BULL. AUSTRAL. MATH. SOC. 46C05, 26D15, 26D10
VOL. 70 (2004) [451–462]

# QUADRATIC REVERSES OF THE TRIANGLE INEQUALITY FOR BOCHNER INTEGRAL IN HILBERT SPACES

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Some quadratic reverses of the continuous triangle inequality for the Bochner integral of vector-valued functions in Hilbert spaces are given. Applications for complex-valued functions are provided as well.

## 1. INTRODUCTION

Let $f : [a,b] \to \mathbb{K}$, $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$ be a Lebesgue integrable function. The following inequality is the continuous version of the triangle inequality

$$\left| \int_a^b f(x)\, dx \right| \leqslant \int_a^b \left| f(x) \right| dx \tag{1.1}$$

and plays a fundamental role in Mathematical Analysis and its applications.

It seems, see [**6**, p. 492], that the first reverse inequality for (1.1) was obtained by Karamata in his book from 1949, [**4**]:

$$\cos\theta \int_a^b \left| f(x) \right| dx \leqslant \left| \int_a^b f(x)\, dx \right| \tag{1.2}$$

provided

$$\left| \arg f(x) \right| \leqslant \theta, \quad x \in [a,b],$$

where $\theta$ is a given angle in $(0, \pi/2)$.

This integral inequality is the continuous version of a reverse inequality for the generalised triangle inequality

$$\cos\theta \sum_{i=1}^n |z_i| \leqslant \left| \sum_{i=1}^n z_i \right|, \tag{1.3}$$

provided

$$a - \theta \leqslant \arg(z_i) \leqslant a + \theta, \quad \text{for} \quad i \in \{1, \dots, n\},$$

Received 15th June, 2004





https://doi.org/10.1017/S0004972700034699 Published online by Cambridge University Press

where $a \in \mathbb{R}$ and $\theta \in (0, \pi/2)$, which, as pointed out in [**6**, p. 492], was first discovered by Petrovich in 1917, [**7**], and, subsequently rediscovered by other authors, including Karamata [**4**, p. 300–301], Wilf [**8**], and in an equivalent form by Marden [**5**].

The first to consider the problem for sums in the more general case of Hilbert and Banach spaces, were Diaz and Metcalf [**1**].

In our previous work [**2**], we pointed out some continuous versions of Diaz and Metcalf results providing reverses of the generalised triangle inequality in Hilbert spaces. We mention here some results from [**2**] which may be compared with the new ones obtained in Sections 2 and 3 below.

We recall that $f \in L([a,b];H)$, the space of *Bochner integrable* functions defined on $[a,b]$ and with values in the Hilbert space $H$, if and only if the function $f : [a,b] \to H$ is Bochner measurable on $[a,b]$ and $\|f\|$ is Lebesgue integrable on $[a,b]$ (see for instance [**9**, pp. 132 et seq.]).

**Theorem 1.** *If $f \in L([a,b];H)$ and there exists a constant $K \geqslant 1$ and a vector $e \in H$, $\|e\| = 1$ such that*

$$\|f(t)\| \leqslant K \operatorname{Re}\langle f(t), e\rangle \quad \text{for almost all } t \in [a,b], \tag{1.4}$$

*then we have the inequality:*

$$\int_a^b \|f(t)\|\, dt \leqslant K \left\| \int_a^b f(t)\, dt \right\|. \tag{1.5}$$

*The case of equality holds in* (1.5) *if and only if*

$$\int_a^b f(t)\, dt = \frac{1}{K}\left(\int_a^b \|f(t)\|\, dt\right) e. \tag{1.6}$$

As particular cases of interest that may be applied in practice, we note the following corollaries established in [**2**].

**Corollary 1.** *Let $e$ be a unit vector in the Hilbert space $(H; \langle\cdot,\cdot\rangle)$, $\rho \in (0,1)$ and $f \in L([a,b];H)$ so that*

$$\|f(t) - e\| \leqslant \rho \quad \text{for almost every } t \in [a,b]. \tag{1.7}$$

*Then we have the inequality*

$$\sqrt{1-\rho^2} \int_a^b \|f(t)\|\, dt \leqslant \left\| \int_a^b f(t)\, dt \right\|, \tag{1.8}$$

*with equality if and only if*

$$\int_a^b f(t)\, dt = \sqrt{1-\rho^2}\left(\int_a^b \|f(t)\|\, dt\right) \cdot e. \tag{1.9}$$

**COROLLARY 2.** *Let $e$ be a unit vector in $H$ and $M \geqslant m > 0$. If $f \in L([a,b];H)$ is such that*

$$\operatorname{Re}\langle Me - f(t), f(t) - me\rangle \geqslant 0 \tag{1.10}$$

*or, equivalently,*

$$\left\| f(t) - \frac{M+m}{2} e \right\| \leqslant \frac{1}{2}(M-m) \tag{1.11}$$

*for almost every $t \in [a,b]$, then we have the inequality*

$$\frac{2\sqrt{mM}}{M+m} \int_a^b \|f(t)\|\,dt \leqslant \left\| \int_a^b f(t)\,dt \right\|, \tag{1.12}$$

*or, equivalently*

$$0 \leqslant \int_a^b \|f(t)\|\,dt - \left\| \int_a^b f(t)\,dt \right\| \leqslant \frac{(\sqrt{M}-\sqrt{m})^2}{M+m} \left\| \int_a^b f(t)\,dt \right\|. \tag{1.13}$$

*The equality holds in (1.12) (or in the second part of (1.13)) if and only if*

$$\int_a^b f(t)\,dt = \frac{2\sqrt{mM}}{M+m} \left( \int_a^b \|f(t)\|\,dt \right) e.$$

The case of additive reverse inequalities for the continuous triangle inequality has been considered in [**3**].

We recall here the following general result.

**THEOREM 2.** *If $f \in L([a,b];H)$ is such that there exists a vector $e \in H$, $\|e\| = 1$ and $k : [a,b] \to [0,\infty)$ a Lebesgue integrable function such that*

$$\|f(t)\| - \operatorname{Re}\langle f(t), e\rangle \leqslant k(t) \quad \text{for almost every } t \in [a,b], \tag{1.14}$$

*then we have the inequality:*

$$(0 \leqslant) \int_a^b \|f(t)\|\,dt - \left\| \int_a^b f(t)\,dt \right\| \leqslant \int_a^b k(t)\,dt. \tag{1.15}$$

*The equality holds in (1.15) if and only if*

$$\int_a^b \|f(t)\|\,dt \geqslant \int_a^b k(t)\,dt \tag{1.16}$$

*and*

$$\int_a^b f(t)\,dt = \left( \int_a^b \|f(t)\|\,dt - \int_a^b k(t)\,dt \right) e. \tag{1.17}$$

https://doi.org/10.1017/S0004972700034699 Published online by Cambridge University Press

This general result has some particular cases of interest that may be easily applied [3].

**Corollary 3.** *If $f \in L([a,b];H)$ is such that there exists a vector $e \in H$, $\|e\| = 1$ and $\rho \in (0,1)$ such that*

$$\|f(t) - e\| \leqslant \rho \quad \text{for almost every } t \in [a,b], \tag{1.18}$$

*then*

$$0 \leqslant \int_a^b \|f(t)\|\,dt - \left\|\int_a^b f(t)\,dt\right\| \leqslant \frac{\rho^2}{\sqrt{1-\rho^2}\left(1+\sqrt{1-\rho^2}\right)} \operatorname{Re}\left\langle \int_a^b f(t)\,dt, e\right\rangle. \tag{1.19}$$

**Corollary 4.** *If $f \in L([a,b];H)$ is such that there exists a vector $e \in H$, $\|e\| = 1$ and $M \geqslant m > 0$ such that either*

$$\operatorname{Re}\left\langle Me - f(t), f(t) - me\right\rangle \geqslant 0 \tag{1.20}$$

*or, equivalently,*

$$\left\|f(t) - \frac{M+m}{2}e\right\| \leqslant \frac{1}{2}(M-m) \tag{1.21}$$

*for almost every $t \in [a,b]$, then*

$$0 \leqslant \int_a^b \|f(t)\|\,dt - \left\|\int_a^b f(t)\,dt\right\| \leqslant \frac{(\sqrt{M}-\sqrt{m})^2}{2\sqrt{mM}} \operatorname{Re}\left\langle \int_a^b f(t)\,dt, e\right\rangle. \tag{1.22}$$

**Corollary 5.** *If $f \in L([a,b];H)$ and $r \in L_2([a,b])$, $e \in H$, $\|e\| = 1$ are such that*

$$\|f(t) - e\| \leqslant r(t) \quad \text{for almost every } t \in [a,b], \tag{1.23}$$

*then*

$$(0 \leqslant) \int_a^b \|f(t)\|\,dt - \left\|\int_a^b f(t)\,dt\right\| \leqslant \frac{1}{2}\int_a^b r^2(t)\,dt. \tag{1.24}$$

The main aim of this paper is to point out some quadratic reverses for the continuous triangle inequality, namely, upper bounds for the nonnegative difference

$$\left(\int_a^b \|f(t)\|\,dt\right)^2 - \left\|\int_a^b f(t)\,dt\right\|^2$$

under various assumptions on the functions $f \in L([a,b];H)$. Some related results are also pointed out. Applications for complex-valued functions are provided as well.

## 2. Quadratic Reverses of the Triangle Inequality

The following lemma holds.

**Lemma 1.** *Let $f \in L([a,b];H)$ be such that there exists a function $k : \Delta \subset \mathbb{R}^2 \to \mathbb{R}$, $\Delta := \{(t,s) \mid a \leqslant t \leqslant s \leqslant b\}$ with the property that $k \in L(\Delta)$ and*

$$(0 \leqslant)\|f(t)\|\|f(s)\| - \operatorname{Re}\langle f(t), f(s)\rangle \leqslant k(t,s), \tag{2.1}$$

*for almost every $(t,s) \in \Delta$. Then we have the following quadratic reverse of the continuous triangle inequality:*

$$\left(\int_a^b \|f(t)\|\,dt\right)^2 \leqslant \left\|\int_a^b f(t)\,dt\right\|^2 + 2\iint_\Delta k(t,s)\,dt\,ds. \tag{2.2}$$

*The case of equality holds in* (2.2) *if and only if it holds in* (2.1) *for almost every $(t,s) \in \Delta$.*

Proof: We observe that the following identity holds

$$\begin{aligned}(2.3)\qquad &\left(\int_a^b \|f(t)\|\,dt\right)^2 - \left\|\int_a^b f(t)\,dt\right\|^2 \\ &= \int_a^b\int_a^b \|f(t)\|\|f(s)\|\,dt\,ds - \left\langle \int_a^b f(t)\,dt, \int_a^b f(s)\,ds\right\rangle \\ &= \int_a^b\int_a^b \|f(t)\|\|f(s)\|\,dt\,ds - \int_a^b\int_a^b \operatorname{Re}\langle f(t), f(s)\rangle\,dt\,ds \\ &= \int_a^b\int_a^b \left[\|f(t)\|\|f(s)\| - \operatorname{Re}\langle f(t), f(s)\rangle\right]dt\,ds := I.\end{aligned}$$

Now, observe that for any $(t,s) \in [a,b]\times[a,b]$, we have

$$\|f(t)\|\|f(s)\| - \operatorname{Re}\langle f(t), f(s)\rangle = \|f(s)\|\|f(t)\| - \operatorname{Re}\langle f(s), f(t)\rangle$$

and thus

$$I = 2\iint_\Delta \left[\|f(t)\|\|f(s)\| - \operatorname{Re}\langle f(t), f(s)\rangle\right]dt\,ds. \tag{2.4}$$

Using the assumption (2.1), we deduce

$$\iint_\Delta \left[\|f(t)\|\|f(s)\| - \operatorname{Re}\langle f(t), f(s)\rangle\right]dt\,ds \leqslant \iint_\Delta k(t,s)\,dt\,ds,$$

and, by the identities (2.3) and (2.4), we deduce the desired inequality (2.2).

The case of equality is obvious and we omit the details. ∎

Remark 1. From (2.2) one may deduce a coarser inequality that can be useful in some applications. It is as follows:

$$(0 \leqslant)\int_a^b \|f(t)\|\,dt - \left\|\int_a^b f(t)\,dt\right\| \leqslant \sqrt{2}\left(\iint_\Delta k(t,s)\,dt\,ds\right)^{1/2}.$$

REMARK 2. If the condition (2.1) is replaced with the following refinement of the Schwarz inequality

$$(0 \leqslant) k(t,s) \leqslant \|f(t)\| \|f(s)\| - \operatorname{Re}\langle f(t), f(s)\rangle \tag{2.5}$$

for almost every $(t,s) \in \Delta$, then the following refinement of the quadratic triangle inequality is valid

$$\left(\int_a^b \|f(t)\|\,dt\right)^2 \geqslant \left\|\int_a^b f(t)\,dt\right\|^2 + 2\iint_\Delta k(t,s)\,dt\,ds \tag{2.6}$$
$$\left(\geqslant \left\|\int_a^b f(t)\,dt\right\|^2\right).$$

The equality holds in (2.6) if and only if the case of equality holds in (2.5) for almost every $(t,s) \in \Delta$.

The following result holds.

**THEOREM 3.** *Let $f \in L([a,b];H)$ be such that there exists $M \geqslant 1 \geqslant m \geqslant 0$ such that either*

$$\operatorname{Re}\langle Mf(s) - f(t), f(t) - mf(s)\rangle \geqslant 0 \quad \text{for almost every } (t,s) \in \Delta, \tag{2.7}$$

*or, equivalently,*

$$\left\|f(t) - \frac{M+m}{2} f(s)\right\| \leqslant \frac{1}{2}(M-m)\|f(s)\| \quad \text{for almost every } (t,s) \in \Delta. \tag{2.8}$$

*Then we have the inequality:*

$$\left(\int_a^b \|f(t)\|\,dt\right)^2 \leqslant \left\|\int_a^b f(t)\,dt\right\|^2 + \frac{1}{2}\cdot\frac{(M-m)^2}{M+m}\int_a^b (s-a)\|f(s)\|^2\,ds. \tag{2.9}$$

*The case of equality holds in (2.9) if and only if*

$$\|f(t)\| \|f(s)\| - \operatorname{Re}\langle f(t), f(s)\rangle = \frac{1}{4}\cdot\frac{(M-m)^2}{M+m}\|f(s)\|^2 \tag{2.10}$$

*for almost every $(t,s) \in \Delta$.*

PROOF: Firstly, observe that, in an inner product space $(H; \langle\cdot,\cdot\rangle)$ and for $x, z, Z \in H$, the following statements are equivalent

(i) $\operatorname{Re}\langle Z - x, x - z\rangle \geqslant 0$ and

(ii) $\|x - (Z+z)/2\| \leqslant \|Z - z\|/2$.

This shows that (2.7) and (2.8) are obviously equivalent.

Now, taking the square in (2.8), we get

$$\|f(t)\|^2 + \left(\frac{M+m}{2}\right)^2 \|f(s)\|^2 \leqslant 2\operatorname{Re}\left\langle f(t), \frac{M+m}{2} f(s)\right\rangle + \frac{1}{4}(M-m)^2\|f(s)\|^2,$$

https://doi.org/10.1017/S0004972700034699 Published online by Cambridge University Press

for almost every $(t,s) \in \Delta$, and obviously, since

$$2\Big(\frac{M+m}{2}\Big)\|f(t)\|\|f(s)\| \leqslant \|f(t)\|^2 + \Big(\frac{M+m}{2}\Big)^2\|f(s)\|^2,$$

we deduce that

$$2\Big(\frac{M+m}{2}\Big)\|f(t)\|\|f(s)\| \leqslant 2\operatorname{Re}\Big\langle f(t), \frac{M+m}{2}f(s)\Big\rangle + \frac{1}{4}(M-m)^2\|f(s)\|^2,$$

giving the much simpler inequality:

$$\|f(t)\|\|f(s)\| - \operatorname{Re}\langle f(t), f(s)\rangle \leqslant \frac{1}{4}\cdot\frac{(M-m)^2}{M+m}\|f(s)\|^2 \tag{2.11}$$

for almost every $(t,s) \in \Delta$.

Applying Lemma 1 for $k(t,s) := 1/4 \cdot (M-m)^2/(M+m)\|f(s)\|^2$, we deduce

$$\Big(\int_a^b \|f(t)\|\,dt\Big)^2 \leqslant \Big\|\int_a^b f(t)\,dt\Big\|^2 + \frac{1}{2}\cdot\frac{(M-m)^2}{M+m}\iint_\Delta \|f(s)\|^2\,ds \tag{2.12}$$

with equality if and only if (2.11) holds for almost every $(t,s) \in \Delta$.

Since

$$\iint_\Delta \|f(s)\|^2\,ds = \int_a^b \Big(\int_a^s \|f(s)\|^2\,dt\Big)\,ds = \int_a^b (s-a)\|f(s)\|^2\,ds,$$

then by (2.12) we deduce the desired result (2.9). □

Another result which is similar to the one above is incorporated in the following theorem.

**Theorem 4.** *With the assumptions of Theorem 3, we have*

$$\Big(\int_a^b \|f(t)\|\,dt\Big)^2 - \Big\|\int_a^b f(t)\,dt\Big\|^2 \leqslant \frac{(\sqrt{M}-\sqrt{m})^2}{2\sqrt{Mm}}\Big\|\int_a^b f(t)\,dt\Big\|^2 \tag{2.13}$$

*or, equivalently,*

$$\int_a^b \|f(t)\|\,dt \leqslant \Big(\frac{M+m}{2\sqrt{Mm}}\Big)^{1/2}\Big\|\int_a^b f(t)\,dt\Big\|. \tag{2.14}$$

*The case of equality holds in* (2.13) *or* (2.14) *if and only if*

$$\|f(t)\|\|f(s)\| = \frac{M+m}{2\sqrt{Mm}}\operatorname{Re}\langle f(t), f(s)\rangle, \tag{2.15}$$

*for almost every* $(t,s) \in \Delta$.

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PROOF: From (2.7), we deduce

$$\|f(t)\|^2 + Mm\|f(s)\|^2 \leqslant (M+m)\operatorname{Re}\langle f(t), f(s)\rangle$$

for almost every $(t,s) \in \Delta$. Dividing by $\sqrt{Mm} > 0$, we deduce

$$\frac{\|f(t)\|^2}{\sqrt{Mm}} + \sqrt{Mm}\|f(s)\|^2 \leqslant \frac{M+m}{\sqrt{Mm}}\operatorname{Re}\langle f(t), f(s)\rangle$$

and, obviously, since

$$2\|f(t)\|\|f(s)\| \leqslant \frac{\|f(t)\|^2}{\sqrt{Mm}} + \sqrt{Mm}\|f(s)\|^2,$$

hence

$$\|f(t)\|\|f(s)\| \leqslant \frac{M+m}{\sqrt{Mm}}\operatorname{Re}\langle f(t), f(s)\rangle$$

for almost every $(t,s) \in \Delta$, giving

$$\|f(t)\|\|f(s)\| - \operatorname{Re}\langle f(t), f(s)\rangle \leqslant \frac{(\sqrt{M}-\sqrt{m})^2}{2\sqrt{Mm}}\operatorname{Re}\langle f(t), f(s)\rangle.$$

Applying Lemma 1 for $k(t,s) := (\sqrt{M}-\sqrt{m})^2/\sqrt{Mm}\operatorname{Re}\langle f(t), f(s)\rangle$, we deduce

$$\left(\int_a^b \|f(t)\|\,dt\right)^2 \leqslant \left\|\int_a^b f(t)\,dt\right\|^2 + \frac{(\sqrt{M}-\sqrt{m})^2}{2\sqrt{Mm}}\operatorname{Re}\langle f(t), f(s)\rangle. \tag{2.16}$$

On the other hand, since

$$\operatorname{Re}\langle f(t), f(s)\rangle = \operatorname{Re}\langle f(s), f(t)\rangle \text{ for any } (t,s) \in [a,b]^2,$$

hence

$$\begin{aligned}\iint_\Delta \operatorname{Re}\langle f(t), f(s)\rangle\,dt\,ds &= \frac{1}{2}\int_a^b\int_a^b \operatorname{Re}\langle f(t), f(s)\rangle\,dt\,ds \\ &= \frac{1}{2}\operatorname{Re}\left\langle \int_a^b f(t)\,dt, \int_a^b f(st)\,ds\right\rangle \\ &= \frac{1}{2}\left\|\int_a^b f(t)\,dt\right\|^2\end{aligned}$$

and thus, from (2.16), we get (2.13).

The equivalence between (2.13) and (2.14) is obvious and we omit the details. ∎

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## 3. Related Results

The following result also holds.

**Theorem 5.** *Let $f \in L([a,b];H)$ and $\gamma, \Gamma \in \mathbb{R}$ be such that either*

$$\operatorname{Re}\langle \Gamma f(s) - f(t), f(t) - \gamma f(s)\rangle \geqslant 0 \quad \text{for almost every } (t,s) \in \Delta, \tag{3.1}$$

*or, equivalently,*

$$\left\| f(t) - \frac{\Gamma+\gamma}{2} f(s) \right\| \leqslant \frac{1}{2}|\Gamma - \gamma| \left\| f(s) \right\| \quad \text{for almost every } (t,s) \in \Delta. \tag{3.2}$$

*Then we have the inequality:*

$$\int_a^b \left[(b-s) + \gamma\Gamma(s-a)\right] \left\| f(s) \right\|^2 ds \leqslant \frac{\Gamma+\gamma}{2} \left\| \int_a^b f(s)\, ds \right\|^2. \tag{3.3}$$

*The case of equality holds in (3.3) if and only if the case of equality holds in either (3.1) or (3.2) for almost every $(t,s) \in \Delta$.*

Proof: The inequality (3.1) is obviously equivalent to

$$\left\| f(t) \right\|^2 + \gamma\Gamma \left\| f(s) \right\|^2 \leqslant (\Gamma+\gamma) \operatorname{Re}\langle f(t), f(s)\rangle \tag{3.4}$$

for almost every $(t,s) \in \Delta$.

Integrating (3.4) on $\Delta$, we deduce

$$\int_a^b \left( \int_a^s \left\| f(t) \right\|^2 dt \right) ds + \gamma\Gamma \int_a^b \left( \left\| f(s) \right\|^2 \int_a^s dt \right) ds \leqslant (\Gamma+\gamma) \int_a^b \left( \int_a^s \operatorname{Re}\langle f(t), f(s)\rangle\, dt \right) ds. \tag{3.5}$$

It is easy to see, on integrating by parts, that

$$\begin{aligned} \int_a^b \left( \int_a^s \left\| f(t) \right\|^2 dt \right) ds &= s \int_a^s \left\| f(t) \right\|^2 dt \Big|_a^b - \int_a^b s \left\| f(s) \right\|^2 ds \\ &= b \int_a^s \left\| f(s) \right\|^2 ds - \int_a^b s \left\| f(s) \right\|^2 ds \\ &= \int_a^b (b-s) \left\| f(s) \right\|^2 ds \end{aligned}$$

and

$$\int_a^b \left( \left\| f(s) \right\|^2 \int_a^s dt \right) ds = \int_a^b (s-a) \left\| f(s) \right\|^2 ds.$$

Since

$$\begin{aligned} \frac{d}{ds}\left( \left\| \int_a^b f(t)\, dt \right\|^2 \right) &= \frac{d}{ds}\left\langle \int_a^s f(t)\, dt, \int_a^s f(t)\, dt \right\rangle \\ &= \left\langle f(s), \int_a^s f(t)\, dt \right\rangle + \left\langle \int_a^s f(t)\, dt, f(s) \right\rangle \\ &= 2 \operatorname{Re}\left\langle \int_a^s f(t)\, dt, f(s) \right\rangle, \end{aligned}$$

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hence

$$\begin{aligned}\int_a^b\left(\int_a^s \operatorname{Re}\langle f(t), f(s)\rangle\,dt\right)ds &= \int_a^b \operatorname{Re}\left\langle \int_a^s f(t)\,dt, f(s)\right\rangle ds\\ &= \frac{1}{2}\int_a^b \frac{d}{ds}\left(\left\|\int_a^s f(t)\,dt\right\|^2\right)ds\\ &= \frac{1}{2}\left\|\int_a^b f(t)\,dt\right\|^2.\end{aligned}$$

Utilising (3.5), we deduce the desired inequality (3.3).

The case of equality is obvious and we omit the details. ∎

REMARK 3. Consider the function $\varphi(s) := (b-s)+\gamma\Gamma(s-a)$, $s\in[a,b]$. Obviously,

$$\varphi(s) = (\Gamma\gamma - 1)s + b - \gamma\Gamma a.$$

Observe that, if $\Gamma\gamma \geqslant 1$, then

$$b-a = \varphi(a) \leqslant \varphi(s) \leqslant \varphi(b) = \gamma\Gamma(b-a), \qquad s\in[a,b]$$

and, if $\Gamma\gamma < 1$, then

$$\gamma\Gamma(b-a) \leqslant \varphi(s) \leqslant b-a, \qquad s\in[a,b].$$

Taking into account the above remark, we may state the following corollary.

COROLLARY 6. *Assume that $f,\gamma,\Gamma$ are as in Theorem 5.*

(a) *If $\Gamma\gamma \geqslant 1$, then we have the inequality*

$$(b-a)\int_a^b \|f(s)\|^2\,ds \leqslant \frac{\Gamma+\gamma}{2}\left\|\int_a^b f(s)\,ds\right\|^2.$$

(b) *If $0<\Gamma\gamma<1$, then we have the inequality*

$$\gamma\Gamma(b-a)\int_a^b \|f(s)\|^2\,ds \leqslant \frac{\Gamma+\gamma}{2}\left\|\int_a^b f(s)\,ds\right\|^2.$$

## 4. Applications for Complex-Valued Functions

Let $f:[a,b]\to\mathbb{C}$ be a Lebesgue integrable function and $M\geqslant 1\geqslant m\geqslant 0$. The condition (2.7) from Theorem 3, which plays a fundamental role in the results obtained above, can be translated in this case as

$$\operatorname{Re}\Big[\big(Mf(s)-f(t)\big)\big(\overline{f(t)}-m\overline{f(s)}\big)\Big] \geqslant 0 \tag{4.1}$$

for almost every $a\leqslant t\leqslant s\leqslant b$.

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Since, obviously

$$\operatorname{Re}\left[\left(Mf(s)-f(t)\right)\left(\overline{f(t)}-m\overline{f(s)}\right)\right]=\left[\left(M\operatorname{Re}f(s)-\operatorname{Re}f(t)\right)\left(\operatorname{Re}f(t)-m\operatorname{Re}f(s)\right)\right]$$
$$+\left[\left(M\operatorname{Im}f(s)-\operatorname{Im}f(t)\right)\left(\operatorname{Im}f(t)-m\operatorname{Im}f(s)\right)\right]$$

hence a sufficient condition for the inequality in (4.1) to hold is

$$m\operatorname{Re}f(s)\leqslant\operatorname{Re}f(t)\leqslant M\operatorname{Re}f(s)\text{ and }m\operatorname{Im}f(s)\leqslant\operatorname{Im}f(t)\leqslant M\operatorname{Im}f(s)\tag{4.2}$$

for almost every $a\leqslant t\leqslant s\leqslant b$.

Utilising Theorems 3,4 and 5 we may state the following results incorporating quadratic reverses of the continuous triangle inequality:

**Proposition 1.** *With the above assumptions for $f, M$ and $m$, and if* (4.1) *holds true, then we have the inequalities*

$$\left(\int_a^b|f(t)|\,dt\right)^2\leqslant\left|\int_a^b f(t)\,dt\right|^2+\frac{1}{2}\cdot\frac{(M-m)^2}{M+m}\int_a^b(s-a)|f(s)|^2\,ds,$$
$$\int_a^b|f(t)|\,dt\leqslant\left(\frac{M+m}{2\sqrt{Mm}}\right)^{1/2}\left|\int_a^s f(t)\,dt\right|,$$

*and*

$$\int_a^b\left[(b-s)+mM(s-a)\right]|f(s)|^2\,ds\leqslant\frac{M+m}{2}\left|\int_a^s f(s)\,ds\right|^2.$$

Remark 4. One may wonder if there are functions satisfying the condition (4.2) above. It suffices to find examples of real functions $\varphi:[a,b]\to\mathbb{R}$ verifying the following double inequality

$$\gamma\varphi(s)\leqslant\varphi(t)\leqslant\Gamma\varphi(s)\tag{4.3}$$

for some given $\gamma,\Gamma$ with $0\leqslant\gamma\leqslant1\leqslant\Gamma<\infty$ for almost every $a\leqslant t\leqslant s\leqslant b$.

For this purpose, consider $\psi:[a,b]\to\mathbb{R}$ a differentiable function on $(a,b)$, continuous on $[a,b]$ and with the property that there exists $\Theta\geqslant0\geqslant\theta$ such that

$$\theta\leqslant\psi'(u)\leqslant\Theta\text{ for any }u\in(a,b).\tag{4.4}$$

By Lagrange's mean value theorem, we have, for any $a\leqslant t\leqslant s\leqslant b$

$$\psi(s)-\psi(t)=\psi'(\xi)(s-t)$$

with $t\leqslant\xi\leqslant s$. Therefore, for $a\leqslant t\leqslant s\leqslant b$, by (4.4), we have the inequality

$$\theta(b-a)\leqslant\theta(s-t)\leqslant\psi(s)-\psi(t)\leqslant\Theta(s-t)\leqslant\Theta(b-a).$$

If we choose the function $\varphi:[a,b]\to\mathbb{R}$ given by

$$\varphi(t):=\exp\left[-\psi(t)\right],\ t\in[a,b],$$

and $\gamma:=\exp\left[\theta(b-a)\right]\leqslant1$, $\Gamma:=\exp\left[\Theta(b-a)\right]\geqslant1$, then (4.3) holds true for any $a\leqslant t\leqslant s\leqslant b$.

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