ON FINITELY GENERATED SUBGROUPS OF FREE PRODUCTS

R. G. BURNS

(Received 15 April 1969; revised 6 August 1969)

Communicated by B. Mond

1. Statement of Results

If H is a subgroup of a group G we shall say that G is *H-residually finite* if for every element g in G, outside H, there is a subgroup of finite index in G, containing H and still avoiding g. (Then, according to the usual definition, G is *residually finite* if it is E-residually finite, where E is the identity subgroup). Definitions of other terms used below may be found in § 2 or in [6].

In this note we obtain the following result, proved in a slightly more general form as Theorem 3.1.

1.1 THEOREM. *Suppose G is the free product of its subgroups A_i indexed by some set I, and let H be a finitely generated subgroup. The following two conclusions hold.*

1.1.1 *If for each $i \in I$, $g \in G$, A_i is $(g^{-1}Hg \cap A_i)$-residually finite, then G is H-residually finite.*

1.1.2 *If the A_i are residually finite and if for each $i \in I$, $g \in G$, $g^{-1}Hg \cap A_i$ is a free factor of a subgroup of finite index in A_i, then H is a free factor of a subgroup of finite index in G.*

This theorem and Theorem 3.1 generalize Theorem 1 of [1], and the idea of the proof is the same.

Statement 1.1.1 is a generalization both of the result of M. Hall, Jr. [4] that a free group is H-residually finite for all finitely generated subgroups H, and of the result (Gruenberg [3]) that a free product of residually finite groups is residually finite. Statement 1.1.2 generalizes the result ([1]) that a finitely generated subgroup of a free group is a free factor of a subgroup of finite index. C.f. also [7].

We make a few further brief observations. The converse of 1.1.2 is true without the condition that the A_i be residually finite. (This is a simple consequence of the Kurosh subgroup theorem (see § 2, Theorem 2.3). A counterexample to show that 1.1.2 is not true without some such hypothesis is provided by the free product of a 2-cycle and the Prüfer group C_{2^∞}, taking for H the infinite cycle generated by any element of length 2.

Secondly we observe that if we combine the hypotheses of both 1.1.1 and 1.1.2, then given $g \in G \setminus H$, there is a subgroup of finite index in G, containing H as a free factor and avoiding g : i.e. a subgroup satisfying simultaneously the conclusions of 1.1.1 and 1.1.2 can be found. This follows from Lemma 2.5 below.

Finally, if (following a suggestion of S. Meskin) we define a group to be *extended residually finite* [1] if it is H-residually finite for all subgroups H, and *locally-extended residually finite* if it is H-residually finite for all finitely generated subgroups H, then we have the following result as a simple consequence of 1.1.1 and the Kurosh subgroup theorem.

1.2 COROLLARY. *The class of locally-extended residually finite groups is closed under formation of free products.*

If we denote by $\mathscr{F}$ the class of finite groups, and by $R\mathscr{F}$, $ER\mathscr{F}$ and $LER\mathscr{F}$ the classes of residually finite, extended residually finite and locally-extended residually finite groups respectively, it is not difficult to see that

$$\mathscr{F} \subset ER\mathscr{F} \subset LER\mathscr{F} \subset R\mathscr{F},$$

where $\subset$ denotes strict inclusion. Thus by Corollary 1.2 and Gruenberg's result respectively, LER$\mathscr{F}$ and $R\mathscr{F}$ are closed under free product formation. On the other hand $\mathscr{F}$ is trivially not, and, since free groups of rank > 1 do not belong to $ER\mathscr{F}$, neither is $ER\mathscr{F}$.

I thank Drs. S. Meskin and R. Gregorac for helpful comments.

2. Preliminaries

The following more-or-less well-known definitions and results are needed for the proof of our Theorem 3.1. For the sake of precision we include the definition of a free product. The identity will be denoted throughout by e.

2.1 DEFINITION. *Let G be a group and $\{A_i | i \in I\}$ a set of subgroups indexed by I. We say that G is a free product of the A_i if every non-trivial element $g \in G$ can be written uniquely in the form $a_{i_1} \cdots a_{i_n}$ where $e \neq a_{i_k} \in A_{i_k}$ $(k = 1, \cdots, n)$ and $i_k \neq i_{k+1}$ $(k = 1, \cdots, n-1)$. We say that this is the reduced form of g, that g has length n (ascribing length zero to e), and that g ends in a_{i_n} (and e ends in no element). The A_i are called free factors of G and we write $G = \prod^*_{i \in I} A_i$, or briefly $G = \prod^* A_i$.*

Note that every group has at least two free factors, namely itself and E.

In order to state the Kurosh subgroup theorem and a converse of it due to Dey [2], we need the concept of a uniform Schreier system in a free product. It is straightforward to verify that the following definition is interchangeable with that of Dey [2, Definition 2.1] or that on p. 239 of [6], except that ours includes im-

[1] Termed 'a group with finitely distinguishable subgroups' by A. I. Mal'cev. (On homomorphisms onto finite groups, Ivanov. Gos. Ped. Inst. Učen. Zap., 18 (1958), 49—60.)

plicitly the specification of 'a set of admissible functions' (see [2]). The following formulation is convenient for the proof of Theorem 3.1.

2.2 Definition. *Let G be a free product $\prod^* A_i$. A uniform Schreier system is a triple*

2.2.1 $(\{T_i, S_i \mid i \in I\}, \{B(i, \sigma), Z(i, \sigma) \mid i \in I, \sigma \in S_i\}, \{\theta_{ij} \mid i, j \in I\})$

with the following properties:

2.2.2 *For all $i \in I$, $T_i \subseteq G$, $e \in T_i$; if $t \in \bigcup_{i \in I} T_i$ ends in an element $a_j \in A_j$, then t, $ta_j^{-1} \in T_j$.*

2.2.3 *For all $i \in I$, S_i is the subset of those elements of T_i which do not end in an element of A_i; for each pair (i, σ), $i \in I$, $\sigma \in S_i$, $B(i, \sigma)$ is a subgroup of A_i, and $Z(i, \sigma)$ is a right transversal for $B(i, \sigma)$ in A_i such that*

$$(i) \qquad \sigma Z(i, \sigma) = T_i \cap \sigma A_i.$$

2.2.4 *For each ordered pair $(i, j) \in I \times I$, $\theta_{ij} : T_i \to T_j$ is a bijection satisfying: (i) $\theta_{ij} = \theta_{ji}^{-1}$; (ii) θ_{ij} is the identity map on $T_i \cap T_j$; (iii) $\theta_{ik}\theta_{kj} = \theta_{ij}$ for all $k \in I$.*

In the following statements we include explicitly only those details relevant to the present note.

2.3 Theorem. (Kurosh) (cf. [5]) *If $G = \prod^* A_i$ and H is a subgroup of G (briefly $H \leqq G$) then there exists a uniform Schreier system 2.2.1 such that*

$$H = F * \prod_{i \in I}^* \prod_{\sigma \in S_i}^* \sigma B(i, \sigma)\sigma^{-1},$$

where F is free on the set

$$\{t_i(t_i\theta_{i\alpha})^{-1} | \alpha, i \in I, \alpha \text{ fixed}, t_i \in T_i, t_i(t_i\theta_{i\alpha})^{-1} \neq e\}.$$

In addition it follows that for each (i, σ), $i \in I$, $\sigma \in S_i$,

2.3.1 $B(i, \sigma) = \sigma^{-1}H\sigma \cap A_i$.

2.4 Theorem. (Dey [2, Theorem 3.11]) *Given $G = \prod^* A_i$ and any uniform Schreier system 2.2.1 in G, then the set*

$$\{t_i(t_i\theta_{i\alpha})^{-1} | \alpha, i \in I, \alpha \text{ fixed}, t_i \in T_i, t_i(t_i\theta_{i\alpha})^{-1} \neq e\}$$

freely generates a free group F_1, say, and the subgroup closure H_1 of F_1 and the subgroups $\sigma B(i, \sigma)\sigma^{-1}$, $i \in I$, $\sigma \in S_i$, is the free product of F and these subgroups. It then also follows that for each i, T_i is a right transversal for H_1 in G.

Lastly we state the following lemma.

2.5 Lemma. *If a group G is H-residually finite for some subgroup H which is also a free factor of a subgroup of finite index in G, then given any finite subset $S \subseteq G \setminus H$, there is a subgroup of finite index in G, containing H as a free factor and avoiding S (i.e. there is a subgroup serving both purposes at once).*

The proof is trivial once the following simple corollary of the Kurosh subgroup theorem (2.3) is recalled:

If A is a free factor of a group G and $B \leqq G$, then $A \cap B$ is a free factor of B.

3. The Theorem

3.1 THEOREM. *Let H be a subgroup of a free product $G = \prod^* A_i$, with a corresponding uniform Schreier system*

$$(\{T_i, S_i\} | i \in I\}, \{B(i, \sigma), Z(i, \sigma) | i \in I, \sigma \in S_i\}, \{\theta_{ij} | i, j \in I\})$$

yielding the free decomposition

$$H = F * \prod_{i \in I}^* \prod_{\sigma \in S_i}^* \sigma B(i, \sigma)\sigma^{-1}$$

in accordance with Theorem 2.3, such that F has finite rank and the set

$$Q = \{(i, \sigma) | i \in I, \sigma \in S_i, B(i, \sigma) \neq E\}$$

is finite. Suppose further that for all $i \in I$, $g \in G$, A_i is $(g^{-1}Hg \cap A_i)$-residually finite. Then G is H-residually finite.

If in addition to the above assumptions on H, for all $i \in I$, $g \in G$, $g^{-1}Hg \cap A_i$ is a free factor of a subgroup of A_i of finite index, then H is a free factor of a subgroup of G of finite index.

Statement 1.1.1 of Theorem 1.1 follows immediately and 1.1.2 is also easily deduced once the following fact is noted:

If K is a free factor of a subgroup B of finite index in a group A $(B = K * K_1$ say$)$ and K_1 is residually finite, then A is K-residually finite.

The proof of this is as follows. By Theorem 3.1, B is K-residually finite. It is then a straightforward consequence of the definition that since B has finite index in A, the latter is also K-residually finite.

PROOF OF 3.1. By the Kurosh subgroup theorem (2.3) F is freely generated by the set

$$\{t_i(t_i\theta_{i\alpha})^{-1} | \alpha, i \in I\} \ \alpha \text{ fixed } \{t_i \in T_i\} \setminus \{e\}.$$

Define

$$R_1 = \{\sigma | \text{for some } i \in I, (i, \sigma) \in Q\} \cup \{t_i, t_i\theta_{i\alpha} | i \in I, t_i \in T_i, t_i(t_i\theta_{i\alpha})^{-1} \neq e\}.$$

By [5, Lemma 8, equation (20)], if $t_i(t_i\theta_{i\alpha})^{-1} \neq e$, then $t_i(t_i\theta_{i\alpha})^{-1} = t_j(t_j\theta_{j\alpha})^{-1}$ if and only if $t_i = t_j$. (This may also be proved by induction on the length of t_i.) This, together with the hypotheses of the theorem, implies that R_1 is finite.

Let S be a finite subset of G avoiding H. Adjoin to R_1 the identity e and the representatives in T_α of the cosets contained in HS, together with all initial segments of the resulting set, to form R_2, still finite. (If $g = a_{i_1} \cdots a_{i_n}$ is in reduced form in $\prod^* A_i$, then all elements $a_{i_1} \cdots a_{i_r}$ $(1 \leqq r \leqq n)$, and e, are called *initial segments* of g.) The inclusion of e ensures that R_2 is not empty. Clearly $R_2 \subseteq \bigcup_{i \in I} T_i$ by property 2.2.2 in the definition of uniform Schreier system.

For each pair (i, σ) such that $\sigma \in R_2 \cap S_i$, we define the sets

$$X(i, \sigma) = \{a_i | a_i \in A_i, \sigma a_i \in R_2\}$$

and

$$Y(i, \sigma) = X(i, \sigma) X(i, \sigma)^{-1} \setminus \{e\}.$$

We then have

$$Y(i, \sigma) \cap B(i, \sigma) = \phi$$

since by 2.2.3 (i), $X(i, \sigma)$ is a subset of the transversal $Z(i, \sigma)$ for $B(i, \sigma)$ in A_i. Further since R_2 is finite, clearly so is $X(i, \sigma)$ and therefore also $Y(i, \sigma)$. Consider those pairs (i, σ), $\sigma \in R_2 \cap S_i$, for which $Y(i, \sigma)$ is non-empty. By 2.3.1 and hypothesis, $B(i, \sigma)$ is contained in a subgroup $B_1(i, \sigma)$ say, of finite index in A_i and avoiding $Y(i, \sigma)$. Thus $X(i, \sigma)$ can be extended to $Z_1(i, \sigma)$, a (finite) transversal for $B_1(i, \sigma)$ in A_i. For the pairs (i, σ), $\sigma \in R_2 \cap S_i$, such that $Y(i, \sigma) = \phi$, define $B_1(i, \sigma) = A_i$ and $Z_1(i, \sigma) = \{e\}$. For every pair (i, σ) with $\sigma \in R_2 \cap S_i$, adjoin to R_2 all elements of $\sigma Z_1(i, \sigma)$ to obtain finally R. The subset R is finite since the finiteness of R_2 implies that there exist only finitely many pairs (i, σ) for which $Y(i, \sigma)$ is non-empty.

We shall now choose a uniform Schreier system

3.1.1 $$(\{R_i, S_i' | i \in I\}, \{B_1(i, \sigma), Z_1(i, \sigma) | i \in I, \sigma \in S_i'\}, \{\theta_{ij}' | i, j \in I\})$$

in G, such that $\bigcup_{i \in I} R_i = R$. Set

3.1.2 $$R_i = (T_i \cap R_2) \cup (R \setminus R_2).$$

Then S_i' is defined in accordance with 2.2.3 as the subset of those elements of R_i which do not end in an element of A_i. For those pairs (i, σ) with $\sigma \in S_i' \cap R_2$ $(= S_i \cap R_2)$, $B_1(i, \sigma)$ and $Z_1(i, \sigma)$ have been defined above. For those (i, σ) with $\sigma \in S_i' \setminus R_2$, define $B_1(i, \sigma) = A_i$ and $Z_1(i, \sigma) = \{e\}$.

Define θ_{ij}' to agree with θ_{ij} on $T_i \cap R_2$ and as the identity on $R \setminus R_2$. That this definition of $\{\theta_{ij}'\}$ is possible and satisfies 2.2.4, follows from the fact that

3.1.3 $$(T_i \cap R_2)\theta_{ij} = T_j \cap R_2 \qquad \text{for all } i, j \in I.$$

This is established as follows. Let $t_i \in T_i \cap R_2$. If $t_i \theta_{ij} = t_i$, then $t_i \theta_{ij} = t_i \in T_j \cap R_2$. Suppose on the other hand that $t_i \theta_{ij} = t_j \neq t_i$; then at least one of t_i, t_j differs from $t_i \theta_{i\alpha} = t_i \theta_{ij} \theta_{j\alpha} = t_j \theta_{j\alpha}$. Thus at least one of $t_i(t_i \theta_{i\alpha})^{-1}$ and $t_j(t_j \theta_{j\alpha})^{-1}$ is non-trivial, whence by the definition of $R_1 \subseteq R_2$, we have $t_i \theta_{i\alpha}$ $(= t_j \theta_{j\alpha}) \in R_1$, and t_i and t_j must both belong to R_1. For if they both differ from $t_i \theta_{i\alpha}$, the definition of R_1 forces them to be in R_1, while if either equals $t_i \theta_{i\alpha}$, it is trivially in R_1. Thus $(T_i \cap R_2)\theta_{ij} \subseteq T_j \cap R_2$, and 3.1.3 follows by symmetry since $(T_j \cap R_2)\theta_{ij}^{-1} = (T_j \cap R_2)\theta_{ji}$.

https://doi.org/10.1017/S1446788700009824 Published online by Cambridge University Press

Secondly we verify that R_i satisfies 2.2.2. Thus suppose $t \in \bigcup R_i$ ends in an element $a_i \in A_i$. If $t \in R_2$, then since R_2 is closed under taking initial segments, also $ta_i^{-1} \in R_2$. But $R_2 \subseteq \bigcup T_i$. Thus, since $\{T_i\}$ satisfies 2.2.2, both t and ta_i^{-1} are in T_i. Hence $t, ta_i^{-1} \in T_i \cap R_2 \subseteq R_i$. Suppose on the other hand $t \notin R_2$: then $t \in R \setminus R_2$. It follows from the definition of R that $t \in \sigma Z_1(j, \sigma)$ for some element σ in $S_j \cap R_2$. Clearly, since $t \notin R_2$ and t ends in $a_i \in A_i$, we must have $j = i$, $a_i \in Z_1(i, \sigma)$ and $ta_i^{-1} = \sigma$. Hence $ta_i^{-1} \in R_2 \cap T_i \subseteq R_i$. Note that since $e \in T_i \cap R_2$, we have $e \in R_i$ for all i.

There only remains to check that condition 2.2.3 (i) is satisfied; i.e. that

$$\sigma Z_1(i, \sigma) = R_i \cap \sigma A_i$$

for each pair (i, σ) with $\sigma \in S_i'$. If $\sigma \in R_2$, then $\sigma Z_1(i, \sigma) \subseteq R_i$ by definition, whence $\sigma Z_1(i, \sigma) \subseteq R_i \cap \sigma A_i$. If $\sigma \notin R_2$, then by definition, $Z_1(i, \sigma) = \{e\}$ and $\sigma Z_1(i, \sigma) = \{\sigma\} \subseteq R_i \cap \sigma A_i$. It remains to prove that $\sigma Z_1(i, \sigma) \supseteq R_i \cap \sigma A_i$. Let $x \in R_i \cap \sigma A_i$; say $x = \sigma a_i$. If $\sigma \in R_2$, then by construction of R, $\sigma a_i \in \sigma Z_1(i, \sigma)$. If $\sigma \notin R_2$, then $\sigma a_i \notin R_2$ since R_2 is closed under taking initial segments, and therefore $\sigma a_i \in \sigma' Z_1(j, \sigma')$ for some $\sigma' \in R_2 \cap S_j'$ where $j \neq i$. It follows that $a_i = e$, and then trivially $x = \sigma \in \sigma Z_1(i, \sigma)$ since $e \in Z_1(i, \sigma)$. This completes the verification that 3.1.1 is a uniform Schreier system.

Let H_1 be the subgroup determined by this system in accordance with Theorem 2.4. Since R_i is finite, H_1 has finite index in G. We now show that $H_1 \geqq H$. To this end let (i, σ) be an arbitrary element of Q: then $\sigma \in R_2$ by definition of R_2, whence $\sigma \in T_i \cap R_2 \subseteq R_i$, showing that $\sigma \in S_i'$. Now for each $(i, \sigma) \in Q$, $B_1(i, \sigma)$ was chosen to contain $B(i, \sigma)$. We therefore have, for each $(i, \sigma) \in Q$, $\sigma B(i, \sigma)\sigma^{-1} \leqq \sigma B_1(i, \sigma)\sigma^{-1}$. But the latter is a free factor of H_1 by Theorem 2.4, whence

3.1.4 $$\sigma B(i, \sigma)\sigma^{-1} \leqq H_1.$$

Define

$$F_1 = \text{sgp}\,\{t_i(t_i\theta_{i\alpha}')^{-1} | i, \alpha \in I, \alpha \text{ fixed as before}, t_i \in R_i\}.$$

We shall show that

3.1.5 $$F_1 = F.$$

Now $F = \text{sgp}\,\{t_i(t_i\theta_{i\alpha})^{-1} | i, \alpha \in I, \alpha \text{ as above}, t_i \in T_i\}$. By definition of R_1, for $t_i \in T_i$, $t_i(t_i\theta_{i\alpha})^{-1} \neq e$ only if $t_i \in R_1$; i.e. only if $t_i \in R_1 \cap T_i \subseteq R_2 \cap T_i$. Since $\theta_{i\alpha}'$ agrees with $\theta_{i\alpha}$ on $R_2 \cap T_i$ and $(R_2 \cap T_i)\theta_{i\alpha} = R_2 \cap T_\alpha$ by 3.1.3, we have $F_1 \geqq F$. On the other hand if $t_i \in R_i$ is such that $t_i(t_i\theta_{i\alpha}')^{-1} \neq e$, then by the definition (3.1.2) of R_i, and that of θ_{ij}', $t_i \in T_i \cap R_1$. It follows that $F_1 \leqq F$, and 3.1.5 is proved.

We infer from 3.1.4, 3.1.5 and Theorem 2.4 that $H \leqq H_1$.

Next we prove that H_1 avoids S. By the definition of R_2, for each $s \in S$ there is a non-trivial element $r \in R_2$ such that $sr^{-1} \in H$. Suppose $s \in H_1$: then since

https://doi.org/10.1017/S1446788700009824 Published online by Cambridge University Press

$H \lneqq H_1$, $r \in H_1$. However r is a non-trivial member of a right transversal (at least one of the R_i) containing e. Hence $s \notin H_1$ and we have proved that G is H-residually finite.

The second statement of the theorem is proved as follows. By Lemma 2.5 and the hypotheses of the theorem, for each pair $(i, \sigma) \in Q$ there exists a subgroup of A_i, say $D(i, \sigma)$, avoiding $Y(i, \sigma)$ and containing $B(i, \sigma)$ as a free factor. For these (i, σ) we may therefore choose $B_1(i, \sigma) = D(i, \sigma)$. We have, by Theorem 2.4,

$$H_1 = F * \prod_{i \in I}^* \prod_{\sigma \in S_i'}^* \sigma B_1(i, \sigma)\sigma^{-1};$$

whereas

$$H = F * \prod_{i \in I}^* \prod_{\sigma \in M}^* \sigma B(i, \sigma)\sigma^{-1}$$

where $M_i = \{\sigma | (i, \sigma) \in Q\}$. However $M_i \subseteq S_i'$ by the definition of R_i and S_i', and since $\sigma B(i, \sigma)\sigma^{-1}$ is a free factor of $\sigma B_1(i, \sigma)\sigma^{-1}$ for $(i, \sigma) \in Q$, it follows that H is a free factor of H_1. This completes the proof of the theorem.

References

[1] R. G. Burns, 'A note on free groups', *Proc. Amer. Math. Soc.*, 23 (1969), 14—17.

[1] I. M. S. Dey, 'Schreier systems in free products', *Proc. Glasgow Math. Assoc.*, 7 (1965—66), 61—79.

[3] K. W. Gruenberg, 'Residual properties of infinite soluble groups', *Proc. London Math. Soc.* (3) 7 (1957), 29—62.

[4] M. Hall, Jr., 'Coset representation in free groups', *Trans. Amer. Math. Soc.* 67 (1949), 421—432.

[5] S. MacLane, 'A proof of the subgroup theorem for free products', *Mathematika* 5 (1958), 161—183.

[6] W. Magnus, A. Karrass and D. Solitar, *Combinatorial group theory* (Interscience, New York, 1966).

[7] A. Karrass and D. Solitar, 'On finitely generated subgroups of a free group', *Proc. Amer. Math. Soc.*, 22 (1969), 209—213.