\lambda)}{}^{(2)} \in [L'_{\text{loc}}, L_{\text{loc}}]$. But for $\alpha > 2\lambda + 1$ the function $z^m \chi_{(0,1)}(z)$, $\max [0, -\alpha] < m \leqq -(2\lambda + 1)$, is in L'_{loc} while for $0 < y < 1$

(2.15) $$(T_{\lambda,k(\lambda)}{}^{(1)} f)(y) = y^{-2\lambda-1} \int_0^{y/k(\lambda)} z^{m+2\lambda} dz = \infty.$$

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Again, when $\alpha = 2\lambda + 1$, the boundedness of H_λ entails that

$$(2.16) \quad g(t) = \int_0^\infty f(tu)du$$

must be locally m-integrable for all $f \in L'_{\mathrm{loc}}$. This is not so if $f(u) = \chi_{(0,1)}(u)$.

Restricting attention to the case $\rho_1 = \rho_2$, we obtain

THEOREM 2.2. *Let ρ be a μ_α-rearrangement invariant norm on $M(0, \infty)$, where L^ρ has upper index γ and lower index δ. Then $H_\lambda \in [L^\rho]$ if and only if*

(i) $(2\lambda + 1)\alpha^{-1} < \delta \leqq \gamma < -\alpha^{-1}$ *in case* $-1 < \lambda < -\frac{1}{2}$ *and* $\alpha < -1$;

(ii) $(2\lambda + 1)\alpha^{-1} < \delta \leqq \gamma < 1$ *in case* $-1 < \lambda < -\frac{1}{2}$ *and* $-1 \leqq \alpha < 2\lambda + 1$;

(iii) $0 < \delta \leqq \gamma < -\alpha^{-1}$ *in case* $\lambda > -\frac{1}{2}$ *and* $\alpha < -1$;

(iv) $0 < \delta \leqq \gamma < 1$ *in case* $\lambda > -\frac{1}{2}$ *and* $-1 \leqq \alpha \leqq 2\lambda + 1$;

(v) $0 < \delta \leqq \gamma < (2\lambda + 1)\alpha^{-1}$ *in case* $\lambda > -\frac{1}{2}$ *and* $\alpha > 2\lambda + 1$.

Proof. Argue as in Lemma 3.6 of [**2**] using [**4**, Theorem 1].

Remark. The results of Theorems 2.1 and 2.2 can be obtained for the operator $H_{-1/2}$ in the same way as for the other H_λ. In this case, though, the sharper inequalities satisfied by the kernels allow $2\lambda + 1$ (equal to zero in this instance) to be replaced by 2 in all the relevant conditions.

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