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A Central Limit Theorem and Law of the Iterated Logarithm for a Random Field with Exponential Decay of Correlations

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Abstract. In [6], Walter Philipp wrote that "…the law of the iterated logarithm holds for any process for which the Borel-Cantelli Lemma, the central limit theorem with a reasonably good remainder and a certain maximal inequality are valid." Many authors [1], [2], [4], [5], [9] have followed this plan in proving the law of the iterated logarithm for sequences (or fields) of dependent random variables.

We carry on this tradition by proving the law of the iterated logarithm for a random field whose correlations satisfy an exponential decay condition like the one obtained by Spohn [8] for certain Gibbs measures. These do not fall into the ϕ-mixing or strong mixing cases established in the literature, but are needed for our investigations [7] into diffusions on configuration space.

The proofs are all obtained by patching together standard results from [5], [9] while keeping a careful eye on the correlations.

1 Introduction

Our motivation for this paper is a problem from [7] concerning the stochastic dynamics associated with a continuous system of particles from classical statistical physics. In other words, we consider a system of interacting diffusion processes on $\mathbb{R}^d$ whose equilibrium measure is a Gibbs measure μ with potential ϕ. As part of our investigation into the large scale regularity of the distribution of particles, we needed to prove that the law of the iterated logarithm holds in equilibrium. This application will be explained further in Section 5.

If we discretize the problem by letting N_n represent the number of particles in the box $\big(-(n+1/2), n+1/2\big]^d$, what we want to show is that

$$\limsup_n \frac{N_n - E(N_n)}{\sqrt{2\operatorname{Var}(N_n)\log\log n}} = 1, \quad \mu\text{-almost surely.}$$

It is a classical result in probability that this law of the iterated logarithm holds if the number of particles in disjoint sets are independent random variables, that is, if μ describes a Poisson point process. From the statistical mechanics viewpoint, this is the case when the potential function ϕ is identically zero.

Extending the law of the iterated logarithm to dependent fields requires approximate independence, that is, the number of particles in widely separated regions of space should be weakly correlated random variables. The standard proofs [1], [2],

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[4], [5], [9] for dependent fields impose mixing conditions in order to get the result. Unfortunately, for the particular application we have in mind, it is not known whether mixing conditions hold.

Therefore we wrote this paper to give a proof of the law of the iterated logarithm that avoids using mixing conditions, but rather, relies directly on the decay of correlations (Condition 2 below). It is our hope that these results may also be of use to others who study measures where a decay of correlations is known, but not a mixing condition.

2 Notation and Basic Inequalities

We begin with a multiparameter, mean zero, strictly stationary process $(x_i)_{i\in\mathbb{Z}^d}$ with $E(x_0^6) < \infty$. For $I \subseteq \mathbb{Z}^d$ we let $|I|$ denote its cardinality, and we put $\mathcal{F}(I) = \sigma(x_i \mid i \in I)$. All distances in $\mathbb{Z}^d$ will be taken in the ℓ_∞ norm ($|i|_\infty := \sup\{|i_1|, \dots, |i_d|\}$) and for subsets I, J we let $d(I, J) := \inf\{|i - j|_\infty \mid i \in I, j \in J\}$. Define the discrete ℓ_∞ ball of radius n by $B_n = \{i \in \mathbb{Z}^d \mid |i|_\infty \le n\}$ and note that $|B_n| = (2n+1)^d$.

Conditions

1. There is a constant a so that $0 < a|I| \le \operatorname{Var}(\sum_{i\in I} x_i)$.
2. There exist constants $\alpha, c > 0$ such that if Ψ_I, Ψ_J are square integrable real or complex valued random variables with $\Psi_I \in \mathcal{F}(I)$ and $\Psi_J \in \mathcal{F}(J)$, then $|\operatorname{Corr}(\Psi_I, \Psi_J)| \le |I|\,|J| c e^{-\alpha d(I,J)}$.

Comment The factor $|I|\,|J|$ in Condition 2 above means that $(x_i)_{i\in\mathbb{Z}^d}$ does not satisfy the usual ϕ-mixing or strong mixing condition. We lose control over the correlation of very large sets at a fixed distance from each other. On the other hand this is more than compensated for by the fact that $ce^{-\alpha d(I,J)}$ decreases exponentially in $d(I, J)$. It is not hard to see that all the results in this paper hold if we replace $|I|\,|J|$ by $(|I|\,|J|)^p$ for any $p \ge 1$. However, our proofs fail for exponential mixing with exponential factors like $\exp(|I|)\exp(|J|)$. This type of mixing was obtained for Gibbs measures in [3].

Definition 1 *The following explicit constants will prove useful.*

$$\sigma^2 := \sum_{i\in\mathbb{Z}^d} E(x_0 x_i)$$

$$b := \sum_{i\in\mathbb{Z}^d} |E(x_0 x_i)|$$

$$M := \max\{E(x_0^2), E(x_0^4), E(x_0^6)\}$$

$$c_1 := M \sum_{r=0}^{\infty} (2r+1)^{3d} c e^{-\alpha r/3}$$

$$c_2 := 24b^2 + 4c_1.$$

Note that the decay of correlations in Condition 2 gives

$$\sigma^2 \le b \le \mathrm{Var}(x_0) \sum_{i \in \mathbb{Z}^d} c e^{-\alpha |i|_\infty} < \infty.$$

Lemma 3 combined with Condition 1 shows that $\sigma^2 \ge a > 0$.

From the definition of b, it is easy to see that

$$E\Big[\Big(\sum_{i \in I} x_i\Big)^2\Big] \le b|I|. \tag{1}$$

The analogous result for the fourth moment is more difficult, and is given in the following lemma.

Lemma 1 *For any index set $I \subseteq \mathbb{Z}^d$,*

$$E\Big[\Big(\sum_{i \in I} x_i\Big)^4\Big] \le c_2 |I|^2. \tag{2}$$

Proof We first gather some basic facts on the moments of x_i. From stationarity and Cauchy-Schwarz we have $\mathrm{Var}(x_i) \le M$,

$$\mathrm{Var}(x_i x_j) \le E[(x_i x_j)^2] \le E(x_i^4)^{1/2} E(x_j^4)^{1/2} = E(x_0^4) \le M,$$

and $\mathrm{Var}(x_i x_j x_k) \le E[(x_i x_j x_k)^2] \le E(x_0^6) \le M$. Now we analyze the fourth moment of the sum

$$E\Big[\Big(\sum_{i \in I} x_i\Big)^4\Big] = \sum_{(i,j,k,l) \in I^4} E(x_i x_j x_k x_l).$$

For each multiindex $(i, j, k, l) \in I^4$ define the maximum distance between coordinates by

$$r(i, j, k, l) := \max\big\{ |s - t|_\infty \;\big|\; s, t \in \{i, j, k, l\} \big\}.$$

Now divide the index set into pieces accordingly: $I^4 = \bigcup_{r=0}^{\infty} I_r$, where

$$I_r = \{(i, j, k, l) \in I^4 \mid r(i, j, k, l) = r\}.$$

Note that the cardinality of I_r satisfies $|I_r| \le |I|(2r + 1)^{3d}$. The set I_r is, in turn, divided into two pieces depending on whether there is one isolated index, or two pairs of isolated indices. That is,

$$I_r^1 := \Big\{ (i, j, k, l) \in I_r \;\Big|\; \max_{s \in \{i,j,k,l\}} \; \min_{t \in \{i,j,k,l\}, t \ne s} |s - t|_\infty \ge r/3 \Big\},$$

and $I_r^2 := I_r \setminus I_r^1$. For every $(i, j, k, l) \in I_r^2$, the set $\{i, j, k, l\}$ can be divided into two pairs $\{s, t\}$ and $\{u, v\}$ so that $|s - t|_\infty \le r/3$, $|u - v|_\infty \le r/3$ and $d(\{s, t\}, \{u, v\}) \ge r/3$.

If $(i,j,k,l) \in I_r^1$, then supposing i is the isolated index, we get

$$\begin{aligned}|E(x_i x_j x_k x_l)| &= |E(x_i x_j x_k x_l) - E(x_i)E(x_j x_k x_l)| \\ &\le \sqrt{\mathrm{Var}(x_i)}\sqrt{\mathrm{Var}(x_j x_k x_l)}3ce^{-\alpha r/3} \\ &\le M3ce^{-\alpha r/3}.\end{aligned}$$

On the other hand, if $(i,j,k,l) \in I_r^2$, then

$$\begin{aligned}|E(x_i x_j x_k x_l)| &\le |E(x_s x_t)E(x_u x_v)| + \sqrt{\mathrm{Var}(x_s x_t)}\sqrt{\mathrm{Var}(x_u x_v)}4ce^{-\alpha r/3} \\ &\le |E(x_s x_t)E(x_u x_v)| + M4ce^{-\alpha r/3}.\end{aligned}$$

Therefore

$$\begin{aligned}E\Big[\Big(\sum_{i\in I} x_i\Big)^4\Big] &\le 4M\sum_{r=0}^{\infty}|I_r|ce^{-\alpha r/3} + 4!\sum_{(s,t)\in I^2}\sum_{(u,v)\in I^2}|E(x_s x_t)E(x_u x_v)| \\ &= 4M\sum_{r=0}^{\infty}|I_r|ce^{-\alpha r/3} + 24\Big(\sum_{(s,t)\in I^2}|E(x_s x_t)|\Big)^2 \\ &\le |I|4M\sum_{r=0}^{\infty}(2r+1)^{3d}ce^{-\alpha r/3} + 24(b|I|)^2 \\ &\le c_2|I|^2.\end{aligned}$$

■

Definition 2 *Let $\xi_0 = x_0$ and for $r \ge 1$ let $\xi_r = \sum_{i\in B_r\setminus B_{r-1}} x_i$.*

Lemma 2 *For any indices $R \subseteq \{0,1,\dots,n\}$ and any $x > 0$ we have*

$$P\Big(\sum_{r\in R}|\xi_r| \ge x\Big) \le \frac{4d^2 c_2|R|^4(2n+1)^{2(d-1)}}{x^4}.$$

Proof Using Jensen's inequality we find the pointwise bound $(\sum_{r\in R}|\xi_r|)^4 \le |R|^3\sum_{r\in R}\xi_r^4$. Taking expectations and using Lemma 1 gives

$$E\Big[\Big(\sum_{r\in R}|\xi_r|\Big)^4\Big] \le |R|^3\sum_{r\in R}E(\xi_r^4) \le |R|^4\sup_{r\in R}E(\xi_r^4) \le |R|^4 c_2|B_n\setminus B_{n-1}|^2.$$

Now $|B_n\setminus B_{n-1}| = (2n+1)^d - \big(2(n-1)+1\big)^d \le 2d(2n+1)^{d-1}$, so square this and the result follows from Chebyshev's inequality. ■

Definition 3 *Define $S_n = \sum_{r=0}^{n}\xi_r = \sum_{i\in B_n} x_i$.*

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Lemma 3 *For $n \geq 1$ we have*

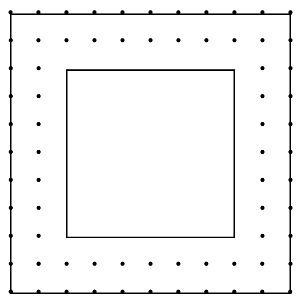
$$\left|\frac{\operatorname{Var}(S_n)}{(2n+1)^d} - \sigma^2\right| = \left|\frac{E(S_n^2)}{(2n+1)^d} - \sigma^2\right| \leq \frac{c_1}{2n+1}. \tag{3}$$

Proof By stationarity we have $(2n+1)^d\sigma^2 = \sum_{i \in B_n} \sum_{j \in \mathbb{Z}^d} E(x_i x_j)$ and by definition we have $E(S_n^2) = \sum_{i \in B_n} \sum_{j \in B_n} E(x_i x_j)$. Taking the difference gives

$$(2n+1)^d \sigma^2 - E(S_n^2) = \sum_{i \in B_n} \sum_{j \notin B_n} E(x_i x_j).$$

We will divide this sum into two pieces and estimate them separately:

$$\underbrace{\sum_{r=1}^{n} \sum_{\substack{i \in B_n, j \notin B_n \\ |j-i|_\infty = r}} E(x_i x_j)}_{I} + \underbrace{\sum_{\substack{i \in B_n, j \notin B_n \\ |j-i|_\infty > n}} E(x_i x_j)}_{II}.$$



In bounding the first sum, we observe that if $i \in B_n$, $j \notin B_n$, and $|j-i|_\infty = r$, then $n - r < |i|_\infty \leq n$. The number of such i's is the cardinality of $B_n \setminus B_{n-r}$, that is, $(2n+1)^d - (2(n-r)+1)^d$ which is less than or equal to $2dr(2n+1)^{d-1}$. For each i, the number of j's with $|j-i|_\infty = r$ is less than or equal to $(2r+1)^d$. This leads to the following bound.

$$\begin{aligned}
|I| &\leq \sum_{r=1}^{n} \sum_{\substack{n-r<|i|_\infty \leq n, \\ |j-i|_\infty = r}} |E(x_i x_j)| \\
&= \sum_{r=1}^{n} 2dr(2n+1)^{d-1}(2r+1)^d M c e^{-\alpha r} \\
&\leq (2n+1)^d (2n+1)^{-1} M \sum_{r=1}^{\infty} d(2r)(2r+1)^d c e^{-\alpha r} \\
&\leq (2n+1)^d (2n+1)^{-1} \frac{c_1}{2}.
\end{aligned}$$

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Using stationarity, we bound the second sum as follows

$$\begin{aligned} |II| &\le \sum_{i\in B_n, |j-i|_\infty > n} |E(x_i x_j)| \\ &= \sum_{i\in B_n} \sum_{|j|_\infty > n} |E(x_0 x_j)| \\ &\le (2n+1)^d M \sum_{r>n} (2r+1)^d c e^{-\alpha r} \\ &\le (2n+1)^d (2n+1)^{-1} \frac{c_1}{2}. \end{aligned}$$

Combining the bounds for I and II, and dividing by $(2n+1)^d$ gives (3). ■

3 Central Limit Theorem and Maximal Inequality

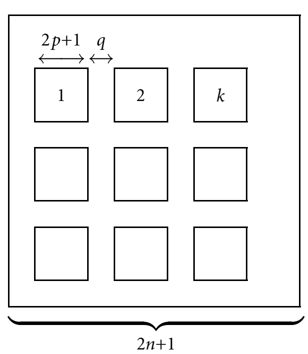
Lemma 4

$$\sup_{z\in\mathbb{R}} \Big| P\big(S_n/\sigma(2n+1)^{d/2} \ge z\big) - \Phi(z)\Big| = O(n^{-1/9}),$$

where Φ is the standard normal error function.

Proof The strategy is first to show that the main contribution to S_n comes from x's whose indices form a collection of reasonably large, but well spread out, subcubes W_i of $[-n,n]^d$. We then use the decay of correlations to show that the contributions from the different W_i's are nearly independent.

Fix $\varepsilon < 1/2$ and for $n \ge 2^{1/\varepsilon}$ define $p(n) := \lfloor n^{1/2} \rfloor$, $q(n) := \lfloor n^{1/2-\varepsilon} \rfloor$, and $k(n) = \lfloor \frac{2n+1}{2p+q+1} \rfloor$. Since n is large enough so that $k(n) \ge 1$, the interval $[-n,n]$ contains k intervals $I_1, \dots, I_k$ of length $2p+1$ with a distance q between them.



For each $i \in \{1,2,\dots,k\}^d$, define the cube $W_i = I_{i_1} \times \cdots \times I_{i_d}$. The collection $(W_i)_{i\in\{1,2,\dots,k\}^d}$ consists of k^d subcubes of $[-n,n]^d$, each with $|W_i| = (2p+1)^d$. The ratio $|\bigcup_i W_i| / |[-n,n]^d|$ of their cardinalities satisfies

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$$\begin{aligned}1 \geq \frac{k^d(2p+1)^d}{(2n+1)^d} &\geq \frac{\left(\frac{2n+1}{2p+q+1}-1\right)^d(2p+1)^d}{(2n+1)^d}\\ &= \left(1-\frac{2p+q+1}{2n+1}\right)^d\left(1-\frac{q}{2p+q+1}\right)^d\\ &\geq \left(1-\frac{4n^{1/2}}{2n+1}\right)^d\left(1-\frac{n^{1/2-\varepsilon}}{n^{1/2}}\right)^d\\ &\geq (1-2n^{-1/2})^d(1-n^{-\varepsilon})^d\\ &\geq (1-2n^{-\varepsilon})^{2d}\\ &\geq 1-2d(2n^{-\varepsilon}),\end{aligned}$$

so that

$$\left|1-\frac{k^d(2p+1)^d}{(2n+1)^d}\right| \leq 4dn^{-\varepsilon}. \tag{4}$$

For every $i \in \{1,2,\dots,k\}^d$ let $\zeta_i := \sum_{j\in W_i} x_j$, and let $(\zeta_i')_{i\in\{1,2,\dots,k\}^d}$ be independent copies of $(\zeta_i)_{i\in\{1,2,\dots,k\}^d}$. Notice that ζ_i has the same distribution as S_p. The central limit theorem for S_n uses the following series of approximations to a standard normal Z:

$$\frac{S_n}{\sigma(2n+1)^{d/2}} \approx \frac{\sum_i \zeta_i}{\sigma(2n+1)^{d/2}} \approx \frac{\sum_i \zeta_i'}{\sigma(2n+1)^{d/2}} \approx \frac{\sum_i \zeta_i'}{(k^d\operatorname{Var}(\zeta))^{1/2}} \approx Z.$$

The first approximation is easiest, so let's begin there. Using (1) and (4) we obtain

$$E\left[\left(\frac{S_n-\sum_i\zeta_i}{\sigma(2n+1)^{d/2}}\right)^2\right] = \frac{\operatorname{Var}(\sum_{j\in[-n,n]^d\setminus\cup_i W_i} x_j)}{\sigma^2(2n+1)^d} \leq \frac{b(4dn^{-\varepsilon})}{\sigma^2}.$$

This gives us

$$E\left(\left|\frac{S_n-\sum_i\zeta_i}{\sigma(2n+1)^{d/2}}\right|\right) \leq \frac{2\sqrt{bd}n^{-\varepsilon/2}}{\sigma}. \tag{5}$$

For the third approximation we first use the independence to get

$$E\left[\left(\frac{\sum_i\zeta_i'}{\sigma(2n+1)^{d/2}}-\frac{\sum_i\zeta_i'}{\left(k^d\operatorname{Var}(\zeta_i)\right)^{1/2}}\right)^2\right] = \left(1-\sqrt{\frac{k^d\operatorname{Var}(\zeta)}{\sigma^2(2n+1)^d}}\right)^2.$$

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Now rewrite the right hand side and use Lemma 3 and (4) to get

$$
\begin{aligned}
\left(1-\sqrt{\frac{k^d\,\mathrm{Var}(\zeta)}{\sigma^2(2n+1)^d}}\right)^2 &\le \left(1-\frac{k^d\,\mathrm{Var}(\zeta)}{\sigma^2(2n+1)^d}\right)^2\\
&= \left(1-\frac{k^d(2p+1)^d}{(2n+1)^d}\times\frac{\mathrm{Var}(\zeta)}{\sigma^2(2p+1)^d}\right)^2\\
&\le \left(\left|1-\frac{k^d(2p+1)^d}{(2n+1)^d}\right|+\left|1-\frac{\mathrm{Var}(S_p)}{\sigma^2(2p+1)^d}\right|\right)^2\\
&\le \left(4dn^{-\varepsilon}+c_1(2p+1)^{-1}\right)^2\\
&\le (4d+c_1)^2n^{-2\varepsilon}.
\end{aligned}
$$

This gives us

$$
E\left(\left|\frac{\sum_i \zeta_i'}{\sigma(2n+1)^{d/2}}-\frac{\sum_i \zeta_i'}{\left(k^d\,\mathrm{Var}(\zeta_i)\right)^{1/2}}\right|\right)\le (4d+c_1)n^{-\varepsilon}. \tag{6}
$$

For the second approximation, we work directly on the characteristic functions. The final bound is obtained by induction, here is the first step, where j is any index in $\{1,2,\ldots,k\}^d$.

$$
\begin{aligned}
\left|E(e^{it\sum_i\zeta_i})-E(e^{it\sum_i\zeta_i'})\right| &\le \left|E(e^{it\sum_{i\neq j}\zeta_i}e^{it\zeta_j})-E(e^{it\sum_{i\neq j}\zeta_i})E(e^{it\zeta_j})\right|\\
&\quad+\left|E(e^{it\sum_{i\neq j}\zeta_i})E(e^{it\zeta_j})-E(e^{it\sum_{i\neq j}\zeta_i'})E(e^{it\zeta_j'})\right|\\
&=\left|\mathrm{Cov}(e^{it\sum_{i\neq j}\zeta_i},e^{-it\zeta_j})\right|+\left|E(e^{it\sum_{i\neq j}\zeta_i})-E(e^{it\sum_{i\neq j}\zeta_i'})\right|\\
&\le |B_n|^2ce^{-\alpha q}+\left|E(e^{it\sum_{i\neq j}\zeta_i})-E(e^{it\sum_{i\neq j}\zeta_i'})\right|\\
&=(2n+1)^{2d}ce^{-\alpha q}+\left|E(e^{it\sum_{i\neq j}\zeta_i})-E(e^{it\sum_{i\neq j}\zeta_i'})\right|.
\end{aligned}
$$

Continuing in this way, peeling off the individual random variables one at a time, we arrive at the uniform bound

$$
\begin{aligned}
|E(e^{it\sum_i\zeta_i})-E(e^{it\sum_i\zeta_i'})| &\le k^d(2n+1)^{2d}ce^{-\alpha q}\\
&\le (2n+1)^{3d}ce^{-\alpha q}.
\end{aligned}
$$

We also have

$$
E\left[\left(\frac{\sum_i\zeta_i}{\sigma(2n+1)^{d/2}}\right)^2\right]\le\frac{\mathrm{Var}(\sum_{r\in\bigcup_i W_i}x_r)}{\sigma^2(2n+1)^d}\le\frac{bk^d(2p+1)^d}{\sigma^2(2n+1)^d}\le\frac{b}{\sigma^2},
$$

and

$$
E\left[\left(\frac{\sum_i\zeta_i'}{\sigma(2n+1)^{d/2}}\right)^2\right]\le\frac{k^d\,\mathrm{Var}(\sum_{r\in W_i}x_r)}{\sigma^2(2n+1)^d}\le\frac{k^db(2p+1)^d}{\sigma^2(2n+1)^d}\le\frac{b}{\sigma^2}.
$$

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This gives us

$$E\left[\left(\frac{\sum_i \zeta_i}{\sigma(2n+1)^{d/2}} - \frac{\sum_i \zeta_i'}{\sigma(2n+1)^{d/2}}\right)^2\right] \leq \frac{4b}{\sigma^2}.$$

Putting these together gives

$$(7)\quad \left|E(e^{it\sum_i \zeta_i/\sigma(2n+1)^{d/2}}) - E(e^{it\sum_i \zeta_i'/\sigma(2n+1)^{d/2}})\right| \leq \min\left\{(2n+1)^{3d}ce^{-\alpha q}, |t|\frac{2\sqrt{b}}{\sigma}\right\}.$$

From Esseen's lemma, there is an absolute constant K so that

$$(8)\qquad \left|e^{-t^2/2} - E(e^{it\sum_i \zeta_i'/\sqrt{k^d E(\zeta_i^2)}})\right| \leq K\frac{E(\zeta_i^4)}{E(\zeta_i^2)^2}k^{-d}|t|^4 e^{-t^2/4},$$

if $|t| \leq \sqrt{k^d}\left(24E(\zeta_i^4)/E(\zeta_i^2)^2\right)^{-1}$. Since $n \geq 4$, we have

$$k \geq \lfloor (2n+1)/(3\sqrt{n}+1)\rfloor \geq \sqrt{n}/4,$$

so by applying (1) and (2) to $E(\zeta_i^4)/E(\zeta_i^2)^2$, we see that the bound (8) is valid for $|t| \leq T := (a^2/48c_2)n^{\varepsilon/4}$. Using (5), (7), (6), and (8) we have

$$\begin{aligned}
&\left|P\left(S_r/\sigma(2n+1)^d \geq z\right) - \Phi(z)\right| \\
&\quad \leq \int_{-T}^{T}\left|\frac{E(e^{itS_n/\sigma(2n+1)^d}) - e^{-t^2/2}}{t}\right| dt + \frac{4}{T} \\
&\quad \leq \int_{-T}^{T}\frac{2\sqrt{bd}n^{-\varepsilon/2}}{\sigma}\,dt + \int_{0\leq|t|\leq T^{-1}}\frac{2\sqrt{b}}{\sigma}\,dt \\
&\qquad + \int_{T^{-1}\leq|t|\leq T}\frac{(2n+1)^{3d}ce^{-\alpha q}}{|t|}\,dt + \int_{-T}^{T}(4d+c_1)n^{-\varepsilon}\,dt \\
&\qquad + \int_{-T}^{T}K\frac{c_2}{a^2}k^{-d}|t|^3e^{-t^2/4}\,dt + \frac{4}{T}.
\end{aligned}$$

The reader may now easily check that each term is $O(n^{-\varepsilon/4})$ and by taking ε close to $1/2$, we may guarantee that $1/9 \leq \varepsilon/4$, which gives the result. ■

Definition 4 *Let* $\chi_n := \left(2\sigma^2|B_n|\log\log(|B_n|)\right)^{1/2}$.

Lemma 5 *For fixed* $\beta > 1$ *and* $\varepsilon > 0$, *we have*

$$P(\max_{1\leq j\leq n}|S_j| \geq \beta\chi_n) \leq 2P\left(|S_n| \geq \beta(1-\varepsilon)\chi_n\right) + O(n^{-1/2}).$$

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Proof Define $r = \lfloor n^{1/6} \rfloor$, $k = \lfloor n/r \rfloor$, and for $j = 1, \dots, n$,

$$E_j = \{|S_i| < \beta\chi_n, i < j\} \cap \{|S_j| \geq \beta\chi_n\}.$$

Now

$$\begin{aligned}
P(&\max_{1\leq j\leq n} |S_j| \geq \beta\chi_n) \\
&\leq P\Big(\bigcup_{1\leq j\leq n} [E_j \cap \{|S_n - S_j| \geq \varepsilon\chi_n\}]\Big) + P\big(|S_n| \geq \beta(1-\varepsilon)\chi_n\big) \\
&\leq \sum_{i=0}^{k-2} P\Big(\bigcup_{j=1}^{r} [E_{ir+j} \cap \{|S_n - S_{ir+j}| \geq \varepsilon\chi_n\}]\Big) \\
&\qquad + P\Big(\bigcup_{l=(k-1)r+1}^{n} [E_l \cap \{|S_n - S_l| \geq \varepsilon\chi_n\}]\Big) + P\big(|S_n| \geq \beta(1-\varepsilon)\chi_n\big) \\
&\leq \sum_{i=0}^{k-2} P\Big(\Big(\bigcup_{j=1}^{r} E_{ir+j}\Big) \cap \Big\{|S_n - S_{(i+2)r}| \geq \frac{\varepsilon}{2}\chi_n\Big\}\Big) \\
&\qquad + \sum_{i=0}^{k-2} P\Big(\bigcup_{j=1}^{r}\Big[E_{ir+j} \cap \Big\{|S_{(i+2)r} - S_{ir+j}| \geq \frac{\varepsilon}{2}\chi_n\Big\}\Big]\Big) \\
&\qquad + P\Big(\bigcup_{l=(k-1)r+1}^{n} [E_l \cap \{|S_n - S_l| \geq \varepsilon\chi_n\}]\Big) + P\big(|S_n| \geq \beta(1-\varepsilon)\chi_n\big) \\
&\leq \sum_{i=0}^{k-2} P\Big(\Big(\bigcup_{j=1}^{r} E_{ir+j}\Big) \cap \Big\{|S_n - S_{(i+2)r}| \geq \frac{\varepsilon}{2}\chi_n\Big\}\Big) \\
&\qquad + \sum_{i=0}^{k-2} P\Big(|\xi_{ir+1}| + \cdots + |\xi_{ir+2r}| \geq \frac{\varepsilon}{2}\chi_n\Big) \\
&\qquad + P\Big(|\xi_{(k-1)r+1}| + \cdots + |\xi_n| \geq \frac{\varepsilon}{2}\chi_n\Big) + P\big(|S_n| \geq \beta(1-\varepsilon)\chi_n\big).
\end{aligned}$$

Applying Lemma 2 (with $x = \frac{\varepsilon}{2}\chi_n$ and $|R| \leq 2r$), for sufficiently large n we get

$$\begin{aligned}
P(\max_{1\leq j\leq n} |S_j| \geq \beta\chi_n) \leq &\sum_{i=0}^{k-2} P\Big(\Big(\bigcup_{j=1}^{r} E_{ir+j}\Big) \cap \Big\{|S_n - S_{(i+2)r}| \geq \frac{\varepsilon}{2}\chi_n\Big\}\Big) \\
&+ k\frac{64^2 c_2 d^2 r^4}{\varepsilon^2\sigma^2(2n+1)^2} + P\big(|S_n| \geq \beta(1-\varepsilon)\chi_n\big)
\end{aligned}$$

From the decay of correlations we get

$$P\Big(\big(\bigcup_{j=1}^{r} E_{ir+j}\big) \cap \Big\{ |S_n - S_{(i+2)r}| \geq \frac{\varepsilon}{2}\chi_n \Big\} \Big)$$
$$\leq P\Big(\bigcup_{j=1}^{r} E_{ir+j}\Big) P\Big(|S_n - S_{(i+2)r}| \geq \frac{\varepsilon}{2}\chi_n \Big) + (2n+1)^{2d} c e^{-\alpha r}.$$

Now for every i we have

$$P\Big(|S_n - S_{(i+2)r}| \geq \frac{\varepsilon}{2}\chi_n \Big) \leq \frac{E\big((S_n - S_{(i+2)r})^2\big)}{(\varepsilon/2)^2\chi_n^2} \leq \frac{2b}{\varepsilon^2\sigma^2 \log\log(|B_n|)} \leq \frac{1}{2},$$

for sufficiently large n. Therefore

$$\sum_{i=0}^{k-2} P\Big(\bigcup_{j=1}^{r} E_{ir+j}\Big) P\Big(|S_n - S_{(i+2)r}| \geq \frac{\varepsilon}{2}\chi_n \Big) \leq \frac{1}{2}\sum_{i=0}^{k-2} P\Big(\bigcup_{j=1}^{r} E_{ir+j}\Big)$$
$$\leq \frac{1}{2} P(\max_{1\leq j\leq n} |S_j| \geq \beta\chi_n),$$

and hence

$$P(\max_{1\leq j\leq n} |S_j| \geq \beta\chi_n) \leq \frac{1}{2} P(\max_{1\leq j\leq n} |S_j| \geq \beta\chi_n) + (k-1)(2n+1)^{2d} c e^{-\alpha r}$$
$$+ k\frac{64^2 c_2 d^2 r^4}{\varepsilon^2\sigma^2(2n+1)^2} + P\big(|S_n| \geq \beta(1-\varepsilon)\chi_n \big).$$

That is,

$$P(\max_{1\leq j\leq n} |S_j| \geq \beta\chi_n) \leq (2n+1)^{2d+1} c e^{-\alpha r} + \frac{64^2 c_2 d^2 r^3}{(2n+1)} + 2P\big(|S_n| \geq \beta(1-\varepsilon)\chi_n \big). \quad \blacksquare$$

Corollary 1 *For fixed $\beta > 1$ there is $\rho > 0$ so that*

$$P(\max_{1\leq j\leq n} |S_j| \geq \beta\chi_n) = O\big(\log(n)^{-(1+\rho)}\big).$$

Proof Combine the central limit theorem (Lemma 4) with the maximal inequality (Lemma 5). ■

4 Law of the Iterated Logarithm

Proposition 1 *The law of the iterated logarithm holds, that is,*

$$\limsup_n \frac{S_n}{\chi_n} = 1 \quad \textit{and} \quad \liminf_n \frac{S_n}{\chi_n} = -1 \quad \textit{P-almost surely.}$$

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Proof The assertion will be proved if we show that for any $\varepsilon > 0$,

$$P\big(|S_n| > (1+\varepsilon)\chi_n \text{ i.o.}\big) = 0 \tag{9}$$

$$P\big(S_n > (1-\varepsilon)\chi_n \text{ i.o.}\big) = 1, \tag{10}$$

and

$$P\big(S_n < -(1-\varepsilon)\chi_n \text{ i.o.}\big) = 1. \tag{11}$$

The proof of (9) is almost identical to [5, Theorem 1]. For $\tau > 0$ and k so large that $(1+\tau)^k/\sigma^2 > 1$, define $n_k = \lfloor (1+\tau)^k/\sigma^2 \rfloor + 1$. Then from the maximal inequality we have

$$\begin{aligned}\sum_k P\big(\max_{1\le n\le n_k} |S_n| > (1+\gamma)\chi_{n_k}\big) &\le K \sum_k \big(\log(n_k)\big)^{-(1+\rho)} \\ &\le K \sum_k \big(k\log(1+\tau) + \log(\sigma^2)\big)^{-(1+\rho)} \\ &< \infty.\end{aligned}$$

For sufficiently large k we have $\chi_{n_k} \le (1+2\tau)^{d/2}\chi_{n_{k-1}}$. Fix $0 < \gamma < \varepsilon$ and choose τ so that $(1+\varepsilon) > (1+\gamma)(1+2\tau)^{d/2}$. The Borel-Cantelli lemma tells us that

$$\begin{aligned}P\big(|S_n| > (1+\varepsilon)\chi_n \text{ i.o.}\big) &\le P\big(\max_{n_{k-1}\le n\le n_k} |S_n| > (1+\varepsilon)\chi_{n_{k-1}} \text{ i.o.}\big) \\ &\le P\big(\max_{1\le n\le n_k} |S_n| > (1+\varepsilon)\chi_{n_{k-1}} \text{ i.o.}\big) \\ &\le P\Big(\max_{1\le n\le n_k} |S_n| > \frac{(1+\varepsilon)}{(1+2\tau)^{d/2}}\chi_{n_k} \text{ i.o.}\Big) \\ &\le P\big(\max_{1\le n\le n_k} |S_n| > (1+\gamma)\chi_{n_k} \text{ i.o.}\big) \\ &= 0,\end{aligned}$$

and this gives us (9).

We proceed to prove (10). For $k \ge 1$ define $n_k = k^{4k}$, $m_k = n_k/k^2$, and for $\lambda > 0$ put $B_k = B_k(\lambda) = \{S_{n_k} - S_{m_k} \ge (1-2\lambda)\chi_{n_k}\}$. The first thing we need to do is show that

$$\sum_k P(B_k) = \infty. \tag{12}$$

We will use the inequality

$$P\big(S_{n_k} \ge (1-\lambda)\chi_{n_k}\big) \le P(B_k) + P(S_{m_k} \ge \lambda\chi_{n_k}). \tag{13}$$

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Using $\operatorname{Var}(S_{m_k}) \le b|B_{m_k}|$ and recalling that $\chi^2_{n_k} \ge \sigma^2 |B_{n_k}|$, Chebyshev's inequality gives us

$$P(S_{m_k} \ge \lambda \chi_{n_k}) \le \frac{b|B_{m_k}|}{\lambda^2\sigma^2|B_{n_k}|} \le \frac{2^d b}{\sigma^2\lambda^2 k^{2d}}. \tag{14}$$

Since this is summable it suffices to show that $\sum_k P\big(S_{n_k} \ge (1-\lambda)\chi_{n_k}\big) = \infty$. From the Central Limit Theorem we have

$$\sum_k \Big| P\big(S_{n_k} \ge (1-\lambda)\chi_{n_k}\big) - \Phi\big((1-\lambda)\chi_{n_k}/\sigma|B_{n_k}|^{1/2}\big)\Big| \le c\sum_k |B_{n_k}|^{-1/9} < \infty.$$

Therefore it suffices to show that

$$\sum_k \Phi\big((1-\lambda)\chi_{n_k}/\sigma|B_{n_k}|^{1/2}\big) = \sum_k \Phi\Big((1-\lambda)\sqrt{2\log\log(|B_{n_k}|)}\Big) = \infty.$$

But this follows in the usual way from the asymptotic relation $\Phi(x) \sim x^{-1}\exp(-x^2/2)$ and this gives us (12).

Let ζ_k be the indicator function of B_k. Considering the distance between m_{k+j} and n_k gives

$$\begin{aligned}(k+j)^{4(k+j)-2} - k^{4k} &\ge k^{4(k+j)-2} + [4(k+j)-2]k^{4(k+j)-3}j - k^{4k}\\ &\ge [4(k+j)-2]k^{4(k+j)-3}j\\ &\ge (k+j)^2,\end{aligned}$$

so by the exponential mixing condition, we see

$$\begin{aligned}\exp(k+j)|\operatorname{Cov}(\zeta_k,\zeta_{k+j})| &\le \exp(k+j)|B_{n_{k+j}}|^2 c\exp\big(-\alpha(k+j)^2\big)\\ &\le \exp(k+j)[2(k+j)^{4(k+j)}+1]^2 c\exp\big(-\alpha(k+j)^2\big)\\ &\le \exp(k+j)3^2[(k+j)^{8(k+j)}]c\exp\big(-\alpha(k+j)^2\big)\\ &\le 9c\exp\big(10(k+j)-\alpha(k+j)^2\big)\\ &\le 9c\exp(25/\alpha).\end{aligned}$$

Adding gives us

$$\begin{aligned}K &:= 2\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}|\operatorname{Cov}(\zeta_k,\zeta_{k+j})|\\ &\le 2\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\exp\big(-(k+j)\big)\,9c\exp(25/\alpha)\\ &< \infty.\end{aligned}$$

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Therefore

$$\begin{aligned}\operatorname{Var}\Big(\sum_{k=1}^{n}\zeta_k\Big) &= \sum_{k=1}^{n}\operatorname{Var}(\zeta_k)+2\sum_{k=1}^{n-1}\sum_{j=1}^{n-k}\operatorname{Cov}(\zeta_k,\zeta_{j+k})\\ &\le \sum_{k=1}^{n}P(B_k)+K.\end{aligned}$$

Thus,

$$\begin{aligned}P\Big(\sum_{k=1}^{\infty}\zeta_k\le\frac12\sum_{k=1}^{n}P(B_k)\Big) &\le P\Big(\sum_{k=1}^{n}\zeta_k\le\frac12\sum_{k=1}^{n}P(B_k)\Big)\\ &\le P\Big(\Big|\sum_{k=1}^{n}\zeta_k-\sum_{k=1}^{n}P(B_k)\Big|\ge\frac12\sum_{k=1}^{n}P(B_k)\Big)\\ &\le \frac{4\operatorname{Var}(\sum_{k=1}^{n}\zeta_k)}{\big(\sum_{k=1}^{n}P(B_k)\big)^2}\\ &\le \frac{4\big(\sum_{k=1}^{n}P(B_k)+K\big)}{\big(\sum_{k=1}^{n}P(B_k)\big)^2}.\end{aligned}$$

Since $\sum_k P(B_k)=\infty$, letting $n\to\infty$ gives $P(\sum_{k=1}^{\infty}\zeta_k<\infty)=0$ so

$$P\big(B_k(\lambda)\text{ i.o.}\big)=1. \tag{15}$$

Note that $B_k(\varepsilon/4)\subset\big(S_{n_k}\ge(1-\varepsilon)\chi_{n_k}\big)\cup\big(-S_{m_k}\ge(\varepsilon/2)\chi_{n_k}\big)$, so from (15)

$$1\le P\big(S_{n_k}\ge(1-\varepsilon)\chi_{n_k}\text{ i.o.}\big)+P\big(-S_{m_k}\ge(\varepsilon/2)\chi_{n_k}\text{ i.o.}\big).$$

But as in (14) we see that $\sum_k P\big(-S_{m_k}\ge(\varepsilon/2)\chi_{n_k}\big)<\infty$ so that $P\big(-S_{m_k}\ge(\varepsilon/2)\chi_{n_k}\text{ i.o.}\big)=0$. From this (10) follows and (11) can be proved similarly. ■

5 An Application

The following example is extracted from [7] to which we refer the reader for complete definitions and more details.

The space of locally finite configurations in $\mathbb{R}^d$ is defined by

$$\Gamma_{\mathbb{R}^d}:=\{\gamma\subset\mathbb{R}^d:|\gamma\cap K|<\infty\text{ for every compact }K\},$$

where the configuration γ is identified with the Radon measure $\sum_{x\in\gamma}\varepsilon_x$. A Gibbs measure μ is a probability measure on $\Gamma_{\mathbb{R}^d}$ that is specified by:

- an activity parameter $z>0$, roughly the average number of particles per unit volume in $\mathbb{R}^d$;

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- and a potential function ϕ, where $\phi(r)$ roughly measures the correlation between particles at a distance r from each other.

It is known that for sufficiently small z, the measure μ is translation invariant with ρ the mean number of particles per unit space.

In the language of Section 4, we take P to be the Gibbs measure μ, and we define the random field for $i \in \mathbb{Z}^d$, by

$$x_i = \gamma\big(i + (-1/2, 1/2]^d\big) - \rho,$$

so that S_n is the number of particles in the cube $C_n := \big(-(n+1/2), n+1/2\big]^d$ minus its mean value. Under Conditions 1 and 2, Proposition 1 gives

$$\limsup_n \frac{S_n}{\sqrt{2\,\mathrm{Var}(S_n)\log\log n}} = 1, \quad \mu\text{-a.s.} \tag{16}$$

For certain of the Gibbs measures we consider, Spohn [8, Lemma 4] proved that there is an exponential decay of correlations, exactly as required in Condition 2. In [7] we show that condition 1 holds as well, and are able to conclude that (16) holds true.

But this is only half of the story. The stochastic dynamics is a $\Gamma_{\mathbb{R}^d}$-valued Markov diffusion process X_t whose invariant measure is μ. Let $X_{n,t} := X_t(C_n)$ denote the number of particles in the cube C_n at time t; then because the process is in equilibrium, equation (16) implies that

$$P\Big(\limsup_n \frac{X_{n,t} - E(X_{n,t})}{\sqrt{2\,\mathrm{Var}(X_{n,t})\log\log n}} = 1\Big) = 1, \quad \text{for all } t \geq 0.$$

Then, under certain conditions, we can use the theory of Dirichlet forms to strengthen this result [7, Proposition 6] to be uniform in time, that is,

$$P\Big(\limsup_n \frac{X_{n,t} - E(X_{n,t})}{\sqrt{2\,\mathrm{Var}(X_{n,t})\log\log n}} = 1 \quad \text{for all } t \geq 0\Big) = 1.$$

This shows that the large scale regularity of the particles is not violated even as they move through space.

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