(6) the segment $E'F'$ is contained in the plane $\hat{\Omega}$ through E' with the normal vector $\vec{r}$.

So the six planes $(\Gamma, \hat{\Gamma})$, $(\Lambda, \hat{\Lambda})$, $(\Omega, \hat{\Omega})$ are pairwise parallel.

Since $\vec{p}\cdot\vec{q} = \vec{q}\cdot\vec{r} = \vec{r}\cdot\vec{p} = 0$, three planes Γ, Λ, Ω are mutually perpendicular. Hence, planes $\Gamma, \hat{\Gamma}, \Lambda, \hat{\Lambda}, \Omega, \hat{\Omega}$ bound a rectangular box. Moreover, we have

$$d\left(\Gamma, \hat{\Gamma}\right) = d\left(\Gamma, A'\right) = \frac{|\varphi^2(-1 - 1)|}{\sqrt{1 + \varphi^2 + \varphi^4}} = \frac{2\varphi^2}{\sqrt{4\varphi^2}} = \varphi,$$

$$d\left(\Lambda, \hat{\Lambda}\right) = d\left(\Lambda, C'\right) = \frac{|2\varphi|}{\sqrt{1 + \varphi^2 + \bar{\varphi}^2}} = \frac{2\varphi}{\sqrt{4}} = \varphi, \text{ and}$$

$$d\left(\Omega, \hat{\Omega}\right) = d\left(\Omega, E'\right) = \frac{|-2|}{\sqrt{1 + \bar{\varphi}^2 + \bar{\varphi}^4}} = \frac{|-2|}{\sqrt{(2\bar{\varphi})^2}} = \left|\frac{-2}{2}(-\varphi)\right| = |\varphi| = \varphi.$$

So, these six planes $\Gamma, \hat{\Gamma}, \Lambda, \hat{\Lambda}, \Omega, \hat{\Omega}$ bound a $\varphi \times \varphi \times \varphi$ cube. From the construction, all vertices of $K(1)$ lie on or inside of the $\varphi \times \varphi \times \varphi$ cubic box. Therefore, the theorem follows.

References

1. J. Kepler, *Epitome of Copernican astronomy*, approximately 1620. Republished by Hutchins, R. M., editor in chief, *Great Books of the Western World*. Volume 16, Ptolemy, Copernicus, Kepler, Encyclopedia Britannica, Inc., (1948).
2. H. S. M. Coxeter, *Introduction to geometry* (2nd edn.), John Wiley & Sons, Inc., (1969).
3. C. Alsina, R. B. Nelsen, *A mathematical space odyssey – solid geometry in the 21st century*, MAA Press (2015).

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107.25 A refinement of Griffiths' formula for the sums of the powers of an arithmetic progression

Consider the sum of k-th powers of the terms of an arithmetic progression with first term a and common difference d

$$S_k^{a,d}(n) = a^k + (a + d)^k + (a + 2d)^k + \ldots + (a + (n - 1)d)^k,$$

where k, a, d and n are assumed to be integer variables with $k, a \geqslant 0$ and $d, n \geqslant 1$. In [1], Griffiths derived the following polynomial formula:

$$S_k^{a,d}(n) = \sum_{m=1}^{k+1} \sum_{r=m}^{k+1} \frac{1}{r} d^{k-m} \left\{ {k \atop r-1} \right\} \overline{\left[{r \atop m} \right]} \left((a + nd)^m - a^m\right), \tag{1}$$

where $\overline{\begin{bmatrix} k \\ j \end{bmatrix}} = (-1)^{k-j} \begin{bmatrix} k \\ j \end{bmatrix}$ are the signed Stirling numbers of the first kind (with $\begin{bmatrix} k \\ j \end{bmatrix}$ being the unsigned Stirling numbers of the first kind), and $\begin{Bmatrix} k \\ r-1 \end{Bmatrix}$ are the unsigned Stirling numbers of the second kind.

The purpose of this Note is to point out that, as can be easily verified by employing the definition of $\overline{\begin{bmatrix} k \\ j \end{bmatrix}}$ given in [1, Equation (2)], and renaming the variable r as j, Griffiths' formula (1) can be expressed equivalently in the more compact form

$$S_k^{a,d}(n) = d^k \sum_{j=0}^{k} j! \begin{Bmatrix} k \\ j \end{Bmatrix} \left[\binom{n + \frac{a}{d}}{j+1} - \binom{\frac{a}{d}}{j+1} \right], \tag{2}$$

where, for real x and integer $i \geqslant 1$, the generalised binomial coefficient $\binom{x}{i}$ is defined as

$$\binom{x}{i} = \frac{x(x-1)(x-2)\dots(x-i+1)}{i!}.$$

In particular, for $a = 0$ and $d = 1$, from (2) we recover the well-known formula for the ordinary power sum $0^k + 1^k + \dots + (n-1)^k$, namely

$$S_k^{0,1}(n) = \sum_{j=0}^{k} j! \begin{Bmatrix} k \\ j \end{Bmatrix} \binom{n}{j+1}.$$

The formula (2) was rediscovered in [2, Equation (3)] and, independently, in [3, Equation (18)].

For completeness, we also note the following 'dual' formula of (2) involving the signed Stirling numbers of the second kind $\overline{\begin{Bmatrix} k \\ j \end{Bmatrix}} = (-1)^{k-j} \begin{Bmatrix} k \\ j \end{Bmatrix}$:

$$S_k^{a,d}(n) = d^k \sum_{j=0}^{k} j! \overline{\begin{Bmatrix} k \\ j \end{Bmatrix}} \left[\binom{n + \frac{a}{d} + j - 1}{j+1} - \binom{\frac{a}{d} + j - 1}{j+1} \right], \tag{3}$$

which can be obtained by following the procedure set forth in [1] in a step-by-step fashion, except that it should be used with the relation

$$x^m = \sum_{r=1}^{m} r! \overline{\begin{Bmatrix} m \\ r \end{Bmatrix}} \binom{x + r - 1}{r},$$

instead of the definition employed in [1], namely

$$x^m = \sum_{r=1}^{m} r! \begin{Bmatrix} m \\ r \end{Bmatrix} \binom{x}{r}.$$

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In particular, for $a = d = 1$, from (3) we obtain

$$S_k^{1,1}(n) = 1^k + 2^k + \dots + n^k = \sum_{j=0}^{k} j! \left\{ {k \atop j} \right\} \binom{n+j}{j+1}.$$

References

1. M. Griffiths, More on sums of powers of an arithmetic progression, *Math. Gaz.* **93** (July 2009) pp. 277–279.
2. D. Laissaoui and M. Rahmani, An explicit formula for sums of powers of integers in terms of Stirling numbers, *J. Integer Seq.* **20** (2017), Article 17.4.8, 6 pp.
3. J. L. Cereceda, Generalized Stirling numbers and sums of powers of arithmetic progressions, *Internat. J. Math. Ed. Sci. Tech.* **51** (2020), pp. 954-966.

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107.26 A difference theorem involving *k*-gonal and centred *k*-gonal numbers

Proposition

Where $p(n, k)$ and $c(n, k)$ denote the k-gonal number of n sides and the centred k-gonal number of n sides, respectively; for $n \in \mathbb{N}$, the following identity holds:

$$p(n, k) - c(n, k - 2) = n - 1.$$

Proof: For $n = 6$, the proof is demonstrated for $k = 10$.

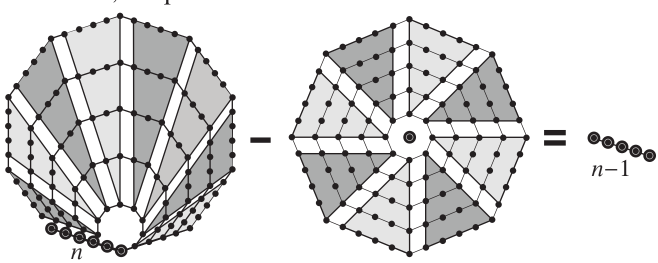

FIGURE 1

Corollary: Using the fact that a star number of n sides is isomorphic to a centred dodecagonal number of n sides [1], we further deduce the following result: where $\tau(n)$ and $\sigma(n)$ denote the tetradecagonal number of n sides and