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TOWARDS MOTIVIC QUANTUM COHOMOLOGY OF $\bar{M}_{0,S}$

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Dedicated to Professor V. V. Shokurov, on his 60th birthday

Abstract We explicitly calculate some Gromov–Witten correspondences determined by maps of labelled curves of genus 0 to the moduli spaces of labelled curves of genus 0. We consider these calculations as the first step towards studying the self-referential nature of motivic quantum cohomology.

Keywords: moduli spaces; Gromov–Witten theory; quantum cohomology

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1. Introduction: motives and quantum cohomology

1.1. A summary

Let Var_k be the category of smooth complete algebraic varieties defined over a field k.

The category of classical motives Mot_k^K, with coefficients in a $\boldsymbol{Q}$-algebra K, is the target of a functor $h\colon \mathrm{Var}_k^{\mathrm{op}} \to \mathrm{Mot}_k^K$, which, in the vision of Alexandre Grothendieck, ought to be *a universal cohomology theory*, with values in a tensor K-linear category.

Morphisms $X \to Y$ in Mot_k^K are represented by *classes of correspondences*, algebraic cycles on $X \times Y$ with coefficients in K. Depending on the equivalence relation imposed on these cycles, one could consider Chow motives, numerical motives, etc.

Besides the objects $h(V)$ for $V \in \mathrm{Var}_k$, the category Mot_k^K contains their *direct summands* ('pieces') and their *twists* by Tate's motive. Formally adding these objects, one turns the category of motives into a Tannakian category. One can then apply the philosophy of the motivic fundamental group to it. Ideally, all inherent structures of cohomology objects can be encoded/replaced by the representations of the respective motivic fundamental group.

What is special about 'total motives' $h(V)$, as opposed to pieces and twists?

For example, the objects $h(V)$ bring with them a natural structure of *commutative algebras* in Mot_k^K. It is not determined only by $h(V)$: isomorphic motives $h(V)$s may well have different multiplications; but, of course, this classical multiplication is motivic in the sense that it is induced by the diagonal map $V \to V \times V$ in Var_k and by the class of its graph in Mot_k^K.

The advent of quantum cohomology from physics to algebraic geometry opened our eyes to the fact that classical cohomology spaces of algebraic varieties, say, over $\boldsymbol{C}$, possess an incomparably richer structure: they (or rather their tensor powers) are acted upon by cohomology of moduli spaces of pointed curves $H^*(\bar{M}_{g,n})$, much as Steenrod operations act in the topological situation. From the physical perspective, these operations encode 'quantum corrections to the classical multiplication'.

Grothendieck's vision, however, turned out to be prophetic. *This new structure is motivic as well* in the same sense: it is induced by canonical Chow correspondences, Gromov–Witten invariants $I^V_{g,n,\beta}$ in $A_*(\bar{M}_{g,n} \times V^n)$ indexed by effective classes β in $A_1(V)$. This was conjectured in [**24**], worked out in more detail in [**7**] and finally proved in [**4**], where the virtual fundamental classes in the Chow groups of spaces of stable maps were constructed by algebraic–geometric techniques.

This construction allowed Behrend to establish a list of universal identities between the Gromov–Witten invariants that were conjectured earlier.

Taken together, these identities imply that, for each total motive $h(V)$, the infinite sum of its copies indexed by the numerical classes β of effective curves on V possesses the canonical structure of an algebra over the cyclic modular operad $\mathcal{HM}$:

$$\mathcal{HM}(n) := \coprod_g h(\bar{M}_{g,n}).$$

This is the motivic core of quantum cohomology.

However, this discovery also stressed an inherent tension between the initial Grothendieck vision and the highly non-Tannakian character of the quantum cohomology expressed in the following observations.

First, these structures of $\mathcal{HM}$-algebras are not functorial in any naive sense with respect to morphisms in Var_k (except isomorphisms). Note that the classical multiplication *is* functorial with respect to morphisms in Var_k; quantum corrections destroy this. However, as was shown in [**27**], quantum multiplication *is functorial* with respect to at least certain isomorphisms in Mot_k^K (flops) that do not agree with classical multiplication: quantum corrections exactly compensate classical discrepancies. This is a remarkable fact, suggesting that motivic functorality might be an important hidden phenomenon.

Second, being total motives, $h(\bar{M}_{g,n})$ *themselves* have quantum cohomology, that is, they define algebras over $\mathcal{HM}$.

One aim of this paper is to draw attention to this self-referentiality and to start studying the quantum cohomology *of* $\mathcal{HM}$ and its relation to the quantum cohomology action of this operad upon other total motives. Analogies with homotopy theory, in particular, A^1-homotopy formalism, might help us to recognise a pattern in algebraic geometry similar to that of iterated loop spaces.

A warning is in order: many meaningful questions cannot be asked, or answers obtained, until one extends both parts of the theory, motives and quantum cohomology, from the category Var_k to at least the category of smooth DM-stacks. Some of the complications arising can be avoided if one restricts the problem to the case of genus 0 quantum cohomology. We adopt this restriction in this paper.

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1.2. Results of this paper

This paper is the first installment in a project whose goal is to understand the Gromov–Witten theory of the moduli spaces of curves, preferably on the motivic level, that is, the level of J- and I-correspondences (see [**6**] for a clear and intuitive introduction).

Specifically, the spaces $\bar{M}_{0,S}$ (with variable S) and their products are interrelated by a host of natural morphisms expressing embeddings of boundary strata, forgetting labelled points, relabelling, etc. (see [**7**, **21**, **22**] for a systematic description).

The Gromov–Witten classes that we study in this paper are certain Chow correspondences

$$I(S, \Sigma, \beta) \in A_*(\bar{M}^{\Sigma}_{0,S} \times \bar{M}_{0,\Sigma}), \tag{1.1}$$

where S, Σ are (disjoint) finite sets of labels and β runs over classes of effective curves in $A_1(\bar{M}_{0,S})$.

Our main motivation is the following vague conjecture.

Guess 1.1. *Classes (1.1) are 'natural' in the sense that they can be functorially expressed through canonical morphisms in the category of moduli spaces of labelled trees of various combinatorial types.*

This guess is a natural first step towards understanding the self-referential nature of Gromov–Witten theory in motivic algebraic geometry: the fact that components of the basic modular operad 'are' algebras over the same operad (if one takes into account twisting and grading by the cones of effective curves).

The main result of this paper is an explicit description, in the spirit of our guess, of those I-correspondences of $\bar{M}_{0,S}$ that correspond *to the classes β of boundary curves* (see Theorem 5.2 in § 5).

The question this answers is quite natural, in particular, because there is a conjecture that boundary curve classes are generators of the Mori cone (see [**10**, **11**, **14**, **17**, **23**] for this and related problems).

1.3. From curves to surfaces and further on?

One can imaginatively say that the quantum cohomology of V reveals hidden geometry that can be seen only when one starts probing V by mapping curves C ('strings') to V. A natural question arises of how to use, say, surfaces ('membranes') in place of curves, and how to do it in algebraic, rather than symplectic or differential, geometry?

If we expect to discover new universal motivic actions in this way, we must first contemplate the case when V is *a point* and pose the following question.

What are the analogues of moduli spaces $\bar{M}_{g,n}$ (or at least $\bar{M}_{g,0}$) for surfaces in this context?

The experience of the stringy case indicates that these analogues must be *rigid* objects, as $\bar{M}_{g,n}$ themselves are (see [**16**]).

In fact, moduli spaces are only rarely rigid, but, according to a brave guess of Kapranov, if one starts with an object $X = X(0)$ of dimension n, produces its moduli space of deformations $X(1)$, then produces the moduli space $X(2)$ of deformations of $X(1)$ etc., then $X(n)$ must be rigid. Quoting [**16**], which summarizes the philosophy expressed in an unpublished manuscript by Kapranov.

> One thinks of $X(1)$ as H^1 of a sheaf of non-abelian groups on $X(0)$. Indeed, at least the tangent space to $X(1)$ at $[X]$ is identified with $H^1(\mathcal{T}_X)$, where $\mathcal{T}_X$ is the tangent sheaf, the sheaf of first-order infinitesimal automorphisms of X. Then one regards $X(m)$ as a kind of non-abelian H^m, and the analogy with the usual definition of abelian H^m suggests the statement above.

Extending this idea, one might guess that an imaginary 'membrane quantum cohomology' should define motivic actions of rigid (iterated) moduli spaces of surfaces (endowed with cycles to keep track of incidence conditions) upon certain total (ind-) motives. One motivation of this paper is to make some propaganda for this idea.

2. Gromov–Witten correspondences

We start with background terminology and notation.

2.1. Moduli stacks

We consider schemes over a fixed field k of characteristic 0.

Any scheme W 'is the same as' the contravariant functor of its T-points $W(T) = \mathrm{Hom}(T, W)$ with values in Sets.

More generally, a stack (of groupoids) $\mathcal{F}$ 'is the same as' the class of its T-points $\mathcal{F}_T$, where T runs over schemes. The main new element of the situation is that each $\mathcal{F}_T$ itself, and their union $\mathcal{F}$ over 'all' T, are not simply sets or classes, but categories. Moreover, they form a sheaf on the étale (or fppf) site of schemes.

So we think about the individual objects of $\mathcal{F}_T$ as schematic T-points of $\mathcal{F}$, whereas the non-trivial morphisms between them are functors subject to a list of restrictions specific to stacks. We recall this list informally below.

As in [**28**, V.3], we imagine that the objects of $\mathcal{F}_T$ are 'families (of something) over T'. In practical terms, one family is usually given by a diagram of schemes and morphisms, in which a part of the data remains fixed, including its 'base' T, and the rest is subject to a list of explicit restrictions.

For example, if $\mathcal{F}$ is represented by a scheme W, then 'one family over T (of points of W)' is a very simple diagram $T \to W$.

The following requirements must be satisfied.

(i) Each object of $\mathcal{F}$ belongs to an $\mathcal{F}_T$ for a unique scheme T, and the map $b\colon \mathcal{F} \to \mathrm{Sch}$, sending a family to its base, is a functor. The groupoid property requires that if $b(f) = \mathrm{id}_T$, then f is an isomorphism between two respective T-points.

(ii) With respect to the morphisms $\varphi\colon T_1 \to T_2$, $\mathcal{F}_T$ must be contravariant: we must be given 'base change' functors $\varphi^*\colon \mathcal{F}_{T_2} \to \mathcal{F}_{T_1}$, together with functor isomorphisms $(\varphi \circ \psi)^* \to \psi^* \circ \varphi^*$ and associativity diagrams for them.

Moreover, if $F \in \mathrm{Ob}\mathcal{F}_{T_2}$, then the lifted family $\varphi^*(F) \in \mathrm{Ob}\mathcal{F}_{T_1}$ must be endowed with a canonical morphism $F \to \varphi^*(F)$ lifting φ and satisfying a set of conditions.

For example, the base change for $T_2 \to W$ is simply the composition $T_1 \to T_2 \to W$.

(iii) $\mathcal{F}$ is a stack, if each T-family is uniquely defined by its restrictions to an étale (or fppf) covering of T and the standard descent data. The same must be true for morphisms of T-families etc.

(iv) Morphisms of stacks are functors between the respective categories of families, identical on bases of families.

Thus, an object $F \in \mathrm{Ob}\mathcal{F}_T$ can also be treated as a stack, and, as such, it is endowed with a morphism of stacks $F \Rightarrow \mathcal{F}$.

2.2. Families of stable maps: preliminaries

We now describe the main classes of families and stacks with which we deal here.

First of all, fix a finite set Σ, a genus $g \geqslant 0$, a smooth projective manifold W over k and an effective class $\beta \in A_1(W)$.

One can then define a (proper DM)-stack $\bar{M}_{g,\Sigma}(W,\beta)$.

For a k-scheme T, one object of the groupoid $\bar{M}_{g,\Sigma}(W,\beta)(T)$ of T-points of this stack consists of a diagram of schemes of the following structure:

$$\begin{tikzcd} \mathcal{C}_T \arrow[r, "f_T"] \arrow[d, "h_T"] & W \\ T & \end{tikzcd} \tag{2.1}$$

and a family of sections $x_{j,T}\colon T \to \mathcal{C}_T$, $j \in \Sigma$, $h_T \circ x_{j,T} = \mathrm{id}_T$.

They must satisfy the following conditions.

(a) $\mathcal{C}_T \to T$ and $(x_{j,T})$ constitute a flat *prestable* T-family of curves of genus g.

(b) $f_T\colon (\mathcal{C}_T; (x_{j,T})) \to W$ is a *stable map* of class β.

For precise definitions of (pre-) stability and the class of the map that we use here, see [**7**] or [**28**].

Given such a diagram with sections, we call (W,β) its *target*, T its *base*, and the whole setup *a T-family of stable maps*. Isomorphisms of families, lifting id_T, must also be identical on W. Base changes are defined in a rather evident way.

The stack $\bar{M}_{g,\Sigma}(W,\beta)$ is defined as the base of the universal family of this type with given target (W,β):

$$\begin{tikzcd} \bar{C}_{g,\Sigma}(W,\beta) \arrow[r, "f"] \arrow[d, "h"] & W \\ \bar{M}_{g,\Sigma}(W,\beta) & \end{tikzcd} \tag{2.2}$$

It is endowed with sections $x_j\colon \bar{M}_{g,\Sigma}(W,\beta) \to \bar{C}_{g,\Sigma}(W,\beta)$, $j \in \Sigma$.

Naturally, $\bar{C}_{g,\Sigma}(W,\beta)$ is a stack as well.

If W is a point, $\beta = 0$, we routinely omit the target and write simply $\bar{M}_{g,\Sigma}$, $\bar{C}_{g,\Sigma}$ etc.

Moreover, (2.2) produces the *evaluation/stabilization* diagram

$$\begin{tikzcd} \bar{M}_{g,\Sigma}(W,\beta) \arrow[r, "\mathrm{st}"] \arrow[d, "\mathrm{ev}"] & \bar{M}_{g,\Sigma} \\ W^{\Sigma} & \end{tikzcd} \tag{2.3}$$

Here,

$$\mathrm{ev} = (\mathrm{ev}_j = f \circ x_j \mid j \in \Sigma)\colon \bar{M}_{g,\Sigma}(W,\beta) \to W^{\Sigma}, \tag{2.4}$$

and, in the case $2g + |\Sigma| \geqslant 3$, the absolute stabilization morphism st discards the map f and stabilizes the remaining prestable family of curves

$$\mathrm{st}\colon \bar{M}_{g,\Sigma}(W,\beta) \to \bar{M}_{g,\Sigma}. \tag{2.5}$$

The *virtual fundamental class*, or the J-class $[\bar{M}_{g,\Sigma}(W,\beta)]^{\mathrm{virt}}$, is a canonical element in the Chow ring $A_*(\bar{M}_{g,\Sigma}(W,\beta))$:

$$J_{g,\Sigma}(W,\beta) \in A_D(\bar{M}_{g,\Sigma}(W,\beta)), \tag{2.6}$$

where D is the virtual dimension (Chow grading degree)

$$(-K_W,\beta) + |\Sigma| + (\dim W - 3)(1-g). \tag{2.7}$$

The respective *Gromov–Witten correspondence*, defined for $2g + |\Sigma| \geqslant 3$, is the proper pushforward

$$I_{g,\Sigma}(W,\beta) := (\mathrm{ev},\mathrm{st})_*(J_{g,\Sigma}(W,\beta)) \in A_D(W^{\Sigma} \times \bar{M}_{g,\Sigma}). \tag{2.8}$$

Understanding these correspondences is the content of *motivic* quantum cohomology.

Example 2.1 ($g = 0$, $\beta = 0$). In this case the universal family (2.2) is

$$\begin{tikzcd} W \times \bar{C}_{0,\Sigma} \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathrm{id}_W \times h"] & W \\ W \times \bar{M}_{0,\Sigma} & \end{tikzcd} \tag{2.9}$$

with structure sections $\mathrm{id}_W \times x_j$.

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The stabilization morphism is simply the projection

$$\mathrm{st} = \mathrm{pr}_2 \colon W \times \bar{M}_{0,\Sigma} \to \bar{M}_{0,\Sigma}. \tag{2.10}$$

The evaluation morphism is the projection followed by the diagonal embedding Δ_Σ:

$$\mathrm{ev} \colon W \times \bar{M}_{0,\Sigma} \to W \to W^\Sigma. \tag{2.11}$$

We have (see [**4**, p. 606]) that

$$J_{0,\Sigma}(W, 0) = [\bar{M}_{0,\Sigma}(W, 0)] = [W] \otimes [\bar{M}_{0,\Sigma}]. \tag{2.12}$$

The virtual dimension (2.7) is

$$|\Sigma| + \dim W - 3 = \dim(W \times \bar{M}_{0,\Sigma}).$$

Thus, finally, the Gromov–Witten correspondence is the class

$$I_{0,\Sigma}(W, 0) = [\Delta_\Sigma(W)] \otimes [\bar{M}_{0,\Sigma}] \in A_*(W^\Sigma \times \bar{M}_{0,\Sigma}). \tag{2.13}$$

2.3. Strategy

In the remaining sections of this paper, we study the Gromov–Witten correspondences *of genus 0* for $W = \bar{M}_{0,S}$ and β a *class of a boundary curve* in $\bar{M}_{0,S}$ (see below). This is the first step of a much more ambitious program in which all components of the stable family diagrams are allowed to be stacks, and in which we take for targets the stacks $\bar{M}_{g,S}$ and arbitrary β.

Our modest goal here allows us to basically restrict ourselves to the case of schemes, whose geometry is already well known. However, some intermediate constructions require the use of stacks.

In particular, we need to understand the relevant J-classes and the diagrams

$$\mathrm{ev} \colon \bar{M}_{0,\Sigma}(\bar{M}_{0,S}, \beta) \to \bar{M}_{0,S}^\Sigma, \qquad \mathrm{st} \colon \bar{M}_{0,\Sigma}(\bar{M}_{0,S}, \beta) \to \bar{M}_{0,\Sigma}. \tag{2.14}$$

We also want to be able to trace various functorialities, in particular, in *both* S and Σ. However, this may result in rather clumsy notation.

In order to postpone its introduction, in the remaining parts of this section we describe a somewhat more general situation. Afterwards, we show that our main problem is contained in it.

2.4. Setup, part I

Consider a morphism of smooth irreducible projective manifolds $b \colon E \to W$. Let β_E be an effective curve class on E, and let $\beta := b_*(\beta_E)$ be its pushforward to W. Any stable map $\mathcal{C}_T/T \to E$, $(x_j \colon T \to \mathcal{C}_T \mid j \in \Sigma)$, of class β_E, induces, after composition with b and stabilization, a stable map with target (W, β). Thus, we get a map

$$\tilde{b} \colon \bar{M}_{0,\Sigma}(E, \beta_E) \to \bar{M}_{0,\Sigma}(W, \beta)$$

that clearly fits into the commutative diagram

$$\begin{tikzcd} \bar{M}_{0,\Sigma}(E,\beta_E) \arrow[r, "\tilde{b}"] \arrow[d, "(\mathrm{ev}_E, \mathrm{st}_E)"] & \bar{M}_{0,\Sigma}(W,\beta) \arrow[d, "(\mathrm{ev}_W, \mathrm{st}_W)"] \\ E^\Sigma \times \bar{M}_{0,\Sigma} \arrow[r, "b^\Sigma \times \mathrm{id}"] & W^\Sigma \times \bar{M}_{0,\Sigma} \end{tikzcd} \tag{2.15}$$

If $|\Sigma| \leqslant 2$, the space $\bar{M}_{0,\Sigma}$ is not a DM-stack; if we discard it and the stabilization morphisms in (2.15), we still get a commutative diagram. Whenever $\bar{M}_{0,\Sigma}$ appears, we assume that $|\Sigma| \geqslant 3$.

Proposition 2.2.

(i) *Assume that*

$$J_{0,\Sigma}(W,\beta) = \tilde{b}_*(J_{0,\Sigma}(E,\beta_E)). \tag{2.16}$$

Then,

$$I_{0,\Sigma}(W,\beta) = (b^\Sigma \times \mathrm{id})_*(I_{0,\Sigma}(E,\beta_E)). \tag{2.17}$$

(ii) *Let $\gamma_j \in H^*(W)$, $j \in \Sigma$, be a family of cohomology classes marked by Σ. Then, from (2.16) it follows that*

$$\mathrm{pr}_W^*(\otimes_{j\in\Sigma}\gamma_j) \cap I_{0,\Sigma}(W,\beta) = (b^\Sigma \times \mathrm{id})_*[\mathrm{pr}_E^*(\otimes_{j\in\Sigma} b^*(\gamma_j)) \cap I_{0,\Sigma}(E,\beta_E)]. \tag{2.18}$$

Here, we denote by $\mathrm{pr}_W\colon W^\Sigma \times \bar{M}_{0,\Sigma} \to W^\Sigma$ and $\mathrm{pr}_E\colon E^\Sigma \times \bar{M}_{0,\Sigma} \to E^\Sigma$ the respective projection morphisms, and H^* can be any standard cohomology theory.

Proof. (i) This follows directly from (2.16) and the commutativity of (2.15).

(ii) We have, using the projection formula, that

$$\begin{aligned} &(b^\Sigma \times \mathrm{id})_*[\mathrm{pr}_E^*(\otimes_{j\in\Sigma} b^*(\gamma_j)) \cap I_{0,\Sigma}(E,\beta_E)] \\ &\qquad = (b^\Sigma \times \mathrm{id})_*[(b^\Sigma \times \mathrm{id})^* \circ \mathrm{pr}_W^*(\otimes_{j\in\Sigma}\gamma_j) \cap I_{0,\Sigma}(E,\beta_E)] \\ &\qquad = \mathrm{pr}_W^*(\otimes_{j\in\Sigma}\gamma_j) \cap (b^\Sigma \times \mathrm{id})_*(I_{0,\Sigma}(E,\beta_E)). \end{aligned}$$

The last expression coincides with the left-hand side of (2.18) in view of (2.17). This completes the proof. □

Remark 2.3. In our applications to the case $W = \bar{M}_{0,S}$ (see § 5), E is a boundary stratum containing the boundary curve representing β, and the virtual fundamental classes $J_{0,\Sigma}$ coincide with the usual fundamental classes, since the relevant deformation problem is unobstructed. Moreover, E has a very special additional structure. We axiomatize the relevant geometry in the next subsections.

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2.5. Setup, part II

Keeping the notation of § 2.4, we make the following additional assumptions.

(a) E is explicitly represented as $E = B \times C$, where C is isomorphic to $\boldsymbol{P}^1$. This identification, including the projections $p = \mathrm{pr}_B \colon E \to B$ and $\mathrm{pr}_C \colon E \to C$, constitutes a part of the structure.

(b) β_E is the (numerical) class of any fibre of p.

(c) The deformation problem for any fibre C_0 of p embedded via b_0 in W is *trivially unobstructed* in the sense of [**6**]:

$$H^1(C_0, b_0^*(\mathcal{T}_W)) = 0. \tag{2.19}$$

(d) The map $\tilde{b}$ in (2.15) is an isomorphism.

These assumptions are quite strong. In particular, from (b)–(d) it follows that (2.16) holds, since the relevant virtual fundamental classes coincide with the ordinary ones. Thus, we can complete the explicit computation of $I_{0,\Sigma}(W, \beta)$ starting with the right-hand side of (2.17). We do so in the remaining part of the section.

First of all, we have that

$$\mathrm{pr}_{B*}(\beta_E) = 0, \qquad \mathrm{pr}_{C*}(\beta_E) = \mathbf{1},$$

where $\mathbf{1}$ is the fundamental class $[C]$ in the Chow ring of C.

Thus, the two projections induce the map

$$(\tilde{\mathrm{pr}}_B, \tilde{\mathrm{pr}}_C) \colon \bar{M}_{0,\Sigma}(E, \beta_E) \to \bar{M}_{0,\Sigma}(B, 0) \times \bar{M}_{0,\Sigma}(C, \mathbf{1}).$$

Stabilization maps embed this morphism into the commutative diagram

$$\begin{tikzcd} \bar{M}_{0,\Sigma}(E, \beta_E) \arrow[r] \arrow[d, "\mathrm{st}_E"'] & \bar{M}_{0,\Sigma}(B, 0) \times \bar{M}_{0,\Sigma}(C, \mathbf{1}) \arrow[d, "\mathrm{st}_B \times \mathrm{st}_C"'] \\ \bar{M}_{0,\Sigma} \arrow[r, "\Delta_{\bar{M}_{0,\Sigma}}"] & \bar{M}_{0,\Sigma} \times \bar{M}_{0,\Sigma} \end{tikzcd} \tag{2.20}$$

where the lower line is the diagonal embedding (see [**5**, Proposition 5]).

Similarly, evaluation maps embed this morphism into the commutative diagram

$$\begin{tikzcd} \bar{M}_{0,\Sigma}(E, \beta_E) \arrow[r] \arrow[d, "\mathrm{ev}_E"'] & \bar{M}_{0,\Sigma}(B, 0) \times \bar{M}_{0,\Sigma}(C, \mathbf{1}) \arrow[d, "\mathrm{ev}_B \times \mathrm{ev}_C"'] \\ E^{\Sigma} \arrow[r, "s"] & B^{\Sigma} \times C^{\Sigma} \end{tikzcd} \tag{2.21}$$

where the lower line is now the evident permutation isomorphism induced by $E = B \times C$.

Combining these two diagrams, we get

$$\begin{tikzcd}
\bar{M}_{0,\Sigma}(E,\beta_E) \arrow[r] \arrow[d, "(\mathrm{ev}_E,\mathrm{st}_E)"'] & \bar{M}_{0,\Sigma}(B,0)\times\bar{M}_{0,\Sigma}(C,\mathbf{1}) \arrow[d, "(\mathrm{ev}_B,\mathrm{st}_B)\times(\mathrm{ev}_C,\mathrm{st}_C)"] \\
E^{\Sigma}\times\bar{M}_{0,\Sigma} \arrow[r, "\tilde{\Delta}"] & B^{\Sigma}\times\bar{M}_{0,\Sigma}\times C^{\Sigma}\times\bar{M}_{0,\Sigma}
\end{tikzcd} \tag{2.22}$$

Here the lower line is an obvious composition of permutations and the diagonal embedding of $\bar{M}_{0,\Sigma}$.

From (2.22) and [**5**] it follows that

$$I_{0,\Sigma}(E,\beta_E) = \tilde{\Delta}^!(I_{0,\Sigma}(B,0)\otimes I_{0,\Sigma}(C,\mathbf{1})). \tag{2.23}$$

(Note that for $x\in A_*(X)$, $y\in A_*(Y)$ we often denote simply by $x\otimes y\in A_*(X\times Y)$ the image of $x\otimes y\in A_*(X)\otimes A_*(Y)$ with respect to the canonical map $A_*(X)\otimes A_*(Y)\to A_*(X\times Y)$.)

Furthermore, according to (2.13),

$$I_{0,\Sigma}(B,0) = [\Delta_\Sigma(B)\times\bar{M}_{0,\Sigma}]\in A_*(B^{\Sigma}\times\bar{M}_{0,\Sigma}). \tag{2.24}$$

Finally, the space $\bar{M}_{0,\Sigma}(C,\mathbf{1})$ and the class $I_{0,\Sigma}(C,\mathbf{1})$ can be described as follows.

Recall a construction from [**12**]. Let V be a smooth complete algebraic manifold. For a finite set Σ, let V^{Σ} be the direct product of a family of Vs labelled by elements of Σ. Denote by $\tilde{V}^{\Sigma}$ the blow-up of the (small) diagonal in V^{Σ}. Finally, define $V^{\Sigma,0}$ as the complement to all partial diagonals in V^{Σ}.

The Fulton–MacPherson *configuration space* $V\langle\Sigma\rangle$ (for curves it was introduced earlier by Beilinson and Ginzburg) is the closure of $V^{\Sigma,0}$ naturally embedded in the product

$$V^{\Sigma}\times\prod_{\Sigma'\subset\Sigma,\,|\Sigma'|\geqslant 2}\tilde{V}^{\Sigma'}.$$

In [**13**], it was shown that $\bar{M}_{0,\Sigma}(C,\mathbf{1})$ can be identified with $C\langle\Sigma\rangle$ in such a way that the birational morphism ev_C becomes the tautological open embedding when restricted to $C^{\Sigma,0}$.

Therefore, denoting by $D_\Sigma\subset C^{\Sigma}\times\bar{M}_{0,\Sigma}$ the closure of the graph of the canonical surjective map $C^{\Sigma,0}\to M_{0,\Sigma}$, we get that

$$I_{0,\Sigma}(C,\mathbf{1}) = [D_\Sigma]. \tag{2.25}$$

We can now state the main result of this section.

Proposition 2.4. *Assuming that (a)–(d) in § 2.5 hold, we have that*

$$I_{0,\Sigma}(E,\beta_E) = \tilde{\Delta}^!([\Delta_\Sigma(B)\times\bar{M}_{0,\Sigma}\times D_\Sigma]). \tag{2.26}$$

and

$$I_{0,\Sigma}(W,\beta) = (b^{\Sigma}\times\mathrm{id})_*\circ\tilde{\Delta}^!([\Delta_\Sigma(B)\times\bar{M}_{0,\Sigma}\times D_\Sigma]). \tag{2.27}$$

This is a straightforward combination of (2.23)–(2.25) and (2.17).

3. Target space $\bar{M}_{0,4}$

3.1. Notation

Stressing functoriality with respect to labelling sets, and having in mind further developments, in this section we denote by S a set *of cardinality* 4, with a marked point $\bullet$. We set $S = P \sqcup \{\bullet\}$. Thus, we are considering the moduli space $\bar{M}_{0,P\sqcup\{\bullet\}}$. It is a projective line endowed with three pairwise distinct points D_σ labelled by unordered partitions $\sigma\colon P \sqcup \{\bullet\} = S_1 \sqcup S_2$, $|S_i| = 2$. These are exactly those points over which the universal stable curve $\bar{C}_{0,P\sqcup\{\bullet\}}$ splits into two components, and labelled points are redistributed among them according to σ. Now, the set of such partitions is naturally bijective to P: $j \in P$ corresponds to the partition $\{\bullet, j\} \sqcup (P \setminus \{\bullet, j\})$. Hence, finally, $\bar{M}_{0,P\sqcup\{\bullet\}}$ is a projective line $\boldsymbol{P}^1$ stabilized by three points labelled by P. This identification is functorial with respect to pointed bijections of S.

The only boundary class of curves in $A_1(\bar{M}_{0,P\sqcup\{\bullet\}})$ is the fundamental class $\beta = \mathbf{1} := [\bar{M}_{0,P\sqcup\{\bullet\}}]$. We have already invoked the description of the universal family of stable maps with this target and the relevant I-class at the end of § 2.5 (see (2.25)). But now, for the sake of a future generalization, we use a slightly different family and an alternative description of $I_{0,\Sigma} \in A_*((\boldsymbol{P}^1)^\Sigma \times \bar{M}_{0,\Sigma})$ that will better fit the passage to target spaces $\bar{M}_{0,S}$ with $|S| > 4$.

3.2. An alternative family

Consider the moduli space $\bar{M}_{0,\Sigma\sqcup P\sqcup\{\bullet\}}$.

Recall that, for any finite set R and its subset $Q \subset R$ with complement of cardinality greater than or equal to 3, the space $\bar{M}_{0,R}$ is the source of the standard *forgetful morphism* $\psi_Q\colon \bar{M}_{0,R} \to \bar{M}_{0,R\setminus Q}$: 'forget the subset of sections labelled by Q and stabilize'.

Thus, we get the diagram

$$\begin{tikzcd} \bar{M}_{0,\Sigma\sqcup P\sqcup\{\bullet\}} \arrow[r, "\psi_\Sigma"] \arrow[d, "\psi_{\{\bullet\}}"] & \bar{M}_{0,P\sqcup\{\bullet\}} \\ \bar{M}_{0,\Sigma\sqcup P} & \end{tikzcd} \tag{3.1}$$

Another standard morphism identifies the vertical arrow in (3.1) with the projection of the universal $(\Sigma\sqcup P)$-labelled curve $\bar{C}_{0,\Sigma\sqcup P}$ to its base (see, for example, [**28**, Chapter V, Theorem 4.5]).

From the explicit form of this identification, one can easily see that the image in $\bar{M}_{0,\Sigma\sqcup P\sqcup\{\bullet\}}$ of the section $x_j\colon \bar{M}_{0,\Sigma\sqcup P} \to \bar{C}_{0,\Sigma\sqcup P}$ for $j \in \Sigma \sqcup P$ is precisely the boundary divisor of $\bar{M}_{0,\Sigma\sqcup P\sqcup\{\bullet\}}$ corresponding to the stable 2-partition

$$\Sigma \sqcup P \sqcup \{\bullet\} = \{\bullet, j\} \sqcup ((\Sigma \sqcup P) \setminus \{j\}). \tag{3.2}$$

Here, we denote this divisor by D_j.

Consider (3.1) now as the family of maps of class $\mathbf{1}$, in which *only the sections x_j for $j \in \Sigma$ are counted as structure sections*, whereas those labelled by P are discarded.

The family will then no longer be stable: if an irreducible component of the fibre curve contains only three special points and one of them corresponds to the section labelled by an element of P, then this component will be contracted by ψ_Σ. We can stabilize this new family and get a diagram

$$\begin{tikzcd} \bar{C} \arrow[r, "\bar{\psi}_\Sigma"] \arrow[d] & \bar{M}_{0,P\sqcup\{\bullet\}} \\ \bar{M}_{0,\Sigma\sqcup P} & \end{tikzcd} \tag{3.3}$$

endowed additionally with sections labelled by Σ and the stabilizing morphism

$$\chi\colon \bar{M}_{0,\Sigma\sqcup P\sqcup\{\bullet\}} \to \bar{C}, \qquad \psi_\Sigma = \bar{\psi}_\Sigma \circ \chi. \tag{3.4}$$

For each $j \in \Sigma$, denote by $\xi_j\colon \bar{M}_{0,\Sigma\sqcup P} \to \bar{M}_{0,\Sigma\sqcup P\sqcup\{\bullet\}}$ the section of $\psi_{\{\bullet\}}$ identifying $\bar{M}_{0,\Sigma\sqcup P}$ with $D_j \subset \bar{M}_{0,\Sigma\sqcup P\sqcup\{\bullet\}}$ from (3.2). Consider the map

$$\bar{\mathrm{ev}} := (\psi_\Sigma \circ \xi_j \mid j \in \Sigma)\colon \bar{M}_{0,\Sigma\sqcup P} \to (\bar{M}_{0,P\sqcup\{\bullet\}})^\Sigma. \tag{3.5}$$

The stable family (3.3) may be obtained by a base change from the universal family of stable maps of class β. Let

$$\mu\colon \bar{M}_{0,\Sigma\sqcup P} \to \bar{M}_{0,P\sqcup\{\bullet\}}\langle\Sigma\rangle \tag{3.6}$$

be the respective morphism of bases.

Dimensions of the two smooth irreducible schemes in (3.6) coincide. It is not difficult to see that the morphism μ is birational and, hence, surjective. In fact, consider a generic fibre of $\bar{C}_{0,\Sigma\sqcup P}$. It is simply $\boldsymbol{P}^1$ with pairwise distinct $(\Sigma\sqcup P(\Pi))$-labelled points. When we discard Σ-labelled points, we land in $\boldsymbol{P}^1$ endowed with three points labelled by $P(\Pi)$ (inverse images of them are just missing sections that we discarded when constructing (3.3) from (3.1)), so in fact at a generic point we neither lose nor gain any information passing from (3.1) to (3.3).

We can now prove the main result of this section.

Proposition 3.1. *We have, for $|\Sigma| \geqslant 3$, that*

$$I_{0,\Sigma}(\bar{M}_{0,P\sqcup\{\bullet\}}, \mathbf{1}) = (\bar{\mathrm{ev}}, \psi_P)_*([\bar{M}_{0,\Sigma\sqcup P}]) \in A_{|\Sigma|}((\bar{M}_{0,P\sqcup\{\bullet\}})^\Sigma \times \bar{M}_{0,\Sigma}). \tag{3.7}$$

Proof. Since μ is birational and surjective, we can identify the relevant J-class with

$$\mu_*([\bar{M}_{0,\Sigma\sqcup P}]) = [\bar{M}_{0,P\sqcup\{\bullet\}}\langle\Sigma\rangle].$$

In order to prove (3.7), it remains to check that

$$\mathrm{ev}\circ\mu = \bar{\mathrm{ev}}, \qquad \mathrm{st}\circ\mu = \psi_P. \tag{3.8}$$

Both facts follow from the discussion in § 3.2. $\square$

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4. Boundary curve classes in $\bar{M}_{0,S}$

In this section, after recalling some basic facts about the boundary of $\bar{M}_{0,S}$ following [**28**] and [**7**], we summarize relevant parts of [**23**] and fix our notation.

4.1. Boundary strata of $\bar{M}_{0,S}$

The main combinatorial invariant of an S-pointed stable curve C is its *dual graph* $\tau = \tau_C$. Its set of vertices V_τ is (bijective to) the set of irreducible components of C. Each vertex v is a boundary point of the set of *flags* $f \in F_\tau(v)$ that is (bijective to) the set consisting of singular points and S-labelled points on this component. We set $F_\tau = \bigcup_{v \in V_\tau} F_\tau(v)$. If two components of C intersect, the respective two vertices carry two flags that are grafted to form an *edge* e connecting the respective vertices; the set of edges is denoted by E_τ. The flags that are not pairwise grafted are called *tails*. They form a set T_τ that is naturally bijective to the set of S-labelled points and, therefore, is itself labelled by S. Stable curves of genus 0 correspond to trees τ whose every vertex carries at least three flags.

The space $\bar{M}_{0,S}$ is a disjoint union of locally closed strata M_τ indexed by stable S-labelled trees. Each such stratum M_τ represents the functor of families consisting of curves of combinatorial type τ. In particular, the open stratum $M_{0,S}$ classifies irreducible smooth curves with pairwise distinct S-labelled points. Its graph is a star: a tree with one vertex, to which all tails are attached, and having no edges.

Generally, a stratum M_τ lies in the closure $\bar{M}_\sigma$ of M_σ, if and only if σ can be obtained from τ by contracting a subset of edges. Closed strata $\bar{M}_\sigma$ corresponding to trees with non-empty sets of edges, are called *boundary* strata. The number of edges is the codimension of the stratum.

4.1.1. *Boundary divisors and $A^1(\bar{M}_{0,S})$*

The classes of boundary divisors generate the whole Chow ring, but are not linearly independent. The following useful basis is constructed in [**11**].

For $s \in S$, let $\mathcal{L}_s$ be the line bundle on $\bar{M}_{0,S}$ whose fibre over a stable curve $(C, (x_t \mid t \in S))$ is $T^*_{x_s}C$. Set $\psi_s := c_1(\mathcal{L}_s)$.

Proposition 4.1. *The classes of boundary divisors D_S with $|S_1|, |S_2| \geqslant 3$ and classes ψ_s, $s \in S$, constitute a basis of the group $A^1(\bar{M}_{0,S})$.*

The rank of this group is $2^{n-1} - n(n-1)/2 - 1$.

This is [**11**, Lemma 2]. An expression of ψ_s through boundary classes is given in [**11**, Lemma 1].

Below, we give some details on one-dimensional strata.

4.2. Boundary curves and $A_1(\bar{M}_{0,S})$: preparatory combinatorics

We start with the following combinatorial construction.

We use here the term *an unordered partition* of a set S as synonymous to *an equivalence relation* on S. A component of a partition is the same as an equivalence class of the respective relation; in particular, all components are non-empty.

Call an unordered partition Π of S *distinguished* if it consists of precisely *four components*. Denote by $S(\Pi)$ the set of the components, that is, the quotient of S with respect to the respective equivalence relation.

Distinguished partitions are in a natural bijection with isomorphism classes of *distinguished stable S-labelled trees* π. By definition, such a tree is endowed with one *distinguished vertex* v_0, the set of flags at this vertex $F_\pi(v_0)$ being (labelled by) elements of $S(\Pi)$. Clearly, this vertex is of multiplicity 4. The flags labelled by one-element components $\{s\}$ of Π are tails, carrying the respective labels $s \in S$. The remaining flags are halves of edges; the second vertex of an edge, one half of which is labelled by a component S_i, carries tails labelled by elements of S_i.

We routinely identify $F_\pi(v_0)$ with $S(\Pi)$.

Definition 4.2.

(i) Given a distinguished partition Π, denote by $P = P(\Pi)$ the set of those stable 2-partitions σ of S, each component of which is a union of two different components of Π. For $|S| \geqslant 4$ we have that $|P(\Pi)| = 3$.

(ii) $N = N(\Pi)$ is the set of those stable 2-partitions of S whose one component coincides with one component of Π. We have, for $|S| \geqslant 5$, that $1 \leqslant |N(\Pi)| \leqslant 4$.

Lemma 4.3. *Π can be uniquely reconstructed from $P(\Pi)$; hence, $P(\Pi)$ uniquely determines $N(\Pi)$ as well.*

Proof. In fact, if $\Pi = (S_1, S_2, S_3, S_4)$ (numeration arbitrary), then by definition $P(\Pi)$ must consist of the partitions

$$\sigma_1 = (S_1 \cup S_2, S_3 \cup S_4), \qquad \sigma_2 = (S_1 \cup S_3, S_2 \cup S_4), \qquad \sigma_3 = (S_1 \cup S_4, S_2 \cup S_3).$$

Hence, conversely, knowing $P(\Pi)$, we can unambiguously reconstruct Π: its components are exactly the non-empty pairwise intersections of components of different $\sigma_i \in P(\Pi)$. □

4.3. Boundary curves and $A_1(\bar{M}_{0,S})$: geometry

Each distinguished partition Π of S determines the following boundary stratum of $\bar{M}_{0,S}$:

$$b_\Pi \colon \bar{M}_\Pi := \bigcap_{\sigma \in N(\Pi)} D_\sigma \hookrightarrow \bar{M}_{0,S}. \tag{4.1}$$

Equivalently, $\bar{M}_\Pi$ is the stratum, corresponding to the special tree π associated with Π. In other words, all components of Π are now indexed by the flags $f \in F_\pi(v_0)$ at the special vertex v_0, whereas components of cardinality $\geqslant 2$ are also naturally indexed by the remaining vertices of π:

$$\bar{M}_\Pi = \bar{M}_{0,F_\pi(v_0)} \times \prod_{v \neq v_0} \bar{M}_{0,F_\pi(v)}. \tag{4.2}$$

Here the equality sign refers to the canonical isomorphism that is defined for any stable marked tree: it produces from such a tree the product of moduli spaces corresponding to the stars of all vertices.

The information about edges determines the embedding morphism (4.1) of such a product as a boundary stratum. On the level of universal curves, it is defined by merging the pairs of sections labelled by halves of an edge.

The codimension of $\bar{M}_\Pi$ is $|N(P)|$, and $1 \leqslant |N(\Pi)| \leqslant 4$. Since $|F_\pi(v_0)| = 4$, the moduli space $\bar{M}_{0,F_\pi(v_0)}$ is $\boldsymbol{P}^1$ with three points naturally labelled by the set of stable partitions of $F_\pi(v_0)$, which in turn is canonically bijective to $P(\Pi)$ (cf. § 3.1).

Hence, the representation (4.2) allows us to define the projection map

$$p = p_\Pi \colon \bar{M}_\Pi \to B_\Pi := \prod_{v \neq v_0} \bar{M}_{0,F_\pi(v)} \tag{4.3}$$

having three canonical disjoint sections canonically labelled by $P(\Pi)$.

Clearly, all fibres of p_Π are rationally equivalent, so they define a class $\beta = \beta(\Pi) \in A_1(\bar{M}_{0,S})$.

Lemma 4.4 (Keel and McKernan [23]).

(i) *For $n := |S| \geqslant 4$, each boundary curve (one-dimensional boundary stratum) C_τ is a fibre of one of the projections (4.3).*

(ii) *$[C_{\tau_1}] = [C_{\tau_2}] \in A_1(\bar{M}_{0,S})$ if and only if these curves are fibres of one and the same projection (4.3).*

We reproduce the proof for further use.

Proof. (i) Since C_τ is a curve, the S-labelled stable tree τ is a tree with $|E_\tau| = n - 4$ and, hence, $|V_\tau| = n - 3$. Since the tree is stable, all but one of its vertices must have multiplicity 3. The exceptional vertex, denoted $v_0 = v_0(\tau)$, has multiplicity 4.

If we delete from the geometric tree τ the vertex v_0, it will break into four connected components. Thus, the set S of labels of tails will be broken into four non-empty subsets. Among them there are $|T_\tau(v_0)|$ one-element sets (labels of tails adjacent to v_0), and $|E_\tau(v_0)|$ sets of cardinality greater than or equal to 2: each part consists of labels of those tails that can be reached from the critical vertex by a path (without backtracks) starting with the respective flag. We denote this partition $\Pi(\tau)$. Hence, if we contract all edges of τ excepting those that are attached to v_0, we get the distinguished tree associated with a distinguished partition $\Pi = \Pi(\tau)$. It determines the required projection.

(ii) Now consider two sets of stable 2-partitions of S produced from $\Pi = \Pi(\tau)$ as in Definition 4.2, and denote them, respectively, by $P(\tau)$ and $N(\tau)$.

First of all, we check that

$$\left.\begin{aligned} (D_\sigma, C_\tau) &= 1 && \text{if } \sigma \in P(\tau), \\ (D_\sigma, C_\tau) &= -1 && \text{if } \sigma \in N(\tau), \\ (D_\sigma, C_\tau) &= 0 && \text{otherwise.} \end{aligned}\right\} \tag{4.4}$$

We now use formulae and facts proved in [**28**, III.3] and [**25**, Appendix]. In particular, we use the notion of *good monomials*, elements of the commutative polynomial ring freely generated by symbols $m(\sigma)$, where σ runs over stable 2-partitions of S. These monomials form a family indexed by stable S-labelled trees τ: $m(\tau) := \prod_{e\in E_\tau} m(\sigma_e)$, where σ_e is the 2-partition of S obtained by cutting e.

Assume first that $m(\sigma)m(\tau)$ is a good monomial, so $(D_\sigma, C_\tau) = 1$. It is then of the form $m(\rho)$, where ρ is a stable S-labelled tree with all vertices of multiplicity 3 and an edge e such that $m(\sigma) = m(\rho_e)$. This edge is unambiguously characterized by the fact that, after collapsing e in ρ to one vertex, we get the labelled tree (canonically isomorphic to) τ. But the vertex to which e collapses must then have multiplicity larger than 3. It follows that e must collapse precisely to the exceptional vertex v_0 of τ. Conversely, the set of ways of putting e back is clearly in a bijection with $P(\tau)$: the four flags adjacent to v_0 must be distributed in two groups, two flags in each, that will be adjacent to two ends of e.

Assume now that $m(\sigma)$ divides $m(\tau)$. Using [**25**, Proposition 1.7.1], one sees that $m(\sigma)m(\tau)$ represents 0 in the Chow ring (and so $(D_\sigma, C_\tau) = 0$) unless $\sigma = \tau_e$, where e is an edge adjacent to v_0. In this latter case, Kaufmann's formula [**25**, (1.9)] implies that $(D_\sigma, C_\tau) = -1$. The set of such σ is in a bijection with $N(\tau)$.

Finally, for any other stable 2-partition σ there exists an $e \in E_\tau$ such that we have $a(\sigma, \tau_e) = 4$ in the sense of [**28**, III.3.4.1]. In this case, $(D_\sigma, C_\tau) = 0$ in view of [**28**, III.3.4.2].

We now have that $[C_{\tau_1}] = [C_{\tau_2}]$ if and only if $(D_\sigma, C_{\tau_1}) = (D_\sigma, C_{\tau_2})$ for all stable 2-partitions σ, because boundary divisors generate A^1. In view of (4.4), this latter condition means precisely that

$$P(\tau_1) = P(\tau_2), \qquad N(\tau_1) = N(\tau_2).$$

But Lemma 4.3 shows that in this case $\Pi(\tau_1) = \Pi(\tau_2)$. This completes the proof. □

Proposition 4.5. *Denote the canonical class of $\bar{M}_{0,S}$ by K_S. Using the notation of § 4.3, we have that*

$$(-K_S, \beta(\Pi)) = 2 - |N(\Pi)|. \tag{4.5}$$

Proof. For $2 \leqslant j \leqslant [n/2]$, denote by B_j the sum of all divisors D_σ such that one part of the partition σ is of cardinality j, and denote by B the sum of all boundary divisors. We have that

$$-K_S = 2B - \sum_{j=2}^{[n/2]} \frac{j(n-j)}{n-1} B_j \tag{4.6}$$

(see [**11**, **23**] and the references therein).

For a stable 2-partition $\sigma = (S_1, S_2)$ of S, set $c(\sigma) := |S_1||S_2|$. Then, combining (4.4) and (4.6), we get that

$$(-K_S, \beta(\Pi)) = 2(|P(\tau)| - |N(\tau)|) - \sum_{\sigma\in P(\tau)} \frac{c(\sigma)}{n-1} + \sum_{\sigma\in N(\tau)} \frac{c(\sigma)}{n-1}. \tag{4.7}$$

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The most straightforward way to pass from (4.7) to (4.5) is to consider the four cases $|N(\Pi)| = 1, 2, 3, 4$ separately. We include here the calculation for $|N(\Pi)| = 3$; it demonstrates the typical cancellation pattern. We leave the remaining cases to the reader.

We have that $2(|P(\Pi)| - |N(\Pi)|) = 0$. Let $(1, a, b, c)$ be the cardinalities of the components of Π, where $a, b, c \geqslant 2$, $a + b + c = n - 1$. Then, $P(\Pi)$ consists of three partitions, of the following cardinalities, respectively:

$$(a+1, b+c), \qquad (b+1, a+c), \qquad (c+1, a+b).$$

Hence,

$$\sum_{\sigma \in P(\Pi)} c(\sigma) = 2(ab + ac + bc) + 2(a + b + c).$$

Similarly, partitions in $N(\Pi)$ produce the list

$$(a, 1+b+c), \qquad (b, 1+a+c), \qquad (c, 1+a+b),$$

so

$$\sum_{\sigma \in N(\Pi)} c(\sigma) = 2(ab + ac + bc) + (a + b + c).$$

Combining all these together, we get that $(-K_S, \beta(\Pi)) = -1 = 2 - |N(\Pi)|$. $\square$

Proposition 4.6. *Each class of a boundary curve β is indecomposable in the cone of effective curves.*

Proof. This follows from (4.5) and [**23**, Lemma 3.6]: $(K_S + B, \beta(\Pi)) = 1$, and the divisor $K_S + B$ is ample. $\square$

Examples 4.7 ($\bar{M}_{0,4}$ and $\bar{M}_{0,5}$). If $|S| = 4$, there exists one distinguished partition Π, with all components of cardinality 1. The respective 'boundary' curve is in fact the total space $\bar{M}_{0,S}$.

If $|S| = 5$, the boundary curves are ten exceptional curves on the del Pezzo surface $\bar{M}_{0,S}$ corresponding to ten different distinguished partitions of S whose components have cardinalities $(1, 1, 1, 2)$. They define ten different Chow classes.

Example 4.8 ($\bar{M}_{0,6}$). There exist two combinatorial types of unlabelled trees τ corresponding to boundary curves. Below, we draw their subgraphs, consisting of all vertices and edges, and mark them with the numbers of tails at each vertex:

$$3 \bullet - \bullet\, 1 - \bullet 2, \qquad 2 \bullet - \bullet\, 2 - \bullet 2.$$

If we take into account possible labellings by S, we get 60 boundary curves of the first type and 45 boundary curves of the second type. They form two different S_6-orbits.

If τ is of the first type, then $c(\sigma) = 8$ for all three partitions $\sigma \in P(\tau)$. The set $N(\tau)$ contains the unique partition σ, with $c(\sigma) = 9$. Applying Proposition 4.5, we get that

$$(-K_6, C_\tau) = 1.$$

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If τ is of the second type, we have, respectively, $c(\sigma) = 8, 9, 9$ for $\sigma \in P(\tau)$. The set $N(\tau)$ consists of two partitions σ, with $c(\sigma) = 8$. Applying Proposition 4.5, we get that

$$(-K_6, C_\tau) = 0.$$

Chow classes of the boundary curves for $n = 6$ are extremal rays of the Mori cone. There are 20 classes of the first type and 45 classes of the second type.

Example 4.9 ($\bar{M}_{0,7}$). Similarly, there are four combinatorial types of unlabelled trees τ corresponding to boundary curves:

$$\mathrm{A}: \quad 3\bullet - \bullet 1 - \bullet 1 - \bullet 2, \qquad \mathrm{B}: \quad 2\bullet - \bullet 2 - \bullet 1 - \bullet 2$$

and

$$\mathrm{C}: \quad 3\bullet - \bullet\!\!\begin{array}{l}\diagup\bullet 2\\ \diagdown\bullet 2\end{array}, \qquad \mathrm{D}: \quad 2\bullet - 1\bullet\!\!\begin{array}{l}\diagup\bullet 2\\ \diagdown\bullet 2\end{array}.$$

Here the numerology looks as follows.

Type A. We have that $c(\sigma) = 10$ for all $\sigma \in P(\tau)$; $|N(\tau)| = 1$, $c(\sigma) = 12$ for $\sigma \in N(\tau)$. Hence,

$$(-K_7, C_\tau) = 1.$$

Finally, there are 420 labelled trees/boundary curves of this type.

Type B. We have that $c(\sigma) = 10, 12, 12$ for $\sigma \in P(\tau)$; $|N(\tau)| = 2$, $c(\sigma) = 10, 12$ for $\sigma \in N(\tau)$. Hence,

$$(-K_7, C_\tau) = 0.$$

There are 630 boundary curves of this type.

Type C. We have that $c(\sigma) = 10$ for all $\sigma \in P(\tau)$; $|N(\tau)| = 1$, $c(\sigma) = 12$ for $\sigma \in N(\tau)$. Hence,

$$(-K_7, C_\tau) = 1.$$

There are 105 boundary curves of this type.

Type D. Finally, here $c(\sigma) = 12$ for all $\sigma \in P(\tau)$; $|N(\tau)| = 3$, $c(\sigma) = 10$ for $\sigma \in N(\tau)$, and

$$(-K_7, C_\tau) = -1.$$

There are 105 boundary curves of this type.

In the Chow group, there are 35 classes of types A and C altogether, 210 classes of type B, and 105 classes of type D.

5. Gromov–Witten correspondences for boundary curves in $\bar{M}_{0,S}$

In this section we state and prove the main theorem of this paper. We start with some preparation.

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5.1. Preparation: combinatorics

In this section, we choose and fix two disjoint finite sets S and Σ. Assume that $|S| \geqslant 4$ and $|\Sigma| \geqslant 3$.

Fix one element $s_0 \in S$. Choose and fix a distinguished partition Π of S into four disjoint non-empty subsets (see § 4.2). Denote by $S(\Pi)$ the set containing elements that are components of Π. Thus, $|S(\Pi)| = 4$. Denote by $\bullet \in S(\Pi)$ the component of Π that contains the marked element $s_0 \in S$.

The sets $P(\Pi)$ and $N(\Pi)$ are defined as in Definition 4.2. In our setup, the three-element set $P(\Pi)$ is canonically bijective to the following two additional sets.

(a) The set of stable unordered partitions of $S(\Pi)$ into two parts (each consisting of two elements).

(b) The set $S(\Pi) \setminus \{\bullet\}$: any $j \in S(\Pi) \setminus \{\bullet\}$ corresponds to the partition $S(\Pi) = (\{\bullet, j\} \sqcup S(\Pi) \setminus \{\bullet, j\})$. We have already used this trick in § 3.1, and here we use it again to translate the results of § 3 to a new context.

Slightly abusing the notation, we sometimes consider these last identifications as identical maps.

Being more fussy, we can say that our constructions are functorial on the category of pointed finite sets S with bijections. Eventually, they must be extended to the category of marked trees (and more general modular graphs) encoding boundary combinatorial types of curves and maps. Dependence of our geometric construction on the target boundary curve class β is reflected in the dependence of its combinatorial side on Π.

5.2. Preparation: geometry

We intend to show that the results of §§ 2.4 and 2.5 are applicable in the present situation.

More precisely, specialize the objects introduced in § 2.4 in the following way (see (4.1)):

$$W := \bar{M}_{0,S}, \qquad E := \bar{M}_{\Pi}, \qquad b := b_{\Pi}, \qquad \beta := \beta(\Pi). \tag{5.1}$$

Furthermore, specialize the objects described in § 2.5 as follows (see (4.2), (4.3)):

$$B := B_{\Pi}, \qquad C := \bar{M}_{0,F_{\pi}(v_0)}, \qquad p := p_{\Pi}. \tag{5.2}$$

Proposition 5.1. *The assumptions (a)–(d) of § 2.5 hold for (5.1)–(5.2).*

5.3. Proof of Proposition 5.1

Assumptions (a) and (b) of § 2.5 hold by definition.

5.3.1. *Assumption (c)*

Let C be a closed fibre of $p\colon \bar{M}_{\Pi} \to B_{\Pi}$. We have already used the fact that it is isomorphic to $\boldsymbol{P}^1$. Let $j\colon C \to \bar{M}_{0,S}$ be the natural closed embedding. We assert that

$$j^*(\mathcal{T}_{\bar{M}_{0,S}}) \cong \mathcal{O}(2) \oplus \mathcal{O}^{n-4-|N(\Pi)|} \oplus \mathcal{O}(-1)^{|N(\Pi)|}, \tag{5.3}$$

where $\mathcal{T}_{\bar{M}_{0,S}}$ is the tangent sheaf and $\mathcal{O} := \mathcal{O}_C$.

In fact, consider the embedding $i\colon C \to \bar{M}_\Pi$ and the natural filtration

$$\{0\} \subset \mathcal{T}_C \subset i^*(\mathcal{T}_{\bar{M}_\Pi}) \subset j^*(\mathcal{T}_{\bar{M}_{0,S}}). \tag{5.4}$$

The consecutive summands in (5.3) correspond to the consecutive quotients of (5.4). Namely, $\mathcal{T}_C \cong \mathcal{O}(2)$; $i^*(\mathcal{T}_{\bar{M}_\Pi})/\mathcal{T}_C$ is trivial of rank

$$\dim B_\Pi = \sum_{v \in V_\pi} (|F_\pi(v)| - 2) = |S| - 4 - |N(\Pi)|.$$

Finally, the last isomorphism follows from (4.1) and (4.4).

From (5.3) we see that $H^1(C, j^*(\mathcal{T}_{\bar{M}_{0,S}})) = 0$.

5.3.2. *Assumption (d): preparation I*

Any curve X in $\bar{M}_{0,S}$ of class $\beta(\Pi)$ is a closed fibre of $p_\Pi\colon \bar{M}_\Pi \to B_\Pi$. In fact, by (4.1) and (4.4) the curve X is contained in $\bar{M}_\Pi$. Below, we show that it is indeed a fibre of p_Π by analysing degeneration patterns of fibres of the universal family $\bar{C}_{0,S}$ over points of X.

Let σ be the dual graph of the curve from the universal family $\bar{C}_{0,S}$ over a generic point of X. We know that σ admits a contraction onto π. If $p_\Pi(X)$ is not a point, then X must contain a point over which the dual graph σ' of the universal family is not isomorphic to σ. In this case it must admit a non-trivial contraction $\sigma' \to \sigma$. Compose it with the canonical contraction $\sigma \to \pi$.

One of the following two alternatives must hold.

(A) There is an edge of σ' that contracts onto one of the vertices $v \neq v_0$ of π.

(B) No edge of σ' contracts onto one of the vertices $v \neq v_0$, but there is an edge contracting to v_0.

Consider the stable 2-partition ρ of S corresponding to the contracting edge, and the respective boundary divisor D_ρ in $\bar{M}_{0,S}$. Geometrically, (A) implies that X is a curve that does not lie in D_ρ, but intersects D_ρ; hence, we must have that

$$(D_\rho, \beta) = (D_\rho, [X]) > 0.$$

But, from (4.4) it follows that if ρ contracts onto a vertex $v \neq v_0$, then $(D_\rho, \beta) = 0$. Hence, this possibility is excluded.

Consider now alternative (B). We must then have $\rho \in P(\Pi)$. This implies the following degeneration pattern of the induced family of curves parametrized by X. At a generic point, the tree of the curve consists of one irreducible component C, to which trees are attached at $|N(\tau)|$ different points of this component. When the degeneration at a point of D_ρ occurs, C breaks down into two components, say, C_1 and C_2, and the attached trees are distributed among them: some become attached to C_1, and the remaining ones become attached to C_2. What is important here is that the *labelled combinatorial type of each of the attached trees does not change*, otherwise we could have used (A), which was already excluded.

But, in this case the image $p_\Pi(X)$ must land in the product of the *open strata* $\prod_{v \neq v_0} M_{F_\pi(v)}$. This is possible only if this image is a point, because such a product is an affine scheme.

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5.3.3. *Assumption (d): preparation II*

Let $\mathrm{Hilb}(\bar{M}_{0,S})$ be the Hilbert scheme of $\bar{M}_{0,S}$. As usual, it can be written as a disjoint union $\mathrm{Hilb}(X) = \coprod_P \mathrm{Hilb}^P(X)$, where P is a Hilbert polynomial (for this, one should fix an ample line bundle) and each $\mathrm{Hilb}^P(\bar{M}_{0,S})$ is a quasi-projective scheme.

Let C be a closed fibre of $p_\Pi\colon \bar{M}_\Pi \to B_\Pi$. It defines a closed point P in $\mathrm{Hilb}(\bar{M}_{0,S})$. The tangent space to $\mathrm{Hilb}(\bar{M}_{0,S})$ at the point P is identified with $H^0(C, N_{C/\bar{M}_{0,S}})$, and the obstruction space is identified with $H^1(C, N_{C/\bar{M}_{0,S}})$. From computations in § 5.3.1, it follows that

$$\dim H^1(C, N_{C/\bar{M}_{0,S}}) = 0.$$

Therefore, P is a smooth point of $\mathrm{Hilb}(\bar{M}_{0,S})$.

Consider the locus in $\mathrm{Hilb}(\bar{M}_{0,S})$ parametrizing closed fibres of $p_\Pi\colon \bar{M}_\Pi \to B_\Pi$. It is a connected component of $\mathrm{Hilb}(\bar{M}_{0,S})$. Denote it by Y, and let $U \to Y$ be the universal family over it. We have just seen that Y is a smooth (in particular, reduced and irreducible) scheme. Its dimension is

$$\dim Y = \dim H^0(C, N_{C/\bar{M}_{0,S}}) = \dim B_\Pi,$$

which follows from computations in § 5.3.1.

Therefore, we can identify the universal family $U \to Y$ with the projection $p_\Pi\colon \bar{M}_\Pi \to B_\Pi$.

5.3.4. *Assumption (d): proof*

We need to show that the canonical morphism of stacks

$$\tilde{b}_\Pi\colon \bar{M}_{0,\Sigma}(\bar{M}_\Pi, \beta_\Pi) \to \bar{M}_{0,\Sigma}(\bar{M}_{0,S}, \beta(\Pi)), \tag{5.5}$$

induced by $b_\Pi\colon \bar{M}_\Pi \to \bar{M}_{0,S}$, is an isomorphism. Here, β_Π is the Chow class of a fibre of $p_\Pi\colon \bar{M}_\Pi \to B_\Pi$.

One T-point of $\bar{M}_{0,\Sigma}(\bar{M}_{0,S}, \beta(\Pi))$ is a family of connected prestable curves

$$p_T\colon \mathcal{C}_T \to T$$

together with a stable map f_T of the class $\beta(\Pi)$ and labelled sections

$$f_T\colon \mathcal{C}_T \to \bar{M}_{0,S}, \qquad x_{j,T}\colon T \to \mathcal{C}_T, \quad j \in \Sigma.$$

Below, we show that any such map f_T can be factored through b_Π. Since b_Π is a closed embedding, such a factorization is unique if it exists.

Consider the diagram

$$\begin{tikzcd} C_T \arrow[rr, "f_T \times p_T"] \arrow[dr, "p_T"'] & & \bar{M}_{0,S} \times T \arrow[dl, "\mathrm{pr}_T"] \\ & T & \end{tikzcd}$$

provided by the T-point. Since $\beta(\Pi)$ is indecomposable (see Proposition 4.6), for any geometric fibre of $\mathcal{C}_T/T$, f_T must contract each of its components, excepting one, to a point. On the uncontracted component it is a closed embedding.

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Irreducible geometric fibres. Assume that all geometric fibres of p_T are irreducible, and, hence, $f_T \times p_T$ induces closed embeddings on all geometric fibres. By faithfully flat descent it is then a closed embedding on all fibres. Therefore, the fibre of $f_T \times p_T$ at a point $s \in \bar{M}_{0,S} \times T$ is either empty or $\kappa(s)$-isomorphic to $\operatorname{Spec}(\kappa(s))$, where $\kappa(s)$ is the residue field at s.

Since p_T and pr_T are proper, the morphism $f_T \times p_T$ is also proper. By [**15**, Proposition 8.11.5], this implies that $f_T \times p_T$ is a closed embedding. Thus, we see that if we forget the sections $(x_{j,T})$ the stable morphism $(C_T, (x_{j,T}), f_T)$ gives us a T-point of the Hilbert scheme of $\bar{M}_{0,S}$.

Therefore, by § 5.3.3 the diagram

$$\begin{tikzcd} C_T \arrow[r, "f_T"] \arrow[d] & \bar{M}_{0,S} \\ T & \end{tikzcd}$$

is obtained from

$$\begin{tikzcd} \bar{M}_\Pi \arrow[r, "b_\Pi"] \arrow[d, "p_\Pi"'] & \bar{M}_{0,S} \\ B_\Pi & \end{tikzcd}$$

by a unique pullback. Hence, the stable map $(C_T, (x_{j,T}), f_T)$ factors through $\bar{M}_\Pi$.

General case. Let $(C_T, (x_{j,T})_{j\in\Sigma}, f_T)$ be an arbitrary Σ-labelled stable map to $\bar{M}_{0,S}$ of class $\beta(\Pi)$, and let $\Sigma' \subset \Sigma$ be the subset that labels sections that land on the non-contracted component of geometric fibres. Consider the induced *prestable* map $(C_T, (x_{j,T})_{j\in\Sigma'}, f_T)$. Stabilizing it, we get a stable map $(\tilde{C}_T, (y_{j,T})_{j\in\Sigma'}, g_T)$ to $\bar{M}_{0,S}$ of class $\beta(\Pi)$, such that $f_T = g_T \circ \mathrm{st}$. In other words, we get the diagram

$$\begin{tikzcd} C_T \arrow[r, "\mathrm{st}"] \arrow[d] & \tilde{C}_T \arrow[r, "g_T"] \arrow[dl] & \bar{M}_{0,S} \\ T & & \end{tikzcd}$$

where $\tilde{C}_T \to T$ has irreducible geometric fibres. As we have seen above, g_T factors through the embedding $b_\Pi \colon \bar{M}_\Pi \to \bar{M}_{0,S}$, and, hence, so does f_T.

We have shown that any family of stable maps to $\bar{M}_{0,S}$ of class $\beta(\Pi)$ can be factorized uniquely via the closed embedding $b_\Pi \colon \bar{M}_\Pi \to \bar{M}_{0,S}$. Therefore, it gives a family of stable maps to $\bar{M}_\Pi$ of class β_Π.

This procedure gives a map of T-points of stacks appearing in (5.5) for any T. One can check that it naturally extends to morphisms of T-points and gives a functor

$$\bar{M}_{0,\Sigma}(\bar{M}_{0,S}, \beta(\Pi)) \to \bar{M}_{0,\Sigma}(\bar{M}_\Pi, \beta_\Pi). \tag{5.6}$$

Moreover, one can easily check that (5.6) is indeed inverse to (5.5). We leave these checks to the reader.

5.4. The final summary

We now briefly restate the results of the stepwise calculations of §§ 2 and 3 in our current situation (5.1), (5.2).

5.4.1. *Step I: Gromov–Witten correspondences for the target space* $\bar{M}_{0,S(\Pi)}$

We reproduce here the main result of § 3 applied to the target space $\bar{M}_{0,S(\Pi)}$ and its fundamental class **1**. Note that the sets denoted S (respectively, P) in § 3 are now $S(\Pi)$ (respectively, $P(\Pi)$) and $S(\Pi) = P(\Pi) \sqcup \{\bullet\}$.

According to Proposition 3.1, we have

$$I_{0,\Sigma}(\bar{M}_{0,P(\Pi)\sqcup\{\bullet\}}, \mathbf{1}) = (\overline{\mathrm{ev}}, \psi_{P(\Pi)})_*([\bar{M}_{0,\Sigma\sqcup P(\Pi)}]) \in A_{|\Sigma|}((\bar{M}_{0,P(\Pi)\sqcup\{\bullet\}})^{\Sigma} \times \bar{M}_{0,\Sigma}). \tag{5.7}$$

5.4.2. *Step II: Gromov–Witten correspondences for the target space* B_Π *and the zero beta-class*

According to Example 2.1, we have that

$$I_{0,\Sigma}(B_\Pi, 0) = [\Delta_\Sigma(B_\Pi) \times \bar{M}_{0,\Sigma}] \in A_*(B_\Pi^\Sigma \times \bar{M}_{0,\Sigma}). \tag{5.8}$$

Here, $\Delta_\Sigma(B_\Pi)$ is the diagonal in the Cartesian product B_Π^Σ of Σ copies of B_Π.

5.4.3. *Step III: Gromov–Witten correspondences for the target space* $\bar{M}_\Pi$ *and the fibre beta-class*

In this subsection, β_Π is the Chow class of a fibre of the projection $\bar{M}_\Pi \to B_\Pi$. We now have the canonical splitting

$$\bar{M}_\Pi = B_\Pi \times \bar{M}_{0,P(\Pi)\sqcup\{\bullet\}},$$

since $F_\pi(v_0)$ is identified with $S(\Pi) = P(\pi) \sqcup \{\bullet\}$ (cf. § 4.3).

Thus, using (5.7) and (5.8), we have that

$$I_{0,\Sigma}(\bar{M}_\Pi, \beta_\Pi) = \tilde{\Delta}^!([\Delta_\Sigma(B_\Pi) \times \bar{M}_{0,\Sigma}] \otimes (\overline{\mathrm{ev}}, \psi_{P(\Pi)})_*([\bar{M}_{0,\Sigma\sqcup P(\Pi)}])). \tag{5.9}$$

To summarize, we have proved our final theorem, a specialization of Proposition 2.4, which is as follows.

Theorem 5.2. *The structure embedding* $b_\Pi\colon \bar{M}_\Pi \to \bar{M}_{0,S}$ *induces a canonical isomorphism*

$$\tilde{b}_\Pi\colon \bar{M}_{0,\Sigma}(\bar{M}_\Pi, \beta_\Pi) \to \bar{M}_{0,\Sigma}(\bar{M}_{0,S}, \beta(\Pi)),$$

where β_Π *is the Chow class of a fibre of* $p_\Pi\colon \bar{M}_\Pi \to B_\Pi$.

This isomorphism $\tilde{b}_\Pi$ *is compatible with evaluation/stabilization morphisms for both moduli spaces, and induces the identity*

$$I_{0,\Sigma}(\bar{M}_{0,S}, \beta(\Pi)) = (b_\Pi^\Sigma \times \mathrm{id})_*(I_{0,\Sigma}(\bar{M}_\Pi, \beta_\Pi)), \tag{5.10}$$

where

$$b_\Pi^\Sigma \times \mathrm{id}\colon \bar{M}_\Pi^\Sigma \times \bar{M}_{0,\Sigma} \to (\bar{M}_{0,S})^\Sigma \times \bar{M}_{0,\Sigma}.$$

The right-hand side of (5.10) is given by (5.9).

5.5. Gromov–Witten numbers

In this subsection, we specialize (2.18) to our situation in order to calculate numerical invariants of Chow classes of boundary curves.

Let $\gamma_j \in H^{2d_j}(\bar{M}_{0,S})$ be a family of cohomology classes indexed by $j \in \Sigma$. If $\sum_{j\in\Sigma} d_j = \dim B_\Pi$, then the correspondence

$$I_{0,\Sigma}(\bar{M}_{0,S}, \beta(\Pi)) \in A_*((\bar{M}_{0,S})^\Sigma \times \bar{M}_{0,\Sigma})$$

maps $\otimes_{j\in\Sigma}\gamma_j \in (H^*(\bar{M}_{0,S}))^{\otimes\Sigma}$ to a class of maximal dimension in $H^*(\bar{M}_{0,\Sigma})$. The degree of this class is denoted by

$$\langle I^{\bar{M}_{0,S}}_{0,\Sigma,\beta(\Pi)}\rangle(\otimes_{j\in\Sigma}\gamma_j).$$

Generally, this degree is the virtual number of stable maps of pointed curves of class $\beta(\Pi)$ satisfying the incidence conditions $f(x_j) \in \Gamma_j$, where (Γ_j) are cycles in general position whose dual classes are γ_j:

$$f\colon (C; (x_j \mid j \in \Sigma)) \to \bar{M}_{0,S},$$

whenever such incidence conditions are strong enough to enforce existence of only a finite (virtual) number of such maps. In our unobstructed case, this virtual number is the actual number of such maps whenever the incidence cycles are in general position.

Recall also that this number is polylinear in (γ_j).

Proposition 5.3. *We have that*

$$\langle I^{\bar{M}_{0,S}}_{0,\Sigma,\beta(\Pi)}\rangle(\otimes_{j\in\Sigma}\gamma_j) = \deg\biggl(\bigcap_{j\in\Sigma} \mathrm{pr}_{B_\Pi *} \circ b^*_\Pi(\gamma_j)\biggr). \tag{5.11}$$

Sketch of proof. Skipping a clumsy but straightforward formal derivation of (5.11) from (2.18), we describe the geometric content of this counting formula in the general situation axiomatized in § 2.5.

First of all, (2.18) reduces the count to the case of an incidence condition represented by some cycles in $E = \bar{M}_\Pi$. In fact, $b^*_\Pi(\gamma_j)$ are represented by $\Gamma_j \cap \bar{M}_\Pi$ in the case of transversal intersections.

Now, in $\bar{M}_\Pi$ the incidence cycles can be replaced by ones of the form $\Delta_j \times c_j + \Delta'_j \times C$, where c_j are points on a projective line C, as in (5.2), corresponding to the decomposition $\bar{M}_\Pi = B_\Pi \times C$.

Assume first that $\Delta'_j \neq 0$ for some $j = j_0$. If, for such an incidence condition, there exists a fibre C_0 of $\bar{M}_\Pi \to B_\Pi$ satisfying it at all, then the number of relevant pointed stable maps must be infinite, because x_{j_0} can be chosen arbitrarily along this fibre. Hence, decomposable cycles containing at least one factor of the form $\Delta'_j \times C$ give zero contributions to (5.11).

Now consider the case of incidence conditions of the form $\Delta_j \times c_j$ for all $j \in \Sigma$. Let $\Delta_j = \mathrm{pr}_{B_\Pi}(\Delta_j \times c_j)$ be in a general position in B_Π, so that the intersection cycle $\bigcap_{j\in\Sigma}\Delta_j$ is a sum of points $y_a \in B_\Pi$, of multiplicity one each. We can also lift Δ_j arbitrarily to

$\bar{M}_{\Pi}$, that is, choose all $c_j \in C$ pairwise distinct, and consider $\Delta_j \times c_j$ as a geometric incidence condition representing the initial cohomological incidence condition (γ_j).

Subsequently, the geometric count becomes straightforward: each point y_a produces one fibre of the class $\beta(\Pi)$ intersecting each $\Delta_j \times c_j$ at one point corresponding to c_j.

The number of (y_a) is the right-hand side of (5.11), and the curve count interprets the left-hand side of (5.11).

6. Examples and remarks

6.1. 'Naturality' of Gromov–Witten correspondences

In this subsection we try to make Guess 1.1 somewhat more precise. To this end, we first recall that natural objects in the relevant category are moduli spaces $\bar{M}_{\tau}$, and natural morphisms/correspondences are those that are produced from morphisms in the category of modular graphs. The latter include contractions, forgetful morphisms, relabelling morphisms etc. (see [**7**]).

The least controllable characteristic of GW-correspondences is their dependence on the argument β in the relevant Mori cone. So far we have considered only boundary βs, and they are, of course, 'natural' by definition.

In this subsection we show that, keeping the notation of § 5, we may naturally encode most of the relevant combinatorial and geometric information in one moduli space $\bar{M}_{0,\Sigma\times(S\setminus\{s_0\})}$ and a configuration of certain of its boundary strata. This is only a tentative suggestion; we do not develop it fully, because we still lack even a conjectural description of the situation for more general βs.

6.1.1. *The tree* $\boldsymbol{T}$

The tree $\boldsymbol{T}$ has one special vertex, called the *central* vertex, denoted by v_c. Its flags are bijectively labelled by Σ: we use the notation

$$F_{\boldsymbol{T}}(v_c) := \{\langle j\rangle \mid j \in \Sigma\}. \tag{6.1}$$

The remaining vertices constitute a set bijective to $\Sigma \times \{s_0\}$. Together with (6.1), this bijection is a part of the structure, and we may refer to a non-central vertex $v \in V_{\boldsymbol{T}}$ as $v_j := \langle j, s_0\rangle$, $j \in \Sigma$.

Furthermore, we set

$$F_{\boldsymbol{T}}(v_j) := \{j\} \times S = \{(j,s) \mid s \in S\}. \tag{6.2}$$

Thus, the standard identification of $\bar{M}_{\boldsymbol{T}}$ with the product of moduli spaces corresponding to stars of all vertices, $\prod_{v\in V_{\boldsymbol{T}}} \bar{M}_{0,F_{\boldsymbol{T}}(v)}$, can be rewritten as

$$\bar{M}_{\boldsymbol{T}} = (\bar{M}_{0,S})^{\Sigma} \times \bar{M}_{0,\Sigma}, \tag{6.3}$$

where the last factor corresponds to the central vertex.

Edges. The flag $\langle j\rangle$ attached to the central vertex (see (6.1)) is grafted to the flag (j, s_0) incident to the vertex v_j (see (6.2)). There are no more edges.

Thus, the central vertex carries no tails, and the set of edges $E_{\boldsymbol{T}}$ is naturally bijective to Σ. The set of tails is

$$T_{\boldsymbol{T}} = \coprod_{j\in\Sigma}(F_{\boldsymbol{T}}(\langle j, s_0\rangle) \setminus (j, s_0)) = \coprod_{j\in\Sigma}(\{j\} \times (S \setminus \{s_0\})) \cong \Sigma \times (S \setminus \{s_0\}). \tag{6.4}$$

If we interpret the last set in (6.4) as the set of labels of tails, then the above-described set of edges of $\boldsymbol{T}$ determines the canonical embedding of $\bar{M}_{\boldsymbol{T}}$ as a boundary stratum:

$$\bar{M}_{\boldsymbol{T}} \hookrightarrow \bar{M}_{0,\Sigma\times(S\setminus\{s_0\})}. \tag{6.5}$$

This embedding corresponds to full contraction of all edges of $\boldsymbol{T}$ to the star with flags T_τ.

We now encode information about Π into another tree $\boldsymbol{T}(\Pi)$, together with its contraction onto $\boldsymbol{T}$.

6.1.2. *The tree* $\boldsymbol{T}(\Pi)$

Briefly, to get $\boldsymbol{T}(\Pi)$, we replace each non-central vertex v_j, $j \in \Sigma$, by a copy π_j of the tree π described in § 4.2.

More precisely, the special vertex of π_j denoted $v_{0,j}$ now carries tails (6.2) distributed among other vertices of π_j according to Π, and its tail (j, s_0) is grafted in $\boldsymbol{T}(\Pi)$ to the same flag $\langle j\rangle$ of its central vertex as it was in $\boldsymbol{T}$.

The contraction $\boldsymbol{T}(\Pi) \to \boldsymbol{T}$ contracts each π_j to the star of v_j, and is identical on the stars of the central vertices. Combining the relevant boundary morphism with (6.5), we get the diagram of strata embedding

$$\bar{M}_{\boldsymbol{T}(\boldsymbol{\Pi})} \hookrightarrow \bar{M}_{\boldsymbol{T}} \hookrightarrow \bar{M}_{0,\Sigma\times(S\setminus\{s_0\})}. \tag{6.6}$$

The intermediate and final correspondences considered in § 5 can be expressed using the geometry of (6.6).

6.2. Using the reconstruction theorems

For a general target W, if the Chow ring $A^*(W)$ (with coefficients in $\boldsymbol{Q}$) coincides with the whole of $H^*(W)$ and is generated by $A^1(W)$, then the total motivic quantum cohomology of W of genus 0 understood as the family of I-correspondences is completely determined by triple correlators (3-point GW-invariants) of codimension 0. This follows from the first and second reconstruction theorems of [**24**]; see also [**26**].

In any case, these triple correlators are precisely the coefficients of small quantum cohomology as a formal series in q^β. Hence, under the same assumptions the total quantum cohomology is completely determined by the small quantum multiplication in $H^*(V)[\![q^\beta]\!]$:

$$\Delta_a \cdot \Delta_b = \Delta_a \cup \Delta_b + \sum_{\beta\neq 0}\sum_{c\neq 0}\langle \Delta_a\Delta_b\Delta_c\rangle_\beta \Delta^c q^\beta.$$

Here, (Δ_a) is a basis of H^* such that Δ_0 is identity, $g_{ab} = (\Delta_a, \Delta_b)$, (g^{ab}) is the inverse matrix to (g_{ab}) and $\Delta^a := \sum_b g^{ab}\Delta_b$.

This is applicable to all $\bar{M}_{0,n}$.

In turn, the associativity equations allow one to express all triple correlators through a part of them. We now make explicit this subset for $\bar{M}_{0,n}$.

6.3. A generating subset of triple correlators

Set $|\Delta| = i$ for $\Delta \in H^{2i}(\bar{M}_{0,n})$. (No confusion with the cardinality $|S|$ of a set S should arise.) All invariants can then be recursively calculated through the 3-point invariants $\langle\Delta_a\Delta_b\Delta_c\rangle_\beta$ with Δ_c divisorial, $|\Delta_a|, |\Delta_b| \geqslant 1$, $\beta \neq 0$ and

$$|\Delta_a| + |\Delta_b| = (-K_n, \beta) + n - 4,$$

where K_n is the canonical class of $\bar{M}_{0,n}$. Hence, β are restricted by

$$2 - (n-3) \leqslant (-K_n, \beta) - 1 \leqslant n - 3.$$

See [**24**, Theorem 3.1], with the following easy complements. If $|\Delta_a|$ or $|\Delta_b| = 0$, $\beta \neq 0$, then the respective GW-invariant is 0 because of [**24**, (2.7)]. If $\beta = 0$, we can use [**24**, (2.8)]. It remains to consider the following list of parameters:

$$6 - n \leqslant (-K_n, \beta) \leqslant n - 2,$$
$$2 \leqslant |\Delta_a| + |\Delta_b| = (-K_n, \beta) + n - 4 \leqslant 2n - 6.$$

Finally, if Δ is a divisorial class with $(\Delta, \beta) = 0$, then $\langle\Delta'\Delta''\Delta\rangle_\beta = 0$ for any Δ', Δ'' due to the divisor axiom.

6.3.1. *Tables for the first values of n*

In Table 1 we collect values of $(-K_n, \beta)$ and $(|\Delta_a|, |\Delta_b|)$ for $n = 5, 6, 7$.

Note that $\bar{M}_{0,5}$ is the del Pezzo surface of degree 5; in particular, its anticanonical class is ample, and, hence, the generating subset of triple correlators is finite. In fact, generating sets for del Pezzo surfaces are collected in [**2**]. It is also known that all del Pezzo surfaces have generically semi-simple quantum cohomology, and, more generally, this remains true for blow-ups of any finite set of points on or over $\boldsymbol{P}^2$ (see [**1**]).

Already for $\bar{M}_{0,6}$ the situation is more mysterious. For 45 out of 105 generators of the cone of βs, we have that $(-K_6, \beta) = 0$. (See also [**9**].) Hence, our generating list above is, in principle, infinite. Semi-simplicity is an open question as well. For $n \geqslant 7$ the difficulties grow.

6.4. Strategies of computation

A possible way to compute some Gromov–Witten invariants of $\bar{M}_{0,n}$ with non-boundary βs consists of choosing a birational morphism $p_n\colon \bar{M}_{0,n} \to X_n$ such that the following hold.

(a) (Sufficiently many) GW-invariants of X_n are known/computable.

(b) Morphism p_n is such that there exist 'naturality' formulae that allow one to compute (some) GW-invariants of $\bar{M}_{0,n}$ through (some) GW-invariants of X_n.

Table 1. *Tables for the first values of n*

$(-K_5,\beta)$	1	2	3
$(\lvert\Delta_a\rvert,\lvert\Delta_b\rvert)$	(1,1)	(1,2)	(2,2)

$(-K_6,\beta)$	0	1	2	3	4
$(\lvert\Delta_a\rvert,\lvert\Delta_b\rvert)$	(1,1)	(1,2)	(2,2) (1,3)	(2,3)	(3,3)

$(-K_7,\beta)$	-1	0	1	2	3	4	5
$(\lvert\Delta_a\rvert,\lvert\Delta_b\rvert)$	(1,1)	(1,2)	(2,2) (1,3)	(2,3) (1,4)	(3,3) (2,4)	(3,4)	(4,4)

For 'naturality' results see, for example, [**8**, **18**, **19**, **27**, **29**–**31**] (where [**8**] contains corrections to [**18**]). We discuss the relevant classes of morphisms below.

6.4.1. *Blowing $\bar{M}_{0,n}$ down*

The following choices of morphisms seem promising for the application of this strategy, at least for small values of n.

(i) $X_n = \boldsymbol{P}^{n-3}$, p_n is Kapranov's morphism, representing $\bar{M}_{0,n}$ as the result of the consecutive blowing up of $n-1$ points, preimages of lines connecting pairs of these points, preimages of planes, passing through triples of these points, etc. (see [**17**]). It involves forgetting the nth point, then fixing $p_1,\dots,p_{n-1}\in\boldsymbol{P}^{n-3}$.

(ii) $X_n = (\boldsymbol{P}^1)^{n-3}$, p_n is a similar morphism that was described explicitly by Tavakol.

(iii) $X_n = \bar{L}_{n-2}$, the Losev–Manin moduli space parametrizing stable chains of $\boldsymbol{P}^1$ with marked points and a specific stability condition, and p_n is the respective stabilization morphism.

It makes sense to not just use $\bar{L}_{n-2}$ in order to help calculate GW-invariants of $\bar{M}_{0,n}$, but to treat these moduli spaces as replacements of $\bar{M}_{0,n}$ in their own right. In fact, one can define GW-invariants based upon $\bar{L}_{n-2}$; essentially, no information is thereby lost (see [**3**]).

The spaces $\bar{L}_{n-2}$ are toric, and have the largest Chow ring of these three examples. These manifolds are not Fano for $n\geqslant 6$, but, according to [**20**], any toric manifold has generically semi-simple quantum cohomology; therefore, it can be more accessible.

(iv) Finally, combining two or more forgetful morphisms, one can birationally map $\bar{M}_{0,n}$ and $\bar{L}_{n-2}$ onto products of similar manifolds, thus opening a way to an inductive calculation of GW-invariants. Here is the simplest example: for $n\geqslant 5$, forgetting at first x_n, and then all points except for (x_1,x_2,x_3,x_n), we get a birational morphism

$$\bar{M}_{0,n}\to\bar{M}_{0,n-1}\times\bar{M}_{0,4},\quad \bar{M}_{0,4}\cong\boldsymbol{P}^1.$$

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GW-invariants of a product can be calculated via the general quantum Künneth formula whenever they are known for lesser values of n.

For our main preoccupation here, that of understanding the motivic properties of quantum cohomology correspondences, versions of this last suggestion are most promising.

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