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LIMIT CIRCLE CRITERIA FOR FOURTH ORDER DIFFERENTIAL OPERATORS WITH AN OSCILLATORY COEFFICIENT

by RONALD I. BECKER

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1. Introduction

A formally self-adjoint operator L is said to be of *limit circle type at infinity* if its highest order coefficients are zero-free and all solutions x of $L(x)=0$ are square-integrable on $[c,\infty)$ for some c. (We will drop "at infinity" in what follows.)

This paper deals with the perturbation of limit circle operators by operators whose coefficients are oscillatory. We deal mainly with the fourth order case, but we will also obtain some results for second order operators.

For the equation

$$x''+(t^{2m}+ct^n\sin dt^p)x=0 \tag{1.1}$$

Eastham (**5**) proves that for c real, $d=1$, $m>1$, $p>0$ and

$$n<\min(p+\tfrac{1}{4}m-\tfrac{5}{4}, 2p-\tfrac{3}{2}m-\tfrac{5}{2}) \tag{1.2}$$

we have (1.1) limit circle. Eastham establishes this without obtaining the asymptotic expansions of (1.1).

Atkinson (**1**) using asymptotic methods proves that (1.1) is limit circle if one of the following conditions holds:

$$n<m-1 \quad \text{and} \quad m>2 \tag{1.3}$$

or

$$d\neq\frac{2}{m+1}, \quad n<m+\tfrac{1}{2}p-1 \quad \text{and} \quad m>2 \tag{1.4}$$

or

$$m>-1, \quad d=\frac{2}{m+1}, \quad n=m-1 \quad \text{and} \quad 0\leqq\tau<2$$

where

$$\tau=\frac{c}{m+1}. \tag{1.5}$$

Atkinson (**1**) (Theorems 6 and 7) also considers the more general equation

$$x''+(q^2+qg+k)x=0$$

where g satisfies certain conditional integrability conditions and $kq^{-1}\in L^1[c,\infty)$. The result (1.5) is a specialization of one of these results. Note that the results (1.2) overlap with (1.3) and (1.4).

In this paper we extend the results of Atkinson to the equation

$$(r^2x')'+(q^2+rq(g+k))x=0 \tag{1.6}$$

where g is oscillatory and $k\in L^1[c,\infty)$. For this we use a result of Becker (**2**) (stated below as Theorem 1 of Section 2) which gives the asymptotic expansions of solutions of (1.6). The results are given in Examples 1 and 2 of Section 4.

The main purpose of this paper is to develop an analogous theory for the fourth order equation

$$(r^4x^{(2)})^{(2)}+((\alpha r^2q^2+hr^3q)x')'+(\beta q^4+rq^3(g+k))x=0 \tag{1.7}$$

where g is oscillatory and $k\in L^1[c,\infty)$. In order to do this we make transformations of type

$$d\theta=\frac{q}{r}\,dt \quad \text{and} \quad x=(rq^3)^{-1/2}y \tag{1.8}$$

and apply an asymptotic theorem of Becker (**3**) (stated as Theorem 2 of Section 2).

As applications, we obtain results analogous to Examples 1 and 2 below for fourth order equations (see Examples 3 and 4 of Section 3).

It should be remarked that if one is not interested in obtaining a wide range of oscillatory phenomena, but would like sharper limit circle criteria, then we could treat (1.7) as a system and make substitutions requiring much weaker differentiability and integrability criteria than (1.8). This has been done in Becker (**4**), where more complete references to the literature on limit circle criteria may be found. In our case, the oscillatory nature of g seems to preclude such an approach. (The problem arises in that Levinson's asymptotic theorem must be applied in the system case where the coefficient matrix is a sum of a bounded variation matrix and an integrable matrix. If an oscillatory g is included, a further transformation by an oscillatory matrix must be made and this could destroy the bounded variation property of the coefficient matrix.)

Section 2 contains the statements of two results used later, Section 3 develops the asymptotic theory for fourth order equations, and Section 4 contains applications.

2. Preliminary Results

We state two theorems from other papers which will be used in the sequel.

Theorem 1. *Let r and q be positive on $[c,\infty)$ and $C^2[c,\infty)$. Let*

$$(rq)^{-1/2}(r^2((rq)^{-1/2})')' \quad \textit{and} \quad k\in L^1[c,\infty), \tag{2.1}$$

let

$$\theta(t)=\gamma+\int_c^t\frac{q}{r}\,ds$$

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and let g *satisfy the conditions*

(i) $g_1(t)=\int_t^\infty g(s)\cos^2\theta(s)\,ds$ *and* $g_2(s)=\int_t^\infty g(s)\sin^2\theta(s)\,ds$ *converge*

(ii) gg_1 *and* $gg_2\in L^1[c,\infty)$

(iii) $\sigma(t,t_0)=\frac{1}{2}\int_{t_0}^t g(s)\sin 2\theta(s)\,ds$ *does not have both* $\limsup_{t\geq t_0}\sigma(t,t_0)=+\infty$ *and* $\liminf_{t\geq t_0}\sigma(t,t_0)=-\infty$.

Then the equation (1.6) *has independent solutions* x_1, x_2 *satisfying*

$$\begin{aligned} x_1(t)&=(rq)^{-1/2}e^{\sigma(t,c)}(\cos\theta(t)+o(1))\\ x_2(t)&=(rq)^{-1/2}e^{-\sigma(t,c)}(\sin\theta(t)+o(1)). \end{aligned} \tag{2.2}$$

[For a proof see Becker (**2**) Example 3].

Theorem 2. *Let* $g(t)$, $h(t)$ *and* $k(t)$ *be integrable on compact subsets of* $[c,\infty)$. *Let* a, b *be real, positive and* $a\neq b$. *Let* $h(t)$ *and* $k(t)\in L^1[c,\infty)$. *Let* $g(t)$ *satisfy the following conditions:*

(i) *The following integrals converge:*

$$g_1(t)=\int_t^\infty g(s)\,ds,\quad g_2(t)=\int_t^\infty g(s)\cos 2as\,ds,$$

$$g_3(t)=\int_t^\infty g(s)\cos 2bs\,ds,\quad g_4^\pm(t)=\int_t^\infty g(s)\cos(a\pm b)s\,ds,$$

$$g_5^\pm(t)=\int_t^\infty g(s)\sin(a\pm b)s\,ds\quad \textit{for } t\in[c,\infty)$$

(ii) $g(t)g_i(t)\in L^1[c,\infty)$ $(i=1,2,3)$ *and* $g(t)\,g_4^\pm(t)$, $g(t)g_5^\pm(t)\in L^1[c,\infty)$.

(iii) *Let*

$$\sigma_1(t,t_0)=\frac{1}{2a(b^2-a^2)}\int_{t_0}^t g(s)\sin 2as\,ds$$

$$\sigma_2(t,t_0)=\frac{1}{2b(a^2-b^2)}\int_{t_0}^t g(s)\sin 2bs\,ds.$$

Suppose that we do not have both

$$\limsup_{t\geq t_0}(\sigma_1(t,t_0)-\sigma_2(t,t_0))=\infty$$

and

$$\liminf_{t\geq t_0}(\sigma_1(t,t_0)-\sigma_2(t,t_0))=-\infty.$$

Then the equation

$$(D^2+a^2)(D^2+b^2)x=-(g+h)x-(kx')'$$

has solutions $x_i(t)$ $(i=1,2,3,4)$ *satisfying*

$$x_1(t)=e^{\sigma_1(t,c)}(\cos at+o(1))$$
$$x_2(t)=e^{-\sigma_1(t,c)}(\sin at+o(1))$$
$$x_3(t)=e^{\sigma_2(t,c)}(\cos bt+o(1))$$
$$x_4(t)=e^{-\sigma_2(t,c)}(\sin bt+o(1))$$

as $t\to\infty$.

[For proof see Becker (**3**) Theorem 2 slightly modified to include $h\in L^1$ as well.]

3. Asymptotic Expansions for Fourth Order Equations

We will consider equations of the form

$$M(x)\equiv(r_1x^{(2)})^{(2)}+(p_1x')'+q_1x=0. \tag{3.1}$$

Let $x(t)=\omega(t)y(t)$. Then (3.1) becomes

$$(r_1\omega^2y^{(2)})^{(2)}+((4r_1\omega\omega''+2r_1'\omega\omega'-2r_1\omega'^2+p_1\omega^2)y')'+\omega M(\omega)y=0. \tag{3.2}$$

By analogy with the second order case (see Becker (**2**) Section 3), we wish to compare (3.1) with an equation of the form

$$(\partial^4+\alpha\partial^2+\beta)y=0 \tag{3.3}$$

where ∂ is an operator of the form

$$\partial=\mu(t)\frac{d}{dt}=\frac{d}{d\tau} \tag{3.4}$$

and

$$\tau(t)=\int_c^t\frac{ds}{\mu(s)}. \tag{3.5}$$

In order to apply Theorem 2, it seems reasonable to look at the case in which the polynomial

$$p(\lambda)=\lambda^2+\alpha\lambda+\beta \tag{3.6}$$

has roots which are real, negative and distinct, i.e.

$$\alpha>0,\quad \beta>0\quad\text{and}\quad \alpha^2>4\beta. \tag{3.7}$$

Substituting (3.5) into (3.3) and simplifying, we get

$$\mu(\mu^3y^{(2)})^{(2)}+\mu((\mu^2\mu''+\mu\mu'^2+\alpha\mu)y')'+\beta y=0. \tag{3.8}$$

Comparing (3.2) and (3.8) we get the same coefficient of $y^{(4)}$ if

$$r_1\omega^2=\mu^3 \tag{3.9}$$

which we will suppose to hold in what follows. From (3.9) it follows that

$$\begin{aligned}3\mu^2\mu' &= r_1'\omega^2+2r_1\omega\omega'\\ 3(\mu^2\mu')' &= r_1''\omega^2+4r_1'\omega\omega'+2r_1\omega'^2+2r_1\omega\omega''\\ &= 3(\mu^2\mu''+\mu\mu'^2)+3\mu\mu'^2.\end{aligned}$$

Hence we may write (3.2) as

$$\begin{aligned}&\mu(\mu^3y^{(2)})^{(2)}+\mu((\mu^2\mu''+\mu\mu'^2+\alpha\mu)y')'+\beta y\\ &\quad+\mu((-\tfrac{1}{3}r_1''\omega^2+\tfrac{2}{3}r_1'\omega\omega'-\tfrac{8}{3}r_1\omega'^2+\tfrac{10}{3}r_1\omega\omega''+\mu\mu'^2+(p_1\omega^2-\alpha\mu))y')'\\ &\quad+\mu\left(\omega((r_1\omega^{(2)})^{(2)}+(p_1\omega')')+\left(q_1\omega^2-\frac{\beta}{\mu}\right)\right)y=0.\end{aligned} \tag{3.10}$$

We could investigate two methods of proceeding:
(a) Set $q_1\omega^2=\beta/\mu$
or
(b) Set $p_1\omega^2=\alpha\mu$.
However they lead to identical results under the same conditions, so we will only deal with the first.

In the following, we suppose that in equation (3.1) we have

$$r_1 \text{ replaced by } r^4 \text{ and } q_1 \text{ replaced by } q^4. \tag{3.11}$$

In addition to (3.9) we assume

$$\mu q^4\omega^2=1. \tag{3.12}$$

This seems to restrict the generality of substitution (a) above by setting $\beta=1$. However an investigation of the substitution with general β shows that the final asymptotic expansions depend only on $\alpha/\beta^{1/2}$, and so there is no loss of generality in setting $\beta=1$. (See Remark 4 below.)

Equation (3.9) reads

$$r^4\omega^2=\mu^3. \tag{3.13}$$

Hence

$$\mu=\frac{r}{q} \quad\text{and}\quad \omega^2=\frac{1}{rq^3}. \tag{3.14}$$

In what follows, we will choose

$$p_1=\frac{\alpha\mu}{\omega^2}+hr^3q=\alpha(r^2q^2)+hr^3q \tag{3.15}$$

so that $p_1\omega^2-\alpha\mu=hr^2q^{-2}=h\mu^2$.

Theorem 3. *Let r and q be positive and four times differentiable on $[c,\infty)$, let h be differentiable on $[c,\infty)$ and let $g\in L^1[c,T]$ for all $T>c$. Let $\alpha>2$ and let $-a^2$ and $-b^2$ $(a,b>0)$ be the zeros of*

$$p(\lambda)=\lambda^2+\alpha\lambda+1.$$

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We will write

$$\mu=\frac{r}{q},\quad \omega^2=\frac{1}{rq^3}\quad \textit{and}\quad \theta(t)=\gamma+\int_c^t\frac{q}{r}\,ds.$$

Hypothesis A. *The following belong to* $L^1[c,\infty)$: $\mu^{-2}(r^4)''\omega^2$, $\mu^{-2}(r^4)'\omega\omega'$, $\mu^{-2}r^4\omega'^2$, $\mu^{-2}r^4\omega\omega''$, $\mu^{-1}\mu'^2$, $\omega(r^4\omega^{(2)})^{(2)}$ *and* $\omega(r^2q^2\omega')'$; h, k *and* $\omega(hr^3q\omega')'$.

Hypothesis B. $g(t)$ *satisfies* (i), (ii) *and* (iii) *of Theorem* 2 *with the trigonometric functions* $\cos(2a\theta(s))$ *etc. in place of* $\cos 2as$ *etc.*

Then the equation

$$(r^4x^{(2)})^{(2)}+((\alpha r^2q^2+hr^3q)x')'+(q^4+rq^3(g+k))x=0 \tag{3.16}$$

has solutions $x_i(t)$ $(i=1,2,3,4)$ *satisfying*

$$\begin{aligned}x_1(t)&=(rq^3)^{-1/2}e^{\sigma_1(t,c)}(\cos a\theta(t)+o(1))\\x_2(t)&=(rq^3)^{-1/2}e^{-\sigma_1(t,c)}(\sin a\theta(t)+o(1))\\x_3(t)&=(rq^3)^{-1/2}e^{\sigma_2(t,c)}(\cos b\theta(t)+o(1))\\x_4(t)&=(rq^3)^{-1/2}e^{-\sigma_2(t,c)}(\sin b\theta(t)+o(1))\end{aligned} \tag{3.17}$$

where

$$\sigma_1(t,c)=\frac{1}{2a(b^2-a^2)}\int_c^t g(s)\sin 2a\theta(s)\,ds$$

and

$$\sigma_2(t,c)=\frac{1}{2b(a^2-b^2)}\int_c^t g(s)\sin 2b\theta(s)\,ds.$$

Further, (3.16) *is of limit circle type if* $(rq^3)^{-1/2}e^{\pm\sigma_i(t,c)}\in L^2[c,\infty)$ $(i=1,2)$.

Remark 1. In the case $r\equiv 1$, Hypothesis A reduces to:

Hypothesis A$_1$. *The following belong to* $L^1[c,\infty)$: $q''q^{-2}$, $\omega\omega^{(4)}$, h, k *and* $\omega(hq\omega')'$; $(\omega=q^{-3/2})$.

Remark 2. It is known that for q real and positive, the fact that $q''q^{-2}\in L^1[c,\infty)$ and $q\notin L^1[c,\infty)$ implies that $q'^2q^{-3}\in L^1[c,\infty)$. See Coppel (**5**) IV.4.

Three. If q is "slowly oscillating", we expect the first two terms of Hypothesis A$_1$ to be the critical ones.

Four. We may derive the expansions for the equation

$$(r^4x^{(2)})^{(2)}+((\alpha r^2q^2+hr^3q)x')+(\beta q^4+rq^3(g+k))x=0 \tag{3.18}$$

(with $0<\beta\neq 1$) from the case with $\beta=1$ as follows. Let $\alpha_1=\alpha/\beta^{1/2}$, $r_2=r/\beta^{1/4}$, $h_1=h/\beta^{1/4}$, $g_1=g/\beta^{3/4}$ and $k_1=k/\beta^{3/4}$. Then (3.18) becomes

$$(r_2^4x^{(2)})^{(2)}+((\alpha_1r_2^2q^2+h_1r_2^3q)x')'+(q^4+r_2q^3(g+k))x=0. \tag{3.19}$$

Supposing that $\alpha_1>2$ i.e. $\alpha>2\beta^{1/2}$, that Hypothesis A holds for r and q (or equivalently for r_2 and q) and that Hypothesis B holds for r_2 and q, we may use Theorem 3 to obtain expansions for (3.18) of the type (3.17) with r_2 replaced by r.

Proof of Theorem 3. Make the substitutions

$$x=(rq^3)^{-1/2}y$$
$$\theta(t)=\gamma+\int_c^t \frac{q}{r}\,ds, \quad \text{so} \quad d\theta=\frac{dt}{\mu}, \tag{3.20}$$
$$\partial=\frac{d}{d\theta}.$$

We write $\partial y=y^{[1]}$.

Using (3.10), we see that (3.16) reduces to

$$\begin{aligned}(\partial^4+\alpha\partial^2+1)y &=\left(\frac{1}{\mu}\left(\tfrac{1}{3}r_1''\omega^2-\tfrac{2}{3}r_1'\omega\omega'+\tfrac{8}{3}r_1\omega'^2-\tfrac{10}{3}r_1\omega\omega''-\mu\mu'^2-h\mu^2\right)y^{[1]}\right)^{[1]}\\ &\quad+\mu(\omega(r^4\omega^{(2)})^{(2)}+\omega((\alpha r^2q^2+hr^3q)\omega')'-(g+k))y\\ &=\left(\frac{1}{\mu}N_1y^{[1]}\right)^{[1]}+\mu N_2y \quad \text{(say)}.\end{aligned} \tag{3.21}$$

We have $\frac{1}{\mu}N_1$ is integrable with respect to θ if and only if $\frac{1}{\mu^2}N_1$ is integrable with respect to t, and μN_2 is integrable with respect to θ if and only if N_2 is integrable with respect to t etc. Also

$$\int^\infty \mu(s)g(s)\cos 2a\theta\,d\theta=\int^\infty g(s)\cos 2a\theta(s)\,ds \text{ etc.}$$

Then apply Theorem 2.

4. Applications

Example 1. We apply Theorem 1 to the equation

$$(t^{2l}x')'+(t^{2m}+ct^n\sin dt^p)x=0 \tag{4.1}$$

for $t\in[0,\infty)$ where $m>-1$ and $p>0$; c, d real. In applying Theorem 1, we set $k=0$, $\gamma=0$ so that

$$\theta(t)=\int_0^t s^{m-l}\,ds=(m-l+1)^{-1}t^{m-l+1}$$
$$g(t)=ct^{n-m-l}\sin dt^p.$$

Condition (2.1) is satisfied if $m>l-1$. Hence if $n<m+l-1$ then $g\in L^1[0,\infty)$ and

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there are two solutions which are $O(t^{-1/2(l+m)})$. Thus if

$$m > l-1, \quad n < m+l-1 \quad \text{and} \quad m+l > 2 \tag{4.2}$$

then (4.1) is limit circle.

On the other hand we do not need $g \in L^1$. We will have $\sigma(t, 0) \to$ limit; g_1 and g_2 converge and gg_1, $gg_2 \in L^1[0, \infty)$ if

$$m > l-1; \quad d \neq 2(m-l+1)^{-1} \quad \text{or} \quad p \neq m-l+1; \quad n < m+l+\tfrac{1}{2}p-1, \quad m+l > 2. \tag{4.3}$$

Then by Theorem 1, if (4.3) holds then (4.1) is limit circle.

A further case is when

$$m > l-1, \quad d = 2(m+l-1)^{-1} \quad \text{and} \quad n = m+l-1. \tag{4.4}$$

In this case we have (for $a > 0$):

$$\exp \sigma(t, a) = \exp \frac{c}{2} \int_a^t \frac{1}{s} \sin^2 2\theta(s)\, ds = (C + o(1))t^{c/4}.$$

Also, (4.4) implies that g_1, g_2 converge and gg_1 and $gg_2 \in L^1[a, \infty)$. Using the asymptotic expansions of Theorem 1, we see that all solutions are L^2 if

$$-(m+l) \pm \frac{c}{2} < -1 \quad \text{i.e. if}$$

$$|c| < 2(m+l-1). \tag{4.5}$$

Thus (4.4) and (4.5) are sufficient for (4.1) to be limit circle.

The conditions (4.2) and (4.3) are generalizations of the results of Atkinson (**1**) quoted in (1.3), (1.4) and (1.5) above.

Example 2. We consider the equation

$$(rx')' + \left(q^2 + \tau\left(\int_t^\infty \frac{ds}{rq}\right)^{-1} \sin 2\theta(t) + rqk\right)x = 0 \tag{4.6}$$

where

$$\theta(t) = \gamma + \int_c^t \frac{q}{r}\, ds.$$

Then

$$g = \tau\left(\int_t^\infty \frac{ds}{rq}\right)^{-1} (rq)^{-1} \sin 2\theta(t)$$

and

$$g_1 = 2 \int_t^\infty \left(\frac{q}{r} \sin^3 \theta \cos \theta\right) h(s)\, ds$$

where

$$h(t) = \tau\left(\int_t^\infty \frac{ds}{rq}\right)^{-1} q^{-2}.$$

Then if $\tau^{-1}h(t)$ is nonincreasing it follows as in Atkinson (**1**) Theorem 6 that g_1 converges. Further

$$\exp \sigma(t,c)=\exp\left(\frac{\tau}{4}\int_c^t\left(\int_s^\infty \frac{du}{rq}\right)^{-1}(rq)^{-1}\,ds+C+o(1)\right)$$
$$=(C'+o(1))\left(\int_t^\infty \frac{ds}{rq}\right)^{\tau/4}.$$

Again following Atkinson, we have

$$|g_1|\leqq h \quad \text{so that} \quad |gg_1|\leqq \frac{q}{r}h^2.$$

Assume now that

$$rq'q^{-2} \text{ is nonincreasing, } (rq)^{-1}\in L^1[c,\infty) \text{ and } q \text{ is unbounded.} \tag{4.7}$$

Then $q'\geqq 0$ and following Atkinson (**1**) Theorem 7 we see that h is nonincreasing, and

$$\int_t^\infty \frac{ds}{rq}\geqq (2rq')^{-1}.$$

Thus

$$|gg_1|\leqq \frac{q}{r}h^2\leqq 4\tau^2 rq'^2q^{-3}.$$

We assume finally that

$$rq'^2q^{-3}\in L^1[c,\infty). \tag{4.8}$$

Then under hypotheses (4.7), (4.8) the conditions of Theorem 1 are satisfied and there are two solutions which are

$$O\left((rq)^{-1/2}\left(\int_t^\infty \frac{ds}{rq}\right)^{\pm\tau/4}\right).$$

Hence all solutions are square-integrable if

$$1\pm\tau/2>0 \quad \text{i.e. iff } |\tau|<2. \tag{4.9}$$

Thus (4.6) with $r,\ q>0$, $r,\ q\in C^2[c,\infty)$ is of limit circle type if (4.7), (4.8) and (4.9) hold.

Example 3. We consider the equation

$$(t^{4l}x^{(2)})^{(2)}+\alpha(t^{2(l+m)}x')'+(\beta t^{4m}+ct^n\sin dt^p)x=0 \tag{4.10}$$

where $\alpha,\ \beta>0$ and $\alpha^2>4\beta$. (See Remark 4 for the technique of dealing with the case $\beta\neq 1$). Then

$$\theta_1=\beta^{1/4}\int_0^t s^{m-l}\,ds=\beta^{1/4}(m-l+1)^{-1}t^{m-l+1};\quad g=\beta^{-1/2}ct^{n-l-3m}\sin dt^p.$$

We proceed as in Example 1 but using Theorem 3 in place of Theorem 1. Hypothesis A is satisfied if $m>l-1$. Hence if $\sigma_1(\infty,c)$ and $\sigma_2(\infty,c)$ are absolutely convergent then there will be two solutions which are $o(t^{-1/2(l+3m)})$. This will be the case if $n-l-3m<-1$. Thus if

$$m>l-1, \quad n<l+3m-1 \quad \text{and} \quad l+3m>2 \tag{4.11}$$

then (4.10) is limit circle.

Let $-a^2$ and $-b^2$ be the two zeros of $\lambda^2+\alpha/\beta^{1/2}\lambda+1=0$. Assume that

$$(d\neq 2a\beta^{1/4}(m-l+1)^{-1} \quad \text{and} \quad d\neq 2b\beta^{1/4}(m-l+1)^{-1} \quad \text{and} \quad d\neq(a\pm b)\beta^{1/4}(m-l+1)^{-1}) \quad \text{or} \quad p\neq m-l+1 \tag{4.12}$$

and

$$m>l-1, \quad n<l+3m+\tfrac{1}{2}p-1, \quad l+3m>2.$$

Then under conditions (4.12) the equation (4.10) is limit circle.

The last case of Example 1 could be treated as before, but it will be covered by the following example which generalizes Example 2.

Example 4. Consider the equation

$$(r^4x^{(2)})^{(2)}+\alpha(r^2q^2x')'+(q^4+q^4(h_1\sin 2a\theta(t)+h_2\sin 2b\theta(t))+rq^3k)x=0 \tag{4.13}$$

where

$$\theta(t)=\gamma+\int_c^t \mu^{-1}\,ds, \quad \mu=\frac{r}{q};$$

$-a^2$ and $-b^2$ are the roots of $\lambda^2+\alpha\lambda+1=0$ (and we have taken $\beta=1$ for simplicity). We assume

$$\alpha>2; \quad k\in L^1[c,\infty); \quad r,q>0; \quad r,q\in C^2[c,\infty) \ \text{and satisfy Hypothesis A} \tag{4.14}$$

$$h_i(t) \text{ has one sign; } \operatorname{sgn} h_i(t)h_i'(t)\leqq 0; \quad \mu^{-1}h_i^2\in L^1[c,\infty) \quad (i=1,2). \tag{4.15}$$

(See Atkinson (**1**) Theorems 6 and 7 for a treatment of the second-order case under similar hypotheses). In the notation of our Theorem 3, we have

$$g=\mu^{-1}(h_1\sin 2a\theta+h_2\sin 2b\theta).$$

Under hypotheses (4.14) and (4.15) we may follow Atkinson (**1**) Theorem 6, but using our Theorem 3 in place of his Theorem 1, to obtain the existence of independent solutions x_i $(i=1,2,3,4)$ satisfying

$$\begin{aligned}
x_1(t)&=(rq^3)^{-1/2}\left\{\exp -\frac{1}{2L}\int_c^t \frac{q}{r}h_1\,ds\right\}(\sin a\theta(t)+o(1))\\
x_2(t)&=(rq^3)^{-1/2}\left\{\exp \frac{1}{2L}\int_c^t \frac{q}{r}h_1\,ds\right\}(\cos a\theta(t)+o(1))\\
x_3(t)&=(rq^3)^{-1/2}\left\{\exp -\frac{1}{2M}\int_c^t \frac{q}{r}h_2\,ds\right\}(\sin b\theta(t)+o(1))\\
x_4(t)&=(rq^3)^{-1/2}\left\{\exp \frac{1}{2M}\int_c^t \frac{q}{r}h_2\,ds\right\}(\cos b\theta(t)+o(1))
\end{aligned} \tag{4.16}$$

where $L=2a(b^2-a^2)$ and $M=2b(a^2-b^2)$. From this we see that if $\frac{q}{r}h_i$ $(i=1,2)$ are integrable and $(rq^3)^{-1}\in L^1[c,\infty)$ then (4.13) is limit circle.

A case when $\frac{q}{r}h_i$ are not integrable will now be discussed. We will set

$$h_1(t)=\tau q^{-4}\left(\int_t^\infty \frac{ds}{rq^3}\right)^{-1} \quad\text{and}\quad h_2(t)=\nu q^{-4}\left(\int_t^\infty \frac{ds}{rq^3}\right)^{-1}.$$

In order to apply the above, we must verify (4.15), which will be done under further hypotheses:

$$\begin{gathered} rq'q^{-2} \text{ is nonincreasing; } (rq^3)^{-1}\in L^1[c,\infty); \\ q \text{ is unbounded and } rq'^2q^{-3}\in L^1[c,\infty) \text{ (so that } q'\geqq 0). \end{gathered} \tag{4.17}$$

We have

$$(\tau^{-1}h_1)'=-\left\{4q^3q'\left(\int_t^\infty \frac{ds}{rq^3}\right)-\frac{q}{r}\right\}q^{-8}\left(\int_t^\infty \frac{ds}{rq^3}\right)^{-2}. \tag{4.18}$$

Also

$$\int_t^\infty \frac{ds}{rq^3}=\int_t^\infty \left(\frac{q'}{q^5}\right)\left(\frac{rq'}{q^2}\right)^{-1} ds \geqq 4\left(\frac{rq'}{q^2}\right)^{-1}q^{-4}=(4rq^2q')^{-1}.$$

Using this in (4.18) we see that $(\tau^{-1}h_1)'\leqq 0$. Further,

$$\mu^{-1}h_1^2=\tau^2\frac{q}{r}q^{-8}\left(\int_t^\infty \frac{ds}{rq}\right)^{-2}\leqq 16\tau^2 rq'^2q^{-3}\in L^1[c,\infty).$$

Thus (4.15) holds. We have

$$\begin{aligned} \sigma_1(t,c)&=(2a(b^2-a^2))^{-1}\int_c^t g(s)\sin 2a\theta(s)\,ds \\ &=(2a(b^2-a^2))^{-1}\frac{1}{2}\int_c^t \mu^{-1}h_1(s)\,ds+C+o(1) \end{aligned}$$

$$\exp\sigma_1(t,c)=C'\left(\int_t^\infty \frac{ds}{rq^3}\right)^{\tau_1} \quad (\tau_1=\tau(4a(b^2-a^2))^{-1}).$$

Similarly

$$\exp\sigma_2(t,c)=C''\left(\int_t^\infty \frac{ds}{rq^3}\right)^{\nu_1} \quad (\nu_1=\nu(4b(a^2-b^2))^{-1}).$$

From (4.16) it follows that there are two solutions which are

$$O\left((rq^3)^{-1/2}\left(\int_t^\infty \frac{ds}{rq^3}\right)^{\pm\tau_1}\right)$$

and two solutions which are

$$O\left((rq^3)^{-1/2}\left(\int_t^\infty \frac{ds}{rq^3}\right)^{\pm\nu_1}\right).$$

In particular under hypotheses (4.14), (4.15) and (4.17) we have (4.13) limit circle if

$$(1\pm 2\tau_1)>0 \quad\text{and}\quad (1\pm 2\nu_1)>0$$

i.e. if $|\tau_1|<\frac{1}{2}$ and $|\nu_1|<\frac{1}{2}$ or

$$|\tau|<2a\,|b^2-a^2| \quad\text{and}\quad |\nu|<2b\,|b^2-a^2|. \tag{4.19}$$

It is clear that if $\tau\neq 0$, $\nu\neq 0$ and (4.14), (4.15) and (4.17) are satisfied then two solutions are always $L^2[c,\infty)$. By using (4.19) we can in some cases choose τ and ν so that exactly 2,3 or 4 solutions are $L^2[c,\infty)$. If we want the limit-3 or limit-4 cases, we may get them with $\nu=0$. We illustrate with an example.

Let $r\equiv 1$, $h_1=2\tau t^{-2}$ and $h_2=2\nu t^{-2}$. Then $q=t$ and the equation becomes

$$x^{(4)}+\alpha(t^2x')'+(t^4+2t^2(\tau\sin 2a\theta(t)+\nu\sin 2b\theta(t)))x=0 \tag{4.20}$$

It is easily seen that (4.14) and (4.15) are satisfied. We have $\theta(t)=\frac{1}{2}t^2$. (Note that Hypothesis A holds if in the notation of Example 3, we have $m>l-1$). By (4.16) there are solutions satisfying

$$\begin{aligned}
x_1&=t^{-3/2-\tau/L}(\sin(a/2)t^2+o(1)),\\
x_2&=t^{-3/2+\tau/L}(\cos(a/2)t^2+o(1)),\\
x_3&=t^{-3/2-\nu/M}(\sin(b/2)t^2+o(1)),\\
x_4&=t^{-3/2+\nu/M}(\cos(b/2)t^2+o(1)).
\end{aligned}$$

Now choosing τ and ν appropriately results in the limit-2, limit-3 or limit circle case.

Atkinson (**1**) uses the theory for second order operators to get limit-1, 2, 3 or 4 cases for their squares, and the above results are a generalization showing how the limit-4 case can be destroyed via lower-order oscillatory perturbations. For results of a different kind on the destruction of the limit-circle case, see Eastham and Thomson (**7**) and Read (**8**).

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CAPE TOWN
7700 RONDEBOSCH, CAPE
REPUBLIC OF SOUTH AFRICA