Hence
$$\frac{x^m-1}{m(x-1)} > \frac{x^n-1}{n(x-1)}.$$

Remove the factor $x-1$ which is positive

$$\therefore \quad (x^m-1)/m > (x^n-1)/n.$$

But if $0<x<1$ the points are arranged from right to left, with the result

$$\frac{x^m-1}{m(x-1)} < \frac{x^n-1}{n(x-1)}.$$

But $x-1$ is now negative, and its removal reverses the sign of inequality.

Hence in both cases

$$\frac{x^m-1}{m} > \frac{x^n-1}{n}$$

if x is positive, and $m>n$.

C. Tweedie

A Rule for Resolving Integral Algebraic Expressions into Factors.—Professor Chrystal remarks (Algebra, Chap. VII. §4) that "for tentative processes no general rule can be given." The tentative processes consist in arranging the terms in groups in such a way as either to manifest a factor common to these groups or aggregates of terms, or to bring the expression under one of the Standard Forms of which the factors are already known, such as a^2-b^2, a^3-b^3, a^3+b^3.

The following Rule seems to me to possess sufficient generality and efficiency to be worth stating :

Let the expression be arranged according to powers of the letter which occurs in the simplest manner, *i.e.* let each group consist of the terms which contain one particular power of that letter.

The phrase "in the simplest manner" is to be interpreted usually as "in the smallest *number* of different powers," and when there is one letter occurring *only* in the first degree, that one ought to be chosen.

In the examples given by Chrystal immediately before his remarks above quoted, viz. :—

$px^2+(1+pq)xy+qy^2$, and $x^3+(m+n+1)x^2a+(m+n+mn)xa^2+mna^3$,

the advantage of the rule is obvious, if we choose p (or q) as the letter in the former, and m (or n) in the latter.

Arranging the latter (*e.g.*) in powers of m we get $m(x^2a+xa^2+nxa^2+na^3)+(x^3+nx^2a+x^2a+nxa^2)$ and factorizing the separate groups by the same rule after separating the common factors (a in the first, x in the second), we find

$$\begin{aligned} &ma\{na(x+a)+x(x+a)\}+x\{na(x+a)+x(x+a)\}\\ &=ma(x+a)(na+x)+x(x+a)(na+x)\\ &=(x+a)(na+x)(ma+x). \end{aligned}$$

Expressions of the type $a^2(b^3-c^3)+b^2(c^3-a^3)+c^2(a^3-b^3)$ are very amenable to this rule if we choose a simplest letter *at each stage* of the work. Thus :—

$$\begin{aligned} &a^2(b^3-c^3)+b^2(c^3-a^3)+c^2(a^3-b^3),\\ &=a^3(c^2-b^2)+a^2(b^3-c^3)+b^2c^2(c-b),\ \text{(acc. to } a)\\ &=(c-b)\{a^3(c+b)-a^2(b^2+bc+c^2)+b^2c^2\}. \end{aligned}$$

Now arrange acc. to b in the latter factor, since a no longer occurs so simply as b, and we get

$$\begin{aligned} &(c-b)\{b^2(c^2-a^2)+ba^2(a-c)+a^2c(a-c)\}\\ &=(c-b)(c-a)\{b^2(c+a)-ba^2-a^2c\} \end{aligned}$$

Now acc. to c, giving

$$\begin{aligned} &(c-b)(c-a)\{c(b^2-a^2)+ab(b-a)\}\\ &=(c-b)(c-a)(b-a)\{c(b+a)+ab\} \end{aligned}$$

which cannot be further factorized.

The power of the Rule lies largely in the fact that any factor not involving the letter chosen must be a factor of *each term* of the expression when arranged according to powers of that letter, so that by such grouping any such factor will be made manifest.

But the rule is useful in other cases. Even such a simple case as $x^2-y^2-6xz+9z^2$ is tractable by the rule. Here y is the letter that occurs in the simplest manner, so that we are led to the form $(x^2 - 6xz + 9z^2) - y^2$, then, factorizing the first group, to $(x-3z)^2 - y^2$ which is of the standard form $a^2 - b^2$.

Sometimes the most effectual grouping of the terms is according to dimensions, counting several or all of the letters. (This is really equivalent to grouping according to powers of a letter of value 1 which may be introduced to render the expression homogeneous with reference to itself and these letters.) We might generalise the original Rule thus:—*Arrange the terms in groups, each of which is homogeneous with regard to a chosen letter or chosen letters, and let this be done in the manner which reduces the expression to the simplest form, and if necessary treat each of the separate groups by the same method in order to factorize it.*

As examples of arranging as to dimensions in *two* letters, we may take these:—

(i)
$$\begin{aligned}&(xy+x+y)^2 - 4xy(x+y)\\ &= x^2y^2 + \{2xy(x+y) - 4xy(x+y)\} + (x+y)^2\\ &= x^2y^2 - 2xy(x+y) + (x+y)^2\\ &= \{xy - (x+y)\}^2\end{aligned}$$

(ii)
$$\begin{aligned}&x^3 - x^2y + xy^2 - y^3 + x^2 + y^2 - x + y - 1\\ &= (x^3 - x^2y + xy^2 - y^3) + (x^2+y^2) - (x-y) - 1\\ &= \{x^2(x-y) + y^2(x-y)\} + (x^2+y^2) - (x-y) - 1\\ &= (x-y)(x^2+y^2) + (x^2+y^2) - (x-y) - 1\\ &= (x-y)(x^2+y^2-1) + (x^2+y^2-1)\\ &= (x^2+y^2-1)(x-y+1).\end{aligned}$$

Examples might be multiplied. The rule here discussed has most value when there are several letters involved. It is to be looked on as a useful supplement to those usually explained in the text-books. I have found it surprisingly general in its application. No doubt it is well known to many—but it has not yet attained the place it deserves.

Apropos of factors, let me refer to Mr Butters' paper on the subject in Vol. XVI. of the *Proceedings* of this Society, where is to be found a discussion of another branch of the topic.

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