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COUNTEREXAMPLES IN INTERSECTIONS FOR C^*-TENSOR PRODUCTS

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Let A and B be C^*-algebras and $A \otimes B$ denote the minimal C^*-tensor product of A and B. T. Huruya [1] gave examples of C^*-tensor products $A \otimes B$ with C^*-subalgebras $A_1 \otimes B_1$ and $A_2 \otimes B_2$ such that $(A_1 \otimes B_1) \cap (A_2 \otimes B_2)$ strictly contains $(A_1 \cap A_2) \otimes (B_1 \cap B_2)$, answering a question of S. Wassermann [3, Remark 23]. In this short note, we show that the same situation can occur even if $A_1 = A_2$.

Let B_0 be a C^*-algebra with a C^*-subalgebra B. Then, the *Fubini product* $F(A, B, A \otimes B_0)$ *of* A *and* B *with respect to* $A \otimes B_0$ is defined by $F(A, B, A \otimes B_0) = \{x \in A \otimes B_0;\ R_f(x) \in B \text{ for each } f \in A^*\}$, where $R_f : A \otimes B_0 \to B_0$ is the uniquely defined bounded linear map satisfying $R_f(a \otimes b) = f(a)b$ for $a \in A$ and $b \in B_0$[2, 3].

Theorem. *Let B be an injective von Neumann algebra which is canonically embedded in its enveloping von Neumann algebra B^{**} and $\bar{J}$ the weak closure for a closed two-sided ideal J in B. Assume that $F(A, J, A \otimes B^{**}) \neq A \otimes J$. Then, we have*

$$A \otimes (B \cap \bar{J}) \subsetneq (A \otimes B) \cap (A \otimes \bar{J}).$$

Proof. Note that there exist projections of norm one from B^{**} onto B and $\bar{J}$ as in [1]. Hence, we have $A \otimes B = F(A, B, A \otimes B^{**})$ and $A \otimes \bar{J} = F(A, \bar{J}, A \otimes B^{**})$ by [2, 3.7]. But, since $B \cap \bar{J} = J$, we obtain:

$$\begin{aligned} A \otimes (B \cap \bar{J}) &= A \otimes J \\ &\subsetneq F(A, J, A \otimes B^{**}) \\ &= F(A, B \cap \bar{J}, A \otimes B^{**}) \\ &= F(A, B, A \otimes B^{**}) \cap (A, \bar{J}, A \otimes B^{**}) \\ &= (A \otimes B) \cap (A \otimes \bar{J}). \end{aligned}$$

S. Wassermann [4, 5] gave two examples of triples (A, B, J) of C^*-algebras satisfying the assumption of the Theorem. For example, we have the formula

$$B(H) \otimes (B(H) \cap \overline{K(H)}) \subsetneq (B(H) \otimes B(H)) \cap (B(H) \otimes \overline{K(H)})$$

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as a required counterexample, where $B(H)$ (resp. $K(H)$) denotes the C^*-algebra of all bounded linear (resp. compact) operators on an infinite dimensional separable Hilbert space H.

A C^*-algebra A is said to have *property S* if $A\otimes D=F(A,D,A\otimes B)$ for any pair (B,D) of C^*-algebras with $D\subset B$. Note that if A has property S then such a pathology does not occur. Indeed, we have

$$\begin{aligned}A\otimes(B_1\cap B_2)&=F(A,B_1\cap B_2,A\otimes B)\\&=F(A,B_1,A\otimes B)\cap F(A,B_2\otimes B)\\&=(A\otimes B_1)\cap(A\otimes B_2),\end{aligned}$$

for any C^*-subalgebras B_1 and B_2 of a C^*-algebra B.

Added in proof. Professor C. J. K. Batty has pointed out that our Theorem implies

$$A\otimes(B(H)\cap\overline{K(H)})\subsetneq(A\otimes B(H))\cap(A\otimes\overline{K(H)})$$

whenever A is not an exact C^*-algebra, by the work of E. Kirchberg [*J. Operator Theory* **10** (1983), 3–8].

REFERENCES

1. T. Huruya, An intersection result for tensor products of C^*-algebras, *Proc. Amer. Math. Soc.* **75** (1979), 186–187.

2. J. Tomiyama, Tensor products and approximation problems of C^*-algebras, *Publ. Res. Inst. Math. Sci.* **11** (1975), 163–183.

3. S. Wassermann, The slice map problem for C^*-algebras, *Proc. London Math. Soc.* (3) **32** (1976), 537–559.

4. S. Wassermann, On tensor products of certain group C^*-algebras, *J. Funct. Anal.* **23** (1976), 239–254.

5. S. Wassermann, A pathology in the ideal space of $L(H)\otimes L(H)$, *Indiana Univ. Math. J.* **27** (1978), 1011–1020.

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