ON SOME THEOREMS OF DOETSCH

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1. Introduction. The spaces $\mathfrak{H}_p(\omega)$, $1 \leqslant p \leqslant \infty$, ω real are defined to consist of those analytic functions $f(s)$, regular for $Re\, s > \omega$ and for which $\mu_p(f;x)$ is bounded for $x > \omega$ where

$$(1.1) \qquad \mu_p(f;x) = \left\{\frac{1}{2\pi}\int_{-\infty}^{\infty} |f(x+iy)|^p\, dy\right\}^{1/p}, \qquad 1 \leqslant p < \infty$$

and

$$(1.2) \qquad \mu_\infty(f;x) = \sup_{-\infty<y<\infty} |f(x+iy)|.$$

These spaces have been extensively studied—for example, see **(2)**, **(4)**.

In particular two results connect these spaces with the theory of Laplace transforms. These are that if $e^{-\omega t}\phi(t) \in L_p(0, \infty)$, $1 \leqslant p \leqslant 2$, and if f is the Laplace transform of ϕ, that is,

$$f(s) = \int_0^\infty e^{-st}\phi(t)dt, \qquad Re\, s > \omega$$

then $f \in \mathfrak{H}_q(\omega)$ where

$$(1.3) \qquad p^{-1} + q^{-1} = 1,$$

and that conversely if $f \in \mathfrak{H}_p(\omega)$, $1 \leqslant p \leqslant 2$, then $f(s)$ is the Laplace transform of a function ϕ such that $e^{-\omega t}\phi(t) \in L_q(0, \infty)$. For $1 < p \leqslant 2$, these two results are due to Doetsch **(2)**, and for $p = 1$ they are trivial. The two results concern the same space if and only if $p = 2$, when they give necessary and sufficient conditions that $f(s)$ be the Laplace transform of a function ϕ such that $e^{-\omega t}\phi(t) \in L_2(0, \infty)$.

Recently the author **(6, 7)** has considered the Laplace transformation of functions of the form $t^\lambda\phi(t)$, $\phi \in L_p(0, \infty)$, $\lambda > -q^{-1}$, and we propose to generalize Doetsch's results so as to deal with functions of this type, though we shall have to restrict λ to be positive. To this end, which is achieved in § 2, we shall first define certain new spaces $\mathfrak{H}_{\lambda,p}(\omega)$, $\lambda \geqslant 0$, $1 \leqslant p \leqslant \infty$, which in a sense are generalizations of the spaces $\mathfrak{H}_p(\omega)$.

In the case $p = 2$ we shall see that we again obtain necessary and sufficient conditions for a representation, and in § 3 we shall relate these results to some previous work of ours and by so doing show that in this case the conditions for the representation can be slightly relaxed.

Doetsch **(2)** has further shown that for $p = 2$ a certain real inversion formula for the Laplace transformation, originally due to Paley and Wiener

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(**5**, p. 39), is very useful. In § 4 we shall show how this formula can be generalized to deal with Laplace transforms of the type mentioned earlier.

2. Generalized spaces. In this section we first define the spaces $\mathfrak{H}_{\lambda,p}$, and then prove two theorems generalizing Doetsch's result.

Definition. $\mathfrak{H}_{0,p}(\omega) \equiv \mathfrak{H}_p(\omega)$. *If* $\lambda > 0$, $\mathfrak{H}_{\lambda,p}(\omega)$ *consists of those functions* $f(s) \in \mathfrak{H}_p(\omega')$ *for every* $\omega' > \omega$ *and such that* $\mu_p^\lambda(f;\omega)$ *is finite, where*

$$\mu_p^\lambda(f;\omega) = \int_\omega^\infty (x-\omega)^{q\lambda-1}(\mu_p(f;x))^q dx, \qquad 1 < p \leqslant \infty \tag{2.1}$$

and

$$\mu_1^\lambda(f;\omega) = \sup_{x>\omega} (x-\omega)^\lambda \mu_1(f;x). \tag{2.2}$$

It is clear that $\mathfrak{H}_{\lambda,p}(\omega)$ is a linear space. It is easy to show that it is a Banach space under the norm

$$||f||_{\lambda,p} = \begin{cases} \{\mu_p^\lambda(f;\omega)/\Gamma(\lambda q)\}^{1/q} & \lambda > 0, p > 1 \\ \mu_1^\lambda(f;\omega) & \lambda > 0, p = 1 \\ \sup\limits_{x>\omega} \mu_p(f;x) & \lambda = 0. \end{cases}$$

Also an easy proof shows that if $||f||_{\lambda,p} \leqslant M$, $0 < \lambda < \lambda_0$, then $||f||_{0,p} \leqslant M$, and $||f||_{\lambda,p} \to ||f||_{0,p}$ as $\lambda \to 0+$. Since these properties are not needed in what ensues, they will not be elaborated further here.

THEOREM 1. *If* $e^{-\omega t}\phi(t) \in L_p(0,\infty)$, $1 \leqslant p \leqslant 2$, $\lambda \geqslant 0$ *and*

$$f(s) = \int_0^\infty e^{-st} t^\lambda \phi(t)dt, \qquad Re\, s > \omega,$$

then

$$f(s) \in \mathfrak{H}_{\lambda,q}(\omega).$$

Proof. If $\lambda = 0$, $1 < p \leqslant 2$, the theorem follows from (**2**, Theorem 2), and if $\lambda = 0$, $p = 1$, the result is trivial.

If $\lambda > 0$ and $\omega' > \omega$, then since

$$t^\lambda e^{-(\omega'-\omega)t}$$

is bounded for $t \geqslant 0$,

$$e^{-\omega' t} t^\lambda \phi(t) \in L_p(0,\infty),$$

and hence by (**2**, Theorem 2) $f(s) \in \mathfrak{H}_q(\omega')$. It remains to show $\mu_q^\lambda(f;\omega)$ is finite.

If $p = 1$, $x > \omega$,

$$|f(x+iy)| \leqslant \int_0^\infty e^{-xt} t^\lambda |\phi(t)|dt$$

so that

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$$\mu_\infty(f;x) \leqslant \int_0^\infty e^{-xt}\, t^\lambda |\phi(t)|dt.$$

Hence,

$$\begin{aligned}\mu_\infty^\lambda(f;\omega) &= \int_\omega^\infty (x-\omega)^{\lambda-1}\,\mu_\infty(f;x)dx\\ &\leqslant \int_\omega^\infty (x-\omega)^{\lambda-1}dx \int_0^\infty e^{-xt}\, t^\lambda|\phi(t)|dt\\ &= \int_0^\infty t^\lambda|\phi(t)|dt \int_\omega^\infty (x-\omega)^{\lambda-1}\, e^{-xt}\,dx\\ &= \Gamma(\lambda)\int_0^\infty e^{-\omega t}|\phi(t)|dt < \infty,\end{aligned}$$

and $f \in \mathfrak{H}_{\lambda,\infty}(\omega)$.

If $1 < p \leqslant 2$, $\lambda > 0$, $x > \omega$,

$$f(x-iy) = \int_0^\infty e^{iyt}(e^{-xt}\, t^\lambda \phi(t))dt$$

is the Fourier transform of a function in $L_p(0, \infty)$, $1 < p \leqslant 2$. Hence by (**8**, Theorem 74), for $x > \omega$,

$$\begin{aligned}\mu_q(f;x) &= \left\{\frac{1}{2\pi}\int_{-\infty}^\infty |f(x+iy)|^q\, dy\right\}^{1/q} = \left\{\frac{1}{2\pi}\int_{-\infty}^\infty |f(x-iy)|^q\, dy\right\}^{1/q}\\ &\leqslant \left\{\int_0^\infty e^{-pxt}\, t^{p\lambda}|\phi(t)|^p\, dt\right\}^{1/p},\end{aligned}$$

so that for $x > \omega$,

$$(\mu_q(f;x))^p \leqslant \int_0^\infty e^{-pxt}\, t^{p\lambda}|\phi(t)|^p\, dt.$$

Hence, we have

$$\begin{aligned}\mu_q^\lambda(f;\omega) &= \int_\omega^\infty (x-\omega)^{p\lambda-1}\,(\mu_q(f;x))^p\, dx\\ &\leqslant \int_\omega^\infty (x-\omega)^{p\lambda-1}\, dx \int_0^\infty e^{-pxt}\, t^{p\lambda}|\phi(t)|^p\, dt\\ &= \int_0^\infty t^{p\lambda}|\phi(t)|^p\, dt \int_\omega^\infty (x-\omega)^{p\lambda-1}\, e^{-pxt}\, dx\\ &= \frac{\Gamma(p\lambda)}{(p)^{p\lambda}}\int_0^\infty e^{-p\omega t}|\phi(t)|^p\, dt < \infty,\end{aligned}$$

and $f \in \mathfrak{H}_{\lambda,q}(\omega)$.

THEOREM 2. *If* $f \in \mathfrak{H}_{\lambda,p}(\omega)$, $1 \leqslant p \leqslant 2$, $\lambda \geqslant 0$, *then there is a function* ϕ *with* $e^{-\omega t}\phi(t) \in L_q(0, \infty)$ *such that*

$$f(s) = \int_0^\infty e^{-st}\, t^\lambda \phi(t)\, dt.$$

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Proof. Without loss of generality we may assume $\omega = 0$, for otherwise we deal with $f(\omega + s)$. We shall consider first the cases $1 < p \leqslant 2$.

By the definition of $\mathfrak{H}_{\lambda,p}$, $f \in \mathfrak{H}_p(\omega')$ for each $\omega' > 0$, and hence, for each fixed $x > 0$, $f(x + iy) \in L_p(-\infty, \infty)$. For $x > 0$ let

$$F_a(t, x) = \frac{1}{2\pi}\int_{-a}^{a} f(x + iy)e^{ity}\, dy.$$

By (**8**, Theorem 74), as $a \to \infty$ F_a converges in mean of order q, as a function of t, to a function $F(t, x) \in L_q(-\infty, \infty)$. Consider, however, the integral

$$\int f(s)e^{ts}ds$$

taken around the rectangle with vertices at $x_1 \pm ia$ and $x_2 \pm ia$ where $0 < x_1 < x_2$.

The integral along the upper side is

$$\int_{x_1}^{x_2} f(x + ia)e^{t(x+ia)}dx = e^{ita}\int_{x_1}^{x_2} f(x + ia)e^{tx}\, dx.$$

But if we let $\Phi(\zeta) = f(\omega' - i\zeta)$, where $0 < \omega' < x_1$, we have that $\Phi(\zeta)$ is an analytic function regular for $\eta = Im\,\zeta > 0$, and for $\eta > 0$,

$$\int_{-\infty}^{\infty} |\Phi(\xi + i\eta)|^p\, d\xi = \int_{-\infty}^{\infty} |f(\omega' + \eta - i\xi)|^p\, d\xi$$
$$= \int_{-\infty}^{\infty} |f(\omega' + \eta + i\xi)|^p\, d\xi = 2\pi(\mu_p(f; \omega' + \eta))^p,$$

and this is bounded for $\eta > 0$ since $f \in \mathfrak{H}_p(\omega')$. Hence, by (**8**, Lemma, p. 125), $\Phi(\xi + i\eta) \to 0$ as $\xi \to -\infty$ uniformly for $\delta \leqslant \eta \leqslant R$ where $R > \delta > 0$. Taking $\xi = -a$, $\eta = x - \omega'$, $R = x_2 - \omega'$, $\delta = x_1 - \omega'$, we have $f(x+ia) \to 0$ as $a \to \infty$ uniformly for $x_1 \leqslant x \leqslant x_2$, and the integral along the upper side of the rectangle tends to zero as $a \to \infty$. Similarly the integral along the lower side tends to zero as $a \to \infty$. Hence, as $a \to \infty$,

$$\int_{-a}^{a} f(x_1 + iy)e^{t(x_1+iy)}dy - \int_{-a}^{a} f(x_2 + iy)e^{t(x_2+iy)}dy \to 0,$$

that is,

$$e^{tx_1}F_a(t, x_1) - e^{tx_2}F_a(t, x_2) \to 0.$$

Thus the mean limit over any finite t-interval is also zero, so that for almost all t

$$e^{tx_1}F(t, x_1) = e^{tx_2}F(t, x_2),$$

and we may write

$$F(t, x) = e^{-tx}F(t).$$

By (**8**, Theorem 74)

$$\int_{-\infty}^{\infty} |F(t)|^q\, e^{-qxt}\, dt \leqslant (\mu_p(f; x))^q. \tag{2.3}$$

Since for any $\delta > 0$ the right hand side of (2.3) is bounded, say by $K(\delta)$, for $x > \delta$, we have

$$\int_{-\infty}^{-\delta} |F(t)|^q \, dt \leqslant e^{-q\delta x} \int_{-\infty}^{\infty} |F(t)|^q \, e^{-qxt} \, dt$$
$$\leqslant K(\delta)\, e^{-q\delta x} \to 0 \quad \text{as} \quad x \to \infty.$$

Thus $F(t) = 0$ a.e. for $t < 0$, and (2.3) becomes

$$\int_0^{\infty} |F(t)|^q \, e^{-qxt} \, dt \leqslant (\mu_p(f;x))^q. \tag{2.4}$$

Multiplying (2.4) by $x^{q\lambda-1}$ and integrating, we obtain

$$\frac{\Gamma(q\lambda)}{q^{q\lambda}} \int_0^{\infty} t^{-q\lambda} |F(t)|^q \, dt \leqslant \int_0^{\infty} x^{q\lambda-1} (\mu_p(f;x))^q \, dx$$
$$= \mu_p^{\lambda}(f;0) < \infty,$$

so that $t^{-\lambda}F(t) \in L_q(0, \infty)$, or $F(t) = t^{\lambda}\phi(t)$, where $\phi \in L_q(0, \infty)$. Finally, from (**8**, Theorem 74), for $x > 0$ and almost all y,

$$f(x+iy) = \frac{d}{dy} \int_0^{\infty} \frac{e^{-iyt} - 1}{-it} e^{-xt} \, t^{\lambda} \phi(t) \, dt$$
$$= \frac{d}{dy} \int_0^{\infty} e^{-xt} \, t^{\lambda} \phi(t) dt \int_0^{y} e^{-itu} \, du$$
$$= \frac{d}{dy} \int_0^{y} du \int_0^{\infty} e^{-(x+iu)t} \, t^{\lambda} \phi(t) \, dt$$
$$= \int_0^{\infty} e^{-(x+iy)t} \, t^{\lambda} \phi(t) \, dt,$$

the interchange of the order of integrations being justified by Fubini's theorem. But since the functions appearing on either side of this equation are continuous, the equation holds for all y and thus, if $Re\, s > 0$,

$$f(s) = \int_0^{\infty} e^{-st} \, t^{\lambda} \phi(t) \, dt.$$

For $p = 1$ we proceed as follows. By the definition of $\mathfrak{H}_{\lambda,1}$, $f \in \mathfrak{H}_1(\omega')$ for any $\omega' > 0$, and thus for each $x > 0$, $f(x + iy) \in L_1(-\infty, \infty)$. For $x > 0$ we let

$$F(t, x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x+iy) e^{ity} \, dy.$$

Then it follows in practically the same manner as previously that for almost all t

$$F(t, x) = e^{-tx} \, F(t).$$

Hence,

$$e^{-tx} |F(t)| \leqslant \frac{1}{2\pi} \int_{-\infty}^{\infty} |f(x+iy)| dy = \mu_1(f;x), \tag{2.5}$$

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and since the right hand side of (2.5) is bounded as $x \to \infty$, we must have $F(t) \equiv 0$ for $t < 0$.

Multiplying both sides of (2.5) by x^λ and taking suprema, we obtain

$$\sup_{x>0} x^\lambda e^{-tx}|F(t)| \leqslant \mu_1^\lambda(f; 0).$$

But

$$\sup_{x>0} x^\lambda e^{-tx} = \lambda^\lambda e^{-\lambda} t^{-\lambda},$$

so that

$$t^{-\lambda}|F(t)| \leqslant M, \qquad t > 0,$$

that is $F(t) = t^\lambda \phi(t)$ with $\phi \in L_\infty(0, \infty)$.

Finally from (**8**, Theorem 3), for $x > 0$

$$f(x + iy) = \lim_{R\to\infty} \int_0^R e^{-iyt} e^{-xt} t^\lambda \phi(t)\, dt = \int_0^\infty e^{-(x+iy)t} t^\lambda \phi(t)\, dt,$$

so that for $Re\, s > 0$

$$f(s) = \int_0^\infty e^{-st} t^\lambda \phi(t)\, dt.$$

3. The case $p = 2$. Theorems 1 and 2 together give for $p = 2$ necessary and sufficient conditions that $f(s)$ be represented as the Laplace transform of a function of the form $t^\lambda \phi(t)$ with $e^{-\omega t}\phi(t) \in L_2(0, \infty)$ and $\lambda \geqslant 0$. However, these conditions can be somewhat relaxed by using a previous result of ours. This is done in the following theorem. For convenience we write here $\lambda = \frac{1}{2}\nu$.

THEOREM 3. *A necessary and sufficient condition that an analytic function $f(s)$, regular for $Re\, s > \omega$ be the Laplace transform of a function of the form $t^{\frac{1}{2}\nu}\phi(t)$, with $e^{-\omega t}\phi(t) \in L_2(0, \infty)$, ω real, $\nu > 0$, is that*

$$\int_\omega^\infty (x - \omega)^{\nu-1} dx \int_{-\infty}^\infty |f(x + iy)|^2 dy < \infty.$$

Proof. We may suppose, without loss of generality, that $\omega = 0$. In (**7**) we showed that a necessary and sufficient condition for such a representation is that

$$\sum_{n=0}^\infty \frac{n!}{\Gamma(\nu + n + 1)} |q_n|^2 < \infty, \tag{3.1}$$

where

$$q_n = \sum_{r=0}^n \binom{n + \nu}{n - r} \frac{1}{r!} f^{(r)}(\tfrac{1}{2}).$$

We shall show here that the two conditions are equivalent.

Now, if $\nu > 0$,

$$\frac{n!}{\Gamma(\nu + n + 1)} = \frac{B(n + 1, \nu)}{\Gamma(\nu)} = \frac{2}{\Gamma(\nu)} \int_0^1 r^{2n+1}(1 - r^2)^{\nu-1} dr,$$

and hence (3.1) becomes

$$\frac{2}{\Gamma(\nu)}\int_0^1 r(1-r^2)^{\nu-1}\left(\sum_{n=0}^{\infty}|q_n|^2 r^{2n}\right)dr < \infty,$$

the interchange of integration and summation being permitted since all summands are positive.

But it was pointed out in (7) that

$$\sum_{n=0}^{\infty} q_n z^n = \frac{F(z)}{(1-z)^{\nu+1}}, \quad |z| < 1,$$

where $F(z) = f(\frac{1}{2}(1+z)/(1-z))$. Hence, from the Parseval theorem for power series (3, p. 245), for $0 \leqslant r < 1$

$$\sum_{n=0}^{\infty}|q_n|^2 r^{2n} = \frac{1}{2\pi}\int_0^{2\pi}\left|\frac{F(re^{i\theta})}{(1-re^{i\theta})^{\nu+1}}\right|^2 d\theta$$

and (3.1) becomes

$$\frac{1}{\pi\Gamma(\nu)}\int_0^1 (1-r^2)^{\nu-1} r\,dr\int_0^{2\pi}\left|\frac{F(re^{i\theta})}{(1-re^{i\theta})^{\nu+1}}\right|^2 d\theta < \infty. \tag{3.2}$$

However, the transformation

$$re^{i\theta} = z = \frac{s-\frac{1}{2}}{s+\frac{1}{2}} = \frac{x+iy-\frac{1}{2}}{x+iy+\frac{1}{2}}$$

maps the interior of the unit circle in the z-plane conformally and univalently onto the half-plane $Re\, s > 0$, and making this change of variable in the integral, (3.2) becomes

$$\frac{2^{\nu-1}}{\pi\Gamma(\nu)}\int_0^{\infty} x^{\nu-1}dx\int_{-\infty}^{\infty}|f(x+iy)|^2 dy < \infty,$$

that is, the condition of the theorem.

It is worth noting the points in which the conditions are relaxed here. Using Theorems 1 and 2 we obtain the condition $f \in \mathfrak{H}_{\lambda,2}(\omega)$ as necessary and sufficient for such a representation. From the definition of $\mathfrak{H}_{\lambda,2}(\omega)$, this implies $f \in \mathfrak{H}_2(\omega')$ for every $\omega' > \omega$, that is, that

$$\int_{-\infty}^{\infty}|f(x+iy)|^2 dy$$

be bounded for $x > \omega'$, for every $\omega' > \omega$, and it is this condition that is dropped. It may also be noted that Theorems 1 and 3 together imply that the condition $f \in \mathfrak{H}_2(\omega')$ for each $\omega' > \omega$, can be dropped from the definition of $\mathfrak{H}_{\lambda,2}(\omega')$.

It is natural to ask whether the condition $f \in \mathfrak{H}_p(\omega')$ for each $\omega' > \omega$ can be dropped from the definition of $\mathfrak{H}_{\lambda,p}(\omega)$ for other values of p. For $p = 1$ and $p = \infty$ this question can be answered affirmatively. In the case $p = 1$, this follows from the fact that for $x > \omega' > \omega$,

$$\mu_1(f;x) \leqslant (\omega'-\omega)^{-\lambda}\mu_1^{\lambda}(f;\omega),$$

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and for $p = \infty$ the affirmative answer can easily be shown to follow from a theorem of Doetsch (**1**) which asserts that $\log \mu_\infty(f;x)$ is a convex function of x. For the remaining values of p the answer is not yet known.

4. Inversion for $p = 2$. The inversion theorem is proved below. We first prove a preliminary lemma.

LEMMA. *Suppose $\phi \in L_2(0, \infty)$, $\lambda > 0$, and*

$$f(s) = \int_0^\infty e^{-st} t^\lambda \phi(t)\, dt, \qquad s > 0.$$

Then for $s > 0$

$$\frac{1}{\Gamma(\lambda)} \int_s^\infty (\sigma - s)^{\lambda-1} f(\sigma) d\sigma = \int_0^\infty e^{-st} \phi(t) dt.$$

Proof.

$$\begin{aligned}\frac{1}{\Gamma(\lambda)} \int_s^\infty (\sigma - s)^{\lambda-1} f(\sigma) d\sigma &= \frac{1}{\Gamma(\lambda)} \int_s^\infty (\sigma - s)^{\lambda-1} d\sigma \int_0^\infty e^{-\sigma t} t^\lambda \phi(t)\, dt \\ &= \frac{1}{\Gamma(\lambda)} \int_0^\infty t^\lambda \phi(t)\, dt \int_s^\infty (\sigma - s)^{\lambda-1} e^{-\sigma t}\, d\sigma \\ &= \int_0^\infty e^{-st} \phi(t)\, dt,\end{aligned}$$

the interchange of the orders of integration being justified by Fubini's theorem.

THEOREM 4. *If $\phi \in L_2(0, \infty)$, $\lambda \geqslant 0$, and*

$$f(s) = \int_0^\infty e^{-st} t^\lambda \phi(t) dt, \qquad Re\, s > 0,$$

then

$$\phi(t) = \underset{\alpha\to\infty}{\text{l.i.m.}} \frac{t^{-\lambda}}{\pi} \int_0^\infty f(s) E_\lambda(st, \alpha)\, ds,$$

where for $x > 0$,

$$E_\lambda(x, \alpha) = \int_0^\alpha Re \left\{ \frac{x^{\lambda - \frac{1}{2} + iy}}{\Gamma(\lambda + \frac{1}{2} + iy)} \right\} dy.$$

Proof. For $\lambda = 0$ the result is given in (**2**, Theorem 6). We shall deduce the result for $\lambda > 0$ from that for $\lambda = 0$. For this suppose $\lambda > 0$. Then by the lemma

$$\frac{1}{\Gamma(\lambda)} \int_s^\infty (\sigma - s)^{\lambda-1} f(\sigma)\, d\sigma$$

is the Laplace transform of ϕ, and hence

$$\phi(t) = \underset{\alpha\to\infty}{\text{l.i.m.}} \frac{1}{\pi\Gamma(\lambda)} \int_0^\infty E_0(st, \alpha)\, ds \int_s^\infty (\sigma - s)^{\lambda-1} f(\sigma)\, d\sigma. \tag{4.1}$$

But since the theorem is true for $\lambda = 0$, it follows that if $g(s)$ is the Laplace transform of a function in $L_2(0, \infty)$, then for all sufficiently large α

$$\int_0^\infty |E_0(st, \alpha) g(s)| ds < \infty.$$

Also, as in the proof of the lemma, if $s > 0$

$$\int_s^\infty (\sigma - s)^{\lambda-1} |f(\sigma)| d\sigma \leqslant \int_s^\infty (\sigma - s)^{\lambda-1} d\sigma \int_0^\infty e^{-\sigma t} t^\lambda |\phi(t)| dt$$
$$= \int_0^\infty e^{-st} |\phi(t)| dt = \tilde{g}(s),$$

and thus since $|\phi(t)| \in L_2(0, \infty)$, we have for all sufficiently large α

$$\int_0^\infty |E_0(st, \alpha)| ds \int_s^\infty (\sigma - s)^{\lambda-1} |f(\sigma)| d\sigma \leqslant \int_0^\infty |E_0(st, \alpha) \tilde{g}(s)| ds < \infty.$$

Hence by Fubini's theorem we may interchange the order of integrations in equation (4.1) and obtain

$$\phi(t) = \underset{\alpha\to\infty}{\text{l.i.m.}} \frac{1}{\pi\Gamma(\lambda)} \int_0^\infty f(\sigma) d\sigma \int_0^\sigma (\sigma - s)^{\lambda-1} E_0(st, \alpha) ds. \tag{4.2}$$

However,

$$\frac{1}{\Gamma(\lambda)} \int_0^\sigma (\sigma - s)^{\lambda-1} E_0(st, \alpha) ds$$
$$= \frac{1}{\Gamma(\lambda)} \int_0^\sigma (\sigma - s)^{\lambda-1} ds \int_0^\alpha Re\left\{\frac{(st)^{-\frac{1}{2}+iy}}{\Gamma(\frac{1}{2} + iy)}\right\} dy$$
$$= \frac{1}{\Gamma(\lambda)} \int_0^\alpha Re\left\{\frac{t^{-\frac{1}{2}+iy}}{\Gamma(\frac{1}{2} + iy)} \int_0^\sigma (\sigma - s)^{\lambda-1} s^{-\frac{1}{2}+iy} ds\right\} dy$$
$$= \frac{1}{\Gamma(\lambda)} \int_0^\alpha Re\left\{\frac{t^{-\frac{1}{2}+iy}}{\Gamma(\frac{1}{2} + iy)} \sigma^{\lambda-\frac{1}{2}+iy} B(\lambda, \tfrac{1}{2} + iy)\right\} dy$$
$$= t^{-\lambda} \int_0^\alpha Re\left\{\frac{(\sigma t)^{\lambda-\frac{1}{2}+iy}}{\Gamma(\lambda + \frac{1}{2} + iy)}\right\} dy = t^{-\lambda} E_\lambda(st, \alpha).$$

Hence (4.2) becomes

$$\phi(t) = \underset{\alpha\to\infty}{\text{l.i.m.}} \frac{t^{-\lambda}}{\pi} \int_0^\infty f(s) E_\lambda(st, \alpha) ds.$$

COROLLARY. *If* $e^{-\omega t}\phi(t) \in L_2(0, \infty)$, $\lambda \geqslant 0$, *and*

$$f(s) = \int_0^\infty e^{-st} t^\lambda \phi(t) dt, \qquad Re\, s > \omega,$$

then

$$\phi(t) = e^{\omega t} \underset{\alpha\to\infty}{\text{l.i.m.}} \frac{t^{-\lambda}}{\pi} \int_\omega^\infty f(s) E_\lambda((s - \omega)t, \alpha) ds.$$

Proof. The result follows on applying the theorem to $f(s + \omega)$, which is the Laplace transform of $t^\lambda e^{-\omega t}\phi(t)$.

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