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A note on the omega lemma

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A class of locally convex spaces, a $\mathcal{B}$-subfamily of finite order, is defined and the omega lemma for spaces belonging to this family is proved.

1. Introduction

Let us denote the family of all finite-dimensional euclidean spaces by $\mathcal{E}$. Let E be a member of $\mathcal{E}$ and X be an open subset of E . We denote by $\mathrm{F}(X, \mathcal{E})$, or more simply, by F , a class of topological linear spaces $\mathrm{F}(X, F)$ for all $F \in \mathrm{F}$, whose elements are maps of X into F . For example, when $\mathrm{F} = \mathrm{C}^k$, $\mathrm{C}^k(X, F)$ is the space of all C^k-maps of X into F .

Let Y be an open subset of F and consider a subset $\mathrm{F}_*(X, Y)$ of $\mathrm{F}(X, Y)$ defined by

$$\mathrm{F}_*(X, Y) = \{f \in \mathrm{F}(X, F) : \overline{f(X)} \subset Y\} .$$

Let G be another member of $\mathcal{E}$, and let

$$\phi : Y \to G$$

be a C^∞-map such that $\phi \circ f$ belongs to $\mathrm{F}(X, G)$ for every $f \in \mathrm{F}_*(X, Y)$. Then we can consider a map

$$\omega_\phi : \mathrm{F}_*(X, Y) \to \mathrm{F}(X, G) : f \to \phi \circ f .$$

The original omega lemma, proved in [1, Corollary 3.8], claims that, when $\mathrm{F} = \mathrm{C}^k$ and X has compact closure, ω_ϕ is a C^∞-map. In this case, $\mathrm{F}_*(X, Y)$ is an open subset of the Banach space $\mathrm{C}^k(X, F)$.

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When F consists of spaces whose elements are C^∞-maps, there are several results of similar type. Among these, the sharpest is the one due to Omori [6, Lemma 2.1.3]. However, from the viewpoint of the theory of differentiation in locally convex spaces, the omega lemma in [2, Satz 21] (see also [4]) has the most general form.

Fischer considers the space $B(X, F)$ of all C^∞-maps $f : X \to F$ such that

$$\|f\|_k = \sup\{|f^{(i)}(x)| : x \in X, 0 \leq i \leq k\} < +\infty$$

for all $k \geq 0$, where $f^{(i)}(x)$ denotes the ith derivative of f at x, and $|\cdot|$ denotes the norms in the spaces in E. With these norms $\{\|\cdot\|_k\}$, $B(X, F)$ is a Fréchet space. Fischer has shown that $B_*(X, Y)$ is an open subset of $B(X, F)$ and

$$\omega_\phi : B_*(X, Y) \to B(X, G)$$

is a HC_0^∞-map. In fact, $B_*(X, Y)$ is open with respect to the 0th norm $\|\cdot\|_0$ on $B(X, F)$.

The HC_0^∞-smoothness, which has been defined by Fischer in the same paper, is equivalent in this case to the C_Γ^∞-smoothness in [8] for a suitably defined calibration Γ. The aim of this note is to present an omega lemma in a more general setting. We shall consider only the Γ-smoothness; we refer to [8] for its definition and its properties.

Before proceeding further, we need to observe the fact that there are locally convex spaces consisting of C^∞-maps for which the omega lemma does not hold or holds only for a special type of the map ϕ. One of such is the space $C^\infty(X, F)$ of all C^∞-maps of X into F, equipped with the calibration consisting of

$$\|f\|_{k,K} = \sup\{|f^{(i)}(x)| : 0 \leq i \leq k, x \in K\}$$

for all $k \geq 0$ and all compact subsets K of X. In this space, which is the biggest among spaces of C^∞-maps, the subset $C_*^\infty(X, Y)$ is not necessarily an open subset of $C^\infty(X, F)$ and, therefore, it is not a suitable domain of a smooth map. The smallest among those spaces of

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C^∞-maps will be the space $\mathcal{D}(X, F)$ of C^∞-maps with compact supports with the usual inductive limit topology. In this space, the smoothness of ω_ϕ, which is regarded as a map of $\mathcal{D}_*(X, Y)$ into $\mathcal{D}(X, G)$, can be meaningfully considered only if ϕ is flat, that is

$$\phi^{(n)}(0) = 0 \quad \text{for all} \quad n \geq 0 ,$$

because $\omega_\phi(f)$ must have compact support whenever f does. An omega lemma in this space will be given in §6.

A typical example of spaces of C^∞-maps to which a general method is applicable is the space $K\{M_m\}(X, F)$ defined by Shilov [3, p. 86] in the following way. Let $\{M_m\}$ be a sequence of functions

$$M_m : X \to R \quad \text{(the reals)}$$

which take on finite or simultaneously infinite values and are continuous where they are finite. It is assumed that

$$1 \leq M_0(x) \leq M_1(x) \leq \dots .$$

Then the space $K\{M_m\}(X, F)$ is the set of all C^∞-maps $f : X \to F$ such that

$$\|f\|_{m,k} = \sup\{M_m(x)|f|_k(x) : x \in X\} < +\infty$$

for all $m \geq 0$ and $k \geq 0$, where

$$|f|_k(x) = \max\{|f^{(i)}(x)| : 0 \leq i \leq k\} .$$

The topology of this space is defined by the calibration

$$\{\|\cdot\|_{m,k} : m \geq 0, k \geq 0\} .$$

When all $M_m(x)$ are equal to a constant, it obviously coincides with $\mathcal{B}(X, F)$. Another example is the space of all rapidly decreasing C^∞-maps.

2. $\mathcal{B}$-subfamilies of finite order

We start with the family $\mathcal{B} = \mathcal{B}(X, E)$ which consists of the spaces $\mathcal{B}(X, F)$ for all $F \in E$. This family has the calibration $\Gamma(\mathcal{B})$, which consists of countable semi-norm maps p_k for $k = 0, 1, 2, \dots$; the

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value of p_k at a member of $\mathcal{B}$ is the kth norm $\|\cdot\|_k$ defined in the previous section. In other words, this is a calibration with "identical components" (see [8, Appendix]).

Now let $\Gamma(\mathsf{F})$ be a set of semi-norm maps on $\mathcal{B}$ with the additional assumption that elements of $\Gamma(\mathsf{F})$ may take the value $+\infty$ on $\mathcal{B}$. For this $\Gamma(\mathsf{F})$, we define for each $F \in E$ a locally convex space $\mathsf{F}(X, F)$ by

$$\mathsf{F}(X, F) = \{f \in \mathcal{B}(X, F) : p(f) < +\infty \text{ for all } p \in \Gamma(\mathsf{F})\} ,$$

where $p(f)$ denotes the value of the $\mathcal{B}(X, F)$-component of p at f. Then we can define a $\Gamma(\mathsf{F})$-family $\mathsf{F} = \mathsf{F}(X, E)$ as the totality of all $\mathsf{F}(X, F)$ for $F \in E$.

A family F defined from $\mathcal{B}$ in this way is called a *$\mathcal{B}$-subfamily* if

$$p(f) \geq \|f\|_0$$

for every $p \in \Gamma(\mathsf{F})$, $f \in \mathsf{F}(X, F)$, and $F \in E$. This last condition ensures that $\mathsf{F}_*(X, Y)$ is an open subset of $\mathsf{F}(X, F)$ for every open subset Y of F, because $\mathcal{B}_*(X, Y)$ is already $\|\cdot\|_0$-open in $\mathcal{B}(X, F)$.

The family $K\{M_m\}$ in the previous section is obviously a $\mathcal{B}$-subfamily. As we shall show in §6, a calibration can be defined for the family $\mathcal{D} = \mathcal{D}(X, E)$ so that it becomes a $\mathcal{B}$-subfamily, but the family $C^\infty(X, E)$ is evidently not a $\mathcal{B}$-subfamily.

An essential difference between the families $K\{M_m\}$ and $\mathcal{D}$ is that $K\{M_m\}$ is of finite order in the following sense. A $\mathcal{B}$-subfamily $\mathsf{F}(X, E)$ is said to be *of finite order* if, for any $p \in \Gamma(\mathsf{F})$, there exists $k \geq 0$, which is called *the order of* p, such that, for each $F \in E$,

(2.1) $p(f) \geq \|f\|_k$ for all $f \in \mathsf{F}(X, F)$,

(2.2) for $f, g \in \mathsf{F}(X, F)$, if

$$|f|_k(x) \leq \alpha_1 |g|_k(x) + \alpha_2 |g|_k(x)^2 + \ldots + \alpha_n |g|_k(x)^n$$

for some $\alpha_i \geq 0$ and every $x \in X$, then

$$p(f) \leq \alpha_1 p(g) + \alpha_2 p(g)^2 + \ldots + \alpha_n p(g)^n .$$

The family $K\{M_m\}$ has the calibration consisting of semi-norm maps $p_{m,k}$ for $m, k = 0, 1, 2, \ldots$, whose components are $\|\cdot\|_{m,k}$ defined in the previous section. The order of $p_{m,k}$ is obviously k. The family $\mathcal{D}$ can not be of finite order.

Except for §6, we shall only be concerned with the $\mathcal{B}$-subfamilies which are of finite order.

3. Some inequalities

Let E, F, and G be members of $\mathcal{E}$, and let X and Y be open subsets of E and F respectively. We take a C^∞-map

$$\phi : Y \to G .$$

For positive integers m and n such that $1 \le n \le m$, we define *the Faa-di-Bruno constants* $\beta(m, n)$ by $\beta(m, 1) = \beta(m, m) = 1$ and

$$\beta(m, n) = \beta(m-1, n-1) + n\beta(m-1, n) .$$

These are coefficients in the Faa-di-Bruno formula (see [7, 1.8.3]). Then, for C^∞-maps

$$f, g : X \to Y ,$$

we have the following inequalities:

(3.1) $$\left|(\phi \circ f)^{(m)}(x)\right| \le |\phi|_m\big(f(x)\big)\left\{\sum_{n=1}^{m} \beta(m, n)|f|_m(x)^n\right\} ;$$

(3.2) $$\left|\big(\phi \circ (f+g) - \phi \circ f\big)^{(m)}(x)\right| \le \sum_{n=1}^{m} \beta(m, n)\Bigg[\left|\phi^{(n)} \circ (f+g) - \phi^{(n)} \circ f\right|_0(x)|f|_m(x)^n + \left|\phi^{(n)} \circ (f+g)\right|_0(x) \sum_{r=0}^{n-1} \binom{n}{r} |f|_m(x)^r |g|_m(x)^{n-r}\Bigg] .$$

(3.3) *For*

$$r\big(\phi^{(n)}, f, g\big) = \phi^{(n)} \circ (f+g) - \phi^{(n)} \circ f - \big(\phi^{(n+1)} \circ f\big) \times g ,$$

we have

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$$\begin{aligned}&\left|\left(r(\phi,\ f,\ g)^{(m)}(x)\right)\right| \\ &\quad \le |r(\phi',\ f,\ g)|_0(x)|f|_m(x) + |\phi' \circ (f+g) - \phi' \circ f|_0(x)|g|_m(x) \\ &\qquad + \sum_{n=2}^{m}\Big[\beta(m,\ n)|r(\phi^{(n)},\ f,\ g)|_0(x)|f|_m(x)^n \\ &\qquad + \beta(m,\ n)|\phi|_m\big(f(x)+g(x)\big)\sum_{r=0}^{n-2}\binom{n}{r}|f|_m(x)^r|g|_m(x)^{n-r} \\ &\qquad + \sum_{r=n-1}^{m-1}\binom{m}{r}\beta(r,\ n-1)|\phi^{(n)} \circ (f+g) - \phi^{(n)} \circ f|_0(x)|f|_m(x)^{n-1}|g|_m(x)\Big] .\end{aligned}$$

In the above, we used the notation

$$(\psi \times g)(x) = \psi(x)[g(x)]$$

for $\psi : X \to L(F,\ G)$ and $g : X \to F$. For this operation, the Leibnitz formula gives the following inequality:

$$(3.4) \qquad |\psi \times g|_m(x) \le 2^m|\psi|_m(x)|f|_m(x) .$$

Here we add two simple consequences of (3.1) and (3.4). We denote by $L^n(F,\ G)$ the space $L\big(F,\ L\big(F,\ \ldots,\ L(F,\ G),\ \ldots\big)\big)$ where F appears n times. Let F be a $\mathcal{B}$-subfamily of finite order.

(3.5) *If* $f \in \mathsf{F}_*(X,\ Y)$, *then* $\phi^{(n)} \circ f \in \mathsf{F}\big(X,\ L^n(F,\ G)\big)$.

Proof. We only need to prove the case when $n = 0$. First we note that, for any $k \ge 0$,

$$\gamma_k(\phi,\ f) = \sup\left\{|\phi|_k\big(f(x)\big) : x \in X\right\} < +\infty ,$$

because $\overline{f(X)}$ is compact. Hence $\phi \circ f \in \mathcal{B}(X,\ G)$ by (3.1). Now let $p \in \Gamma(F)$, and let k be the order of p. Then, by (3.1) and (2.2), we have

$$p(\phi \circ f) = \gamma_k(\phi,\ f)\left\{\sum_{i=1}^{k}\beta(k,\ i)p(f)^i\right\} < +\infty ,$$

which shows that $\phi \circ f$ belongs to $\mathsf{F}(X,\ G)$.

This fact implies in particular that *the* $\mathcal{B}$*-subfamilies of finite order are closed under composition.* The following fact shows that *the* $\mathcal{B}$*-subfamilies of finite order are also closed under products.*

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(3.6) *If* $\psi \in F(X, L(F, G))$ *and* $g \in F(X, F)$ *, then* $\psi \times g \in F(X, G)$ *and, for each* $p \in \Gamma(F)$ *,*

$$p(\psi \times g) \le 2^k p(\psi)p(g)$$

for the order k *of* p *.*

Proof. From (3.4), we have

$$|\psi \times g|_k(x) \le 2^k |\psi|_k(x)\,|g|_k(x) ,$$

and, by (2.1), we have $|\psi|_k(x) \le p(\psi)$. Hence, by (2.2), we have the desired inequality.

4. Γ-smoothness of ω_ϕ

Let $F(X, E)$ be a B-subfamily of finite order, where X is an open convex subset of a space E in E . Let $F, G \in E$, and let Y be a convex open subset of F . The assumption of convexity is required to accommodate the mean-value theorem (see [7, 1.1.3]). Let

$$\phi : Y \to G$$

be a C^∞-map. Then, by (3.5), it is meaningful to consider the $\Gamma(F)$-smoothness of ω_ϕ .

(4.1) ω_ϕ *is of class* $C^\infty_{\Gamma(F)}$ *and* $\omega_\phi^{(n)} = \omega_{\phi^{(n)}}$.

The remainder of this section is devoted to the proof of (4.1). First, we prove that the map $\omega_{\phi'}$, a candidate for the derivative of ω_ϕ , is $\Gamma(F)$-continuous.

(4.2) *For each* $f \in F_*(X, Y)$ *, the linear map*

$$\omega_{\phi'}(f) : F(X, F) \to F(X, G) : g \to (\phi' \circ f) \times g$$

is $\Gamma(F)$*-continuous.*

Proof. Let $p \in \Gamma(F)$, and let k be its order. Then it follows from (3.6) that

$$p\big(\omega_{\phi'}(f)(g)\big) \le 2^k p(\phi' \circ f)p(g)$$

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for every $g \in F(X, F)$, where $p(\phi' \circ f) < +\infty$ by (3.5). This shows the $\Gamma(F)$-continuity of $\omega_{\phi'}(f)$.

The essential part of the remainder of the proof of (4.1) are the following two facts, which can be derived from (3.2) and (3.3) respectively. The fact that $\overline{f(X)}$ is a compact subset of F is an indispensable condition here.

(4.3) *Assume that* $g_n \in F(X, F)$, $x_n \in X$, *and*

$$\lim_{n\to\infty} |g_n|_k(x_n) = 0$$

for some $k \geq 0$. *Then, for any* $f \in F_*(X, Y)$

$$\lim_{n\to\infty} |\omega_{\phi'}(f+g_n)-\omega_{\phi'}(f)|_k(x_n) = 0 .$$

(4.4) *Let* $f \in F_*(X, Y)$, $g_n \in F(X, F)$, *and* $x_n \in X$. *Assume that*

$$\lim_{n\to\infty} |g_n|_k(x_n) = 0$$

for some $k \geq 0$ *and* $|g_n|_k(x_n) \neq 0$ *for all* $n \geq 1$. *Then*

$$\lim_{n\to\infty} |g_n|_k(x_n)^{-1}|\omega_\phi(f+g_n)-\omega_\phi(f)-\omega_{\phi'}(f)(g_n)|_k(x_n) = 0 .$$

Now we show that the $\Gamma(F)$-continuous linear map $\omega_{\phi'}(f)$ is the derivative of ω_ϕ at f .

(4.5) ω_ϕ *is* $\Gamma(F)$-*differentiable on* $F_*(X, Y)$ *and* $\omega_{\phi'}(f)$ *is its* $\Gamma(F)$-*derivative at* $f \in F_*(X, Y)$.

Proof. Assume that $\omega_{\phi'}(f)$ is not the $\Gamma(F)$-derivative of ω_ϕ at $f \in F_*(X, Y)$. Then there exist $p \in \Gamma(F)$, $\varepsilon > 0$, and $g_n \in F(X, F)$ such that $\lim_{n\to\infty} p(g_n) = 0$ and

$$p[\omega_\phi(f+g_n)-\omega_\phi(f)-\omega_{\phi'}(f)(g_n)] > \varepsilon p(g_n)$$

for all $n \geq 1$. Let k be the order of p . Then it follows from (2.2) that there exist $x_n \in X$ such that

$$|\omega_\phi(f+g_n)-\omega_\phi(f)-\omega_{\phi'}(f)(g_n)|_k(x_n) > \varepsilon|g_n|_k(x_n) .$$

Furthermore, by (2.1), we have

$$\lim_{n\to\infty} |g_n|_k(x_n) = 0 ,$$

which is impossible by (4.4).

If we replace ϕ by $\phi^{(n)}$, the above argument gives that $\omega_{\phi^{(n)}}$ is $\Gamma(F)$-differentiable at each $f \in F_*(X, Y)$ with $\omega_{\phi^{(n+1)}}(f)$ as its derivative. This means that ω_ϕ is of class $C^\infty_{\Gamma(F)}$ and $\omega_\phi^{(n)}(f) = \omega_{\phi^{(n)}}(f)$ for all $n \geq 1$.

5. Completional Γ-smoothness of ω_ϕ

We use the same notation as in the previous section. The map $\omega_\phi : F_*(X, Y) \to F(X, G)$ is *completionally* $\Gamma(F)$-*continuous* if, for each $p \in \Gamma(F)$, the following condition is satisfied: *if* $\{f_n\}$ *and* $\{g_n\}$ *are two p-Cauchy sequences in* $F_*(X, Y)$ *such that* $\lim_{n\to\infty} p(f_n-g_n) = 0$, *then* $\lim_{n\to\infty} p(\omega_\phi(f_n)-\omega_\phi(g_n)) = 0$.

If ω_ϕ is of class $C^\infty_{\Gamma(F)}$ and each derivative

$$\omega_\phi^{(n)} : F_*(X, Y) \to F(X, L^n(F, G))$$

is completionally $\Gamma(F)$-continuous, then ω_ϕ is said to be *of class* $CC^\infty_{\Gamma(F)}$. This notion was introduced in [9] in order to describe the smoothness used by Omori [6] in terms of the Γ-differentiation.

The aim of this section is to prove the following fact.

(5.1) *If* $\phi : Y \to G$ *is of class* CC^∞, *then* ω_ϕ *is of class* $CC^\infty_{\Gamma(F)}$.

However, since ω_ϕ is already a $C^\infty_{\Gamma(F)}$-map, this is an immediate

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consequence of the following fact.

(5.2) *Let* $\phi : Y \to G$ *be a* CC^{∞}*-map. Then, for any* $p \in \Gamma(F)$ *, if* $\{f_n\}$ *and* $\{g_n\}$ *are p-Cauchy sequences in* $F_*(X, Y)$ *, there exists a positive constant* γ *such that*

$$p\left(\omega_{\phi}(f_n)-\omega_{\phi}(g_n)\right) \le \gamma p\left(f_n-g_n\right) .$$

Proof. Let $p \in \Gamma(F)$ and k be its order. From (2.2) with $n = 1$, it suffices to show that, under the above assumptions, there exists $\gamma > 0$ such that

$$\left|\omega_{\phi}(f_n)-\omega_{\phi}(g_n)\right|_k(x) \le \gamma\left|f_n-g_n\right|_k(x)$$

for all $x \in X$. But this is an immediate consequence of (3.2), if we use the following fact.

(5.3) *If* $\{z_n\}$ *is a bounded sequence in* Y *, then*

$$\sup\left\{\left|\phi^{(i)}(z_n)\right| : n \ge 1\right\} < +\infty$$

for all $i \ge 0$.

This follows from the fact that a completionally continuous map transforms a Cauchy sequence to a Cauchy sequence.

6. The family $\mathcal{D}(X, E)$

When a $\mathcal{B}$-subfamily is not of finite order, the omega lemma will take a more complex form. As an example, we shall consider the case of the family $\mathcal{D}(X, E)$, where $\mathcal{D}(X, F)$ for $F \in E$ is the space of all C^{∞}-maps with compact supports of X into F, equipped with the usual inductive limit topology. The calibration $\Gamma(\mathcal{D})$ for this family was given in [8] in the following way. First, we take and fix a sequence $\{K_k : k = 0, 1, 2, \ldots\}$ of compact subsets of X such that $K_0 = \emptyset$, $K_k \subset K_{k+1}^0$ (the interior of K_{k+1}), and every compact subset of X is contained in some K_k. Let $\alpha = \{\alpha_k\}$ and $m = \{m_k\}$ be increasing sequences of positive numbers and non-negative integers respectively, and let us define a semi-norm map $p_{\alpha,m}$ by

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$$p_{\alpha,m}(f) = \sup_{k\geq 1} \sup\{\alpha_k |f|_{m_k}(x) : x \in K_{k-1}\}$$

for all $f \in \mathcal{D}(X, F)$ and $F \in E$. The calibration $\Gamma(\mathcal{D})$ consists of all these $p_{\alpha,m}$ for all such sequences α and m.

The notion of *gradings* of a calibration Γ has been introduced in [9]. A grading of Γ is a sequence $\sigma = \{\sigma_n : n = 0, 1, 2, \ldots\}$ of maps

$$\sigma_n : \Gamma \to \Gamma$$

such that

$$\sigma_{n+1}(p) \geq \sigma_n(p) \quad \text{and} \quad \sigma_0(p) = p$$

for all $p \in \Gamma$. The notion of σ-*smooth maps* has also been given in [9] in order to describe the smoothness of the product operations in some groups of C^∞-diffeomorphisms. It is easy to see that every σ-smooth map is a C^∞_Δ-map in the sense of Keller [5, p. 109].

The following fact is the main result of this section. Let E, F, G, X, and Y be as in the previous section, and assume that $0 \in Y$.

(6.1) *For any flat* C^∞*-map* $\phi : Y \to G$, *there exists a calibration* $\Gamma(\phi)$ *for* $\mathcal{D}(X, E)$ *and a grading* $\sigma(\phi)$ *of* $\Gamma(\phi)$ *such that*

$$\omega_\phi : \mathcal{D}_*(X, Y) \to \mathcal{D}(X, G)$$

is a $\sigma(\phi)$*-smooth map.*

We recall the fact that the conditions that $0 \in Y$ and ϕ is flat, that is $\phi^{(n)}(0) = 0$ for all $n \geq 0$, are indispensable.

To prove (6.1), we should first determine the calibration $\Gamma(\phi)$ and the grading $\sigma(\phi)$. Since

$$x \mapsto |\phi|_m(x)$$

is continuous and $|\phi|_m(0) = 0$, there is an increasing sequence $\{\alpha(m)\}$ of positive numbers such that

$$(6.2) \qquad |\phi|_m(x) \leq 1 \quad \text{if} \quad |x| \leq 1/\alpha(m)$$

and

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(6.3) $$2\beta(m) \le \alpha(m) ,$$

where

$$\beta(m) = \sum_{n=1}^{m} \beta(m, n) ,$$

where $\beta(m, n)$ is the Faa-di-Bruno constant defined in §3. It is easy to see that

$$\beta(m) \ge 2^{m-1} \quad \text{for all} \quad m \ge 1 .$$

Now we define $\Gamma(\phi)$ by

$$\Gamma(\phi) = \{p_{\alpha,m} \in \Gamma(\mathcal{D}) : \alpha_k \ge \alpha(m_k) \text{ for all } k \ge 1\} ,$$

which is obviously a calibration for $\mathcal{D}(X, E)$. The grading $\sigma = \{\sigma_n\}$ is defined by the following relations:

$$\sigma_n(p_{\alpha,m}) = p_{\alpha^{(n)},m^{(n)}} ,$$

where

(6.4) $$\alpha^{(n)} = \{\alpha_k + \alpha(m_k+n) - \alpha(m_k)\}$$

and

(6.5) $$m^{(n)} = \{m_k+n\} .$$

As in [7], let $F(X, F)[p]$ be the space $F(X, F)$ regarded as a semi-normed space with respect to a semi-norm p, and let $F_*(X, Y)[p]$ be the set $F_*(X, Y)$ regarded as a subset of $F(X, F)[p]$. Then the map ω_ϕ is $\sigma(\phi)$-smooth if and only if, for each $n \ge 0$ and each $p_{\alpha,m} \in \Gamma(\phi)$, the map ω_ϕ is a C^n-map of $\mathcal{D}_*(X, Y)[\sigma_n(p_{\alpha,m})]$ into $\mathcal{D}(X, G)[p_{\alpha,m}]$.

We start the proof of (6.1) with the following two simple facts.

(6.6) *The map*

$$\omega_\phi : \mathcal{D}_*(X, Y) \to \mathcal{D}(X, G)$$

is infinitely many time Gâteaux-differentiable and, if we denote the n*th*

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Gâteaux-derivative of ω_ϕ *by* $\omega_\phi^{(n)}$, *we have* $\omega_\phi^{(n)} = \omega_{\phi^{(n)}}$.

(6.7) *For each* $f \in \mathcal{D}_*(X, Y)$, *the map* $\omega_\phi^{(n)}(f)$ *is a* $\Gamma(\phi)$*-continuous* n*-linear map of* $\mathcal{D}(X, F)$ *into* $\mathcal{D}(X, G)$.

(6.6) is equivalent to

$$\lim_{i\to\infty} \varepsilon_i^{-1}\left[\omega_{\phi^{(n)}}\left(f+\varepsilon_i g\right)-\omega_{\phi^{(n)}}(f)\right] = \omega_{\phi^{(n+1)}}(f) \times g$$

for each $n \geq 0$ if $\varepsilon_i \to 0$. The limit is in the sense of the usual inductive limit topology; the left-hand side converges uniformly on the compact set that is the union of the supports of f and g .

(6.7) is implied by the following fact, because $\psi = \phi^{(n)} \circ g$ has compact support.

(6.8) *Let* X, Y, E , *and* F *be as above, and let* G *be an arbitrary member of* E . *Assume that* $\psi \in \mathcal{D}(X, L(F, G))$. *Then, for the map*

$$u_\psi : \mathcal{D}(X, F) \to \mathcal{D}(X, G) : g \mapsto \psi \times g$$

and $p_{\alpha,m} \in \Gamma(\phi)$, *we have*

$$p_{\alpha,m}\left(u_\psi(g)\right) \leq p_{\alpha,m}(\psi)p_{\alpha,m}(g) .$$

The proof is a simple application of the Leibnitz formula and the relation $\alpha_k \geq 2^{m_k}$.

It follows from (6.6) and (6.7) that the map

$$\omega_\phi : \mathcal{D}_*(X, Y)\left[\sigma_n\left(p_{\alpha,m}\right)\right] \to \mathcal{D}(X, G)\left[p_{\alpha,m}\right]$$

is infinitely Gâteaux-differentiable and its nth derivative is a continuous n-linear map. If we denote the norm of this n-linear map $\omega_\phi^{(n)}(f)$ by $\left\|\omega_\phi^{(n)}(f)\right\|_{\alpha,m}$, then (6.8) means that

$$\left\|\omega_\phi^{(n)}(f)\right\|_{\alpha,m} \leq p_{\alpha,m}\left(\phi^{(n)} \circ f\right) .$$

Therefore, the proof of (6.1) is completed when the following fact is proved.

(6.9) *Let* E, F, G, X, *and* Y *be as above. Then, for any flat* C^∞-*map* $\phi : Y \to G$, *the map*

$$\omega_{\phi^{(n)}} : \mathcal{D}_*(X, Y)\left[\sigma_n\left(p_{\alpha,m}\right)\right] \to \mathcal{D}\left(X, L^n(F, G)\right)\left[p_{\alpha,m}\right]$$

is continuous for each $n \geq 0$.

Proof. Assume that $\sigma_n\left(p_{\alpha,m}\right)\left(g_i\right) \to 0$ as $i \to \infty$. Let $f \in \mathcal{D}_*(X, Y)$ and its support be contained in K_{k_0}. Then

$$p_{\alpha,m}\left[\omega_{\phi^{(n)}}\left(f+g_i\right)-\omega_{\phi^{(n)}}(f)\right]$$

$$= \max_{0\leq k\leq k_0} \sup\left\{\alpha_k\left|\phi^{(n)} \circ \left(f+g_i\right)-\phi^{(n)} \circ f\right|_{m_k}(x) : x \in K_{k_0} \backslash K_{k-1}\right\}$$

$$+ \sup_{k\geq k_0} \left\{\alpha_k\left|\phi^{(n)} \circ g_i\right|_{m_k}(x) : x \notin K_{k-1}\right\}$$

The second line converges to zero as $i \to \infty$, because

$$\left|g_i\right|_{m_k}(x) \leq p_{\alpha,m}\left(g_i\right)/\alpha_k \quad \text{if} \quad x \notin K_{k-1}$$

and hence the inside of the brackets $\{\ \}$ converges to zero uniformly in the compact set $K_{k_0} \backslash K_{k-1}^0$ for each k. As to the third line, assume that $k \geq k_0$, $x \notin K_{k-1}$, and i is large. Then, by (3.1),

$$\alpha_k\left|\phi^{(n)} \circ g_i\right|_{m_k}(x) \leq \left|\phi^{(n)}\right|_{m_k}\left(g_i(x)\right)\left(\sum_{j=1}^{m_k} \left(\beta\left(m_k, j\right)/\alpha_k^{j-1}\right)\left(\alpha_k\left|g_i\right|_{m_k}(x)^j\right)\right)$$

$$\leq 2p_{\alpha,m}\left(g_i\right) ,$$

because it follows from (6.2) and (6.5) that

$$\left|\phi^{(n)}\right|_{m_k}\left(g_i(x)\right) \leq \left|\phi\right|_{m_k+n}\left(g_i(x)\right) \leq 1 ,$$

since

$$\left|g_i(x)\right| \leq 1/\alpha\left(m_k+n\right)$$

and (6.3) implies

$$\sum_{j=1}^{m_k} \left(\beta(m_k, j)/\alpha_k^{j-1}\right) \leq \beta(m_k, 1) + \sum_{j=2}^{m_k} \left(\beta(m_k, j)/\alpha_k\right) \leq 2 .$$

References

[1] R. Abraham, *Lectures of Smale on differential topology* (Columbia University, New York, 1962).

[2] H.R. Fischer, "Differentialrechnung in lokalkonvexen Räumen und Mannigfaltigkeiten von Abbildungen" (Manuskripte d. Fakultät für Math. und Informatik, Univ. Mannheim, Mannheim [1977]).

[3] I.M. Gel'fand and G.E. Shilov, *Generalized functions*. Volume 2. *Spaces of fundamental and generalized functions* (translated by Morris D. Friedman, Amiel Feinstein, Christian P. Peltzer. Academic Press, New York and London, 1968).

[4] Jürg Gutknecht, "Die C_Γ^∞-Struktur auf der Diffeomorphismengruppe einer kompakten Mannigfaltigkeit" (Doctoral Dissertation, Eidgenössische Technische Hochschule, Zürich, 1977).

[5] H.H. Keller, *Differential calculus in locally convex spaces* (Lecture Notes in Mathematics, 417. Springer-Verlag, Berlin, Heidelberg, New York, 1974).

[6] Hideki Omori, *Infinite dimensional Lie transformation groups* (Lecture Notes in Mathematics, 427. Springer-Verlag, Berlin, Heidelberg, New York, 1974).

[7] Sadayuki Yamamuro, *Differential calculus in topological linear spaces* (Lecture Notes in Mathematics, 374. Springer-Verlag, Berlin, Heidelberg, New York, 1974).

[8] Sadayuki Yamamuro, *A theory of differentiation in locally convex spaces* (Memoirs of the American Mathematical Society, 212. American Mathematical Society, Providence, Rhode Island, 1979).

[9] Sadayuki Yamamuro, "A note on Omori-Lie groups", *Bull. Austral. Math. Soc.* 19 (1978), 333-349 (1979).

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