INJECTIVE HOMOGENEITY AND THE AUSLANDER–GORENSTEIN PROPERTY

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1. Introduction.

1.1. In this paper we refer to [**13**] and [**16**] for the basic terminology and properties of Noetherian rings. For example, an *FBN* ring means a fully bounded Noetherian ring [**13**, p. 132], and a *clique* of a Noetherian ring R means a connected component of the graph of links of R [**13**, p. 178]. For a ring R and a right or left R-module M we use pr.dim.(M) and inj.dim.(M) to denote its projective dimension and injective dimension respectively. The right global dimension of R is denoted by r.gl.dim.(R).

Let R be a ring and let M be a finitely generated right R-module. We define the *upper grade* and *grade* of M, denoted by u.gr.$_R(M)$ and $j_R(M)$ respectively, or simply by u.gr.(M) and $j(M)$, as

$$\text{u.gr.}(M) = \sup\{n \mid \operatorname{Ext}^n_R(M, R) \neq 0\},$$

and

$$j(M) = \inf\{n \mid \operatorname{Ext}^n_R(M, R) \neq 0\},$$

see [**4**], [**6**], [**7**] and [**15**].

1.2. Suppose that R is a Noetherian ring with finite (right and left) injective dimensions, which are equal by [**29**]. If for every finitely generated right or left R-module M, for every integer i, and for every submodule N of $\operatorname{Ext}^i_R(M, R)$, $j(N) \geq i$, then R is called an *Auslander–Gorenstein* ring. (This is a generalization of the concept of a commutative Gorenstein ring, since a commutative Noetherian ring of finite injective dimension is always Auslander–Gorenstein [**2**].) An Auslander–Gorenstein ring R is called *Macaulay* if $j(M) + \text{K.dim.}(M) = \text{K.dim.}(R)$ holds for every finitely generated right or left R-module M, where K.dim.( ) denotes the (Gabriel–Rentschler) Krull dimension. See [**25**, Section 1]. The research of Stafford and Zhang [**25**] shows that the Auslander–Gorenstein and Macaulay properties are closely related to some other homological properties, which are explained in the following definition.

Definition. Let R be an FBN ring

(i) If R has finite injective dimension and for each pair of maximal ideals P and Q in the same clique

$$\text{u.gr.}(R/P) = \text{u.gr.}(R/Q),$$

then R is called a *right injectively homogeneous* ring, *right inj.hom.* ring for short. If for every maximal ideal P

$$\text{u.gr.}(R/P) = \text{inj.dim.}(R),$$

then R is called *right injectively smooth*, *right inj. smooth* for short.

(ii) If R has finite global dimension and for each pair of maximal ideals P and Q in the same clique

$$\text{pr.dim.}(R/P) = \text{pr.dim.}(R/Q),$$

then R is called a *right homologically homogeneous* ring, *right hom.hom.* ring for short. If further for each maximal ideal P of R

$$\text{pr.dim}(R/P) = \text{gl.dim.}(R),$$

then R is called a *right homologically smooth* ring, a *right hom. smooth* ring for short.

In the above definition R/P and R/Q are considered as right R-modules. We also have their symmetric left hand sided concepts.

1.3. Hom. hom. rings and inj.hom. rings were first introduced and studied by K. A. Brown and C. R. Hajarnavis in [**6**] and [**7**]. There the definitions are slightly different and the rings are assumed to be integral over their centres. Our above definition is adopted from Stafford and Zhang [**25**]. Stafford and Zhang have proved that right inj. smooth Noetherian P.I. rings are also left inj. smooth, and these rings are Auslander–Gorenstein and Macaulay [**25**, Theorem 1.3]. We point out that the converse of this result is also true, see 3.2. Proposition, and that right inj.hom. Noetherian P.I. rings are also left inj.hom., so in this case we may simply call then inj.hom rings. [**25**, Theorem 5.6] shows that hom.hom. Noetherian P.I. rings are Auslander-regular. We generalize this result and obtain the following theorem.

Suppose that R is a right inj.hom. Noetherian P.I. ring. Then R is Auslander–Gorenstein, and R is also left inj.hom.

In fact there are many Auslander–Gorenstein rings which are not injectively homogeneous $\left(\text{for example } R = \begin{bmatrix} k & k \\ 0 & k \end{bmatrix}, \text{ where } k \text{ is an arbitrary field}\right)$. However, for a P.I. ring whose cliques of maximal ideals are localizable (and this includes many Noetherian P.I. rings; see 3.5 below) one can be more precise. Such a ring R is called *locally Macaulay* if the localized ring of R at each clique of maximal ideals is Auslander–Gorenstein and Macaulay. We show in 3.8 Corollary that a Noetherian P.I. ring R with each clique of maximal ideals localizable is inj.hom. if and only if R is Auslander–Gorenstein and locally Macaulay.

In Section 2, we first study the injective homogeneity of crossed products, and then use the duality machinery of smash products to transfer our results to strongly group graded rings, and prove the following theorem.

Let G be a finite group and let $S = R(G)$ be a strongly G-graded ring with coefficient ring R. Then R is a right inj.hom. (respectively right inj. smooth) FBN ring if and only if so is S.

A hom.hom. version of 2.9 Theorem (2.11 Corollary) is also given.

2. Injective homogeneity of crossed products and strongly group graded rings. Let G be a group and let R be a ring. We use $R*G$ to denote a skew group ring or crossed product of G over R. Suppose S is a G-graded ring and suppose the component of S corresponding to the identity element of G is R. We call R the coefficient ring of S and denote $S=R(G)$. We refer the reader to [**17**], [**19**] and [**21**] for the definitions and properties of skew group rings, crossed products and group graded rings.

For the proof of our main results, we first give some lemmas.

2.1. Lemma. *Let R be a right Noetherian ring and let M be a finitely generated right R-module. Then* $\text{u.gr.}(M)\le \text{pr.dim.}(M)$. *If* $\text{pr.dim}(M)$ *is finite, then* $\text{u.gr.}(M)=\text{pr.dim.}(M)$. *In particular, if* $\text{r.gl.dim.}(R)<\infty$, *then* $\text{r.inj.dim.}(R)=\text{r.gl.dim.}(R)$.

Proof. The inequality is obvious.

Suppose that $\text{pr.dim.}(M)=n<\infty$. Then $\text{Ext}_R^{n+1}(M,-)=0$ and there exists a finitely generated right R-module N such that $\text{Ext}_R^n(M,N)\neq 0$ [**20**, Proposition 9, p. 147]. Let

$$0\to K\to R^{(m)}\to N\to 0$$

be an exact sequence. Then we have an exact sequence

$$\text{Ext}_R^n(M,R^{(m)})\to \text{Ext}_R^n(M,N)\to 0.$$

Since $\text{Ext}_R^n(M,N)\neq 0$, $\text{Ext}_R^n(M,R^{(m)})\neq 0$. Thus $\text{Ext}_R^n(M,R)\neq 0$. Therefore $\text{u.gr.}(M)=\text{pr.dim.}(M)$. The final statement follows easily.

Note. It is possible that $\text{pr.dim.}(M)=\infty$, but $\text{u.gr.}(M)$ is finite. For example, let k be a field of characteristic $p>0$, let G be the cyclic group of order p and let $R=kG$ be the group ring. Let k be the principal kG-module. Then $\text{pr.dim.}(k)=\infty$, but $\text{u.gr.}(k)=0$.

From 2.1 Lemma we have the following result.

2.2. Lemma. *Let R be an FBN ring. Then R is right hom.hom. (respectively right hom. smooth) if and only if R is right inj.hom. (respectively right inj. smooth) and has finite global dimension.*

The following fact will be used later. We list it as a lemma, but leave its proof to the reader.

2.3. Lemma. *Let R be an FBN ring and let S be a ring Morita equivalent to R. Then S is also FBN and*

(i) *if R is right inj.hom. (resp. right inj. smooth), then so is S;*

(ii) *if R is right hom.hom. (resp. right hom. smooth), then so is S.*

2.4. Let R be a ring with an automorphism σ and let M be a right R-module. We can construct a new R-module, denoted by M^σ, as follows. The underlying Abelian group of M^σ, is that of M, but with the elements labelled by m^σ rather than m; and multiplication is defined by $m^\sigma r=(m\sigma^{-1}(r))^\sigma$. (See [**16**, 7.3.4] for details.) It is easy to see that $R\cong R^\sigma$ as right R-modules and for any two right R-mdoules M and N we have an isomorphism of Abelian groups

$$\text{Ext}_R^n(M,N)\cong \text{Ext}_R^n(M^\sigma,N^\sigma).$$

Therefore for any finitely generated right R-module M

$$\text{u.gr.}(M) = \text{u.gr.}(M^\sigma), \tag{1}$$

and

$$j(M) = j(M^\sigma). \tag{2}$$

Suppose that I is a right ideal of R. Then we can see that $(R/I)^\sigma \cong R/\sigma(I)$ as right R-modules. Therefore

$$\text{u.gr.}(R/I) = \text{u.gr.}(R/\sigma(I)). \tag{3}$$

2.5. Since the following fact will be used quite often, we list it as a lemma.

Lemma. *Let G be a finite group and let $S = R(G)$ be a strongly G-graded ring with coefficient ring R. Suppose M is a finitely generated right S-module. Then* $\text{u.gr.}_S(M) = \text{u.gr.}_R(M)$, *and* $j_S(M) = j_R(M)$.

Proof. By [**18**, Theorem 2.1], $\text{Ext}^i_R(M, R) \cong \text{Ext}^i_S(M, S)$ for each non-negative integer i, so our lemma follows directly.

2.6. Let S be a ring with a subring R, let P be a prime ideal of S and let p be a prime ideal of R. We say that the prime ideal P of S is *lying over* p if p is a miminal prime over $P \cap R$.

Lemma. *Let G be a finite group, let R be an FBN ring and let $S = R * G$ be a crossed product. Suppose P is a maximal ideal of S lying over a prime ideal p of R. Then*

$$\text{u.gr.}_S(S/P) = \text{u.gr.}_R(R/p).$$

Proof. By [**21**, Theorem 16.6], p is a maximal ideal of R and $P \cap R = \bigcap_{g \in G} p^g$, where p^g is the image of p under the automorphism of R induced by g. (The context should prevent any ambiguity with the notation of 2.4.) Since R is FBN, S is also FBN by [**14**, Proposition 4.9]. By [**13**, Proposition 8.4] S/P and R/p are simply Artinian rings. Suppose

$$S/P \cong V^{(n)}, \quad \text{where } V \text{ is a simple right } S\text{-module.} \tag{4}$$

Being a finitely generated module over the semisimple Artinian ring $R/(P \cap R)$, V as a right R-module is semisimple Artinian, and we may suppose

$$V \cong V_1 \oplus V_2 \oplus \ldots \oplus V_m \tag{5}$$

as right R-modules, where V_i are simple right R-modules. Then $P \cap R = \text{ann}_R(V) = \bigcap_{i=1}^{m} \text{ann}_R(V_i)$. Thus for each i, $\text{ann}(V_i) = p^{g_i}$, for some $g_i \in G$. As right R-modules

$$R/p^{g_i} \cong V_i^{(k_i)}, \quad \text{for some positive integer } k_i. \tag{6}$$

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Therefore by (3), (4), (5) and (6) we have

$$\text{u.gr.}_R(S/P) = \text{u.gr.}_R(V) = \max\{\text{u.gr.}_R(V_i)\} = \max\{\text{u.gr.}_R(R/p^{g_i})\} = \text{u.gr.}_R(R/p). \quad (7)$$

By 2.5 Lemma,

$$\text{u.gr.}_S(S/P) = \text{u.gr.}_R(S/P). \quad (8)$$

Combining (7) and (8) we have $\text{u.gr.}_S(S/P) = \text{u.gr.}_R(R/p)$.

2.7. We refer the reader to [**13**] for the definition and basic properties of links between two prime ideals of a Noetherian ring. The following lemma is motivated by [**5**, Lemma 2.2].

LEMMA. *Let R be an FBN ring, let G be a finite group and let $S = R * G$ be a crossed product. Suppose that P and Q are two maximal ideals of S such that there is a link from Q to P. Let $Q \cap R = \bigcap_{g \in G} q^g$ and $P \cap R = \bigcap_{g \in G} p^g$. Then there exists an $h \in G$ such that either $q = p^h$ (in which case $P \cap R = Q \cap R$), or there is a link from q to p^h.*

Proof. Suppose that R, G, S, P and Q are as stated. By [**21**, Theorem 16.6] there exist maximal ideals p and q of R such that $Q \cap R = \bigcap_{g \in G} q^g$ and $P \cap R = \bigcap_{g \in G} p^g$. Since there is a link from Q to P, by [**13**, Theorem 11.2] there exists a finitely generated uniform right S-module M with an affiliated series $0 < U < M$ such that U is isomorphic to a uniform right ideal of S/P and M/U is isomorphic to a uniform right ideal of S/Q, $\text{ann}_S(U) = P$ and $\text{ann}_S(M/U) = Q$.

Let $V = \text{ann}_M(P \cap R)$. If $U \subsetneqq V$, then $P \cap R \subseteq Q$. Therefore $P \cap R \subseteq Q \cap R$. Thus there exists an $h \in G$ such that $p^h \subseteq q$. Because p and q are maximal ideals of R, $p^h = q$.

Thus we may suppose that

$$U = V. \quad (9)$$

Let $H = \{g \in G \mid p^g = p\}$ and let T be a right transversal set of H in G. For $g \in T$, let E_g be the R-injective hull of the right R-module R/p^g. It is easy to see that $(R/p^g) \otimes_R S$ is essential in $E_g \otimes_R S$ both as right R-modules and hence as right S-modules. By [**18**, Corollary 2.6], $E_g \otimes_R S$ is an injective right S-module. Therefore $E_g \otimes_R S$ is the injective hull of $(R/p^g) \otimes_R S$. Let $E = \bigoplus_{g \in T} (E_g \otimes_R S)$ and let $A = \bigoplus_{g \in T} ((R/p^G) \otimes_R S)$. Then E is the S-injective hull of A.

Since p^g, for all $g \in T$, are maximal ideals of R,

$$R\Big/\Big(\bigcap_{g \in T} p^g\Big) \cong \bigoplus_{g \in T} (R/p^g)$$

as right R-modules. Thus as right R-modules

$$A = \bigoplus_{g \in T} ((R/p^g) \otimes_R S) \cong \Big(\bigoplus_{g \in T} (R/p^g)\Big) \otimes_R S \cong \Big(R\Big/\Big(\bigcap_{g \in T} p^g\Big)\Big) \otimes_R S \cong \Big(R\Big/\Big(\bigcap_{g \in T} p^g\Big)\Big) * G.$$

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Let $\bar{S} = \left(R\Big/\left(\bigcap_{g\in T} p^g\right)\right) * G$ and $\bar{P} = P\Big/\left(\bigcap_{g\in T} p^g\right)S$. Because $R\Big/\left(\bigcap_{g\in T} p^g\right)$ is a semisimple Artinian ring and G is finite, $\bar{S}$ is a quasi-Frobenius ring by [**18**, Corollary 2.10]. By [**26**, p. 276 Definition and Proposition 3.1], $\bar{P}$ is a right annihilator ideal of $\bar{S}$; that is, there exists a non-zero right ideal $\bar{I}$ of $\bar{S}$ such that $\bar{I}\bar{P} = 0$. Therefore $\bar{I}$ is a right $\bar{S}/\bar{P}$-module, so it is a right S/P-module. Because S/P is a simple Artinian ring and U is isomorphic to a uniform right ideal of S/P, U is a simple right S/P-module, so we may suppose that $U \subseteq \bar{I}$ as right S-modules. Since U is essential in M, we may suppose that $M \subseteq E$. It is obvious that $A(P \cap R) = 0$. By (9) $A \cap M = U$. Therefore

$$M/U = M/(A \cap M) = (M + A)/A \subseteq E/A.$$

Choose a uniform R-submodule C/A of E/A such that $C/A \subseteq M/U$ and $\mathrm{ann}_R(C/A) = L$, say, is a prime ideal of R. Since $Q \cap R = \mathrm{ann}_R(M/U) \subseteq L$, $q^f \subseteq L$, for some $f \in G$, so $L = q^f$. As R-modules

$$C/A \subseteq E/A \cong \bigoplus_{\substack{g\in T\\ h\in G}} (E(R/p^g) \otimes_R h)/((R/p^g) \otimes_R h).$$

Since C/A is uniform, there exist $g \in T$ and $h \in G$ such that C/A isomorphic to a submodule of $(E(R/p^g) \otimes_R h)/((R/p^g) \otimes_R h)$. Let D be a submodule of $E(R/p^g) \otimes_R h$ containing $(R/p^g) \otimes_R h$ such that

$$C/A \cong D/((R/p^g) \otimes_R h).$$

Let $F = \mathrm{ann}_D(p^{gh})$. If $F = D$ then $p^{gh} \subseteq L$ and so $p^{gh} = L = q^f$. Therefore $q = p^{ghf^{-1}}$. Thus we may suppose that $F \neq D$. Choose $D' \subseteq D$ such that $0 \subset F \subset D'$ is an affiliated series. It is easy to see that its affiliated primes are p^{gh} and L. By Jategaonkar's main Lemma [**13**, Theorem 11.1] there is a link from L to P^{gh}; that is, from q^f to p^{gh}. Therefore there is a link from q to $p^{ghf^{-1}}$.

2.8. Proposition. *Let R be an FBN ring, let G be a finite group and let $S = R * G$ be a crossed product. Then R is right inj.hom.* (*resp. right inj. smooth*) *if and only if so is S.*

Proof. Suppose that R, G and S are as stated. Since R is FBN, by [**14**, Proposition 4.9] S is also FBN. By [**18**, Corollary 2.7]

$$\text{inj.dim.}(S) = \text{inj.dim.}(R),$$

so R has finite injective dimension if and only if S has finite injective dimension.

($\Rightarrow$) Suppose that R is right inj.hom. Let P and Q be two maximal ideals of S in the same clique. By [**21**, Theorem 16.6] we may suppose that

$$P \cap R = \bigcap_{g\in G} p^g, \quad \text{and} \quad Q \cap R = \bigcap_{g\in G} q^g,$$

where p and q are two maximal ideals of R. By 2.7 Lemma $q \in \bigcup_{g\in G} \mathrm{cl}(p^g)$, so there exists an $h \in G$ such that $q \in \mathrm{cl}(p^h)$. Since R is right inj.hom. we have $\mathrm{u.gr.}_R(R/q) = \mathrm{u.gr.}_R(R/p^h)$. By (3) we have

$$\mathrm{u.gr.}_R(R/p) = \mathrm{u.gr.}_R(R/q). \tag{10}$$

By 2.6 Lemma and (10), we obtain $\text{u.gr.}_S(S/P) = \text{u.gr.}_S(S/Q)$. Therefore S is right inj.hom.

($\Leftarrow$) Suppose that S is right inj.hom. Let p and q be two maximal ideals of R such that there is a link from q to p. By [**14**, Theorem 5.3], there exist prime ideals Q and P of S with Q lying over q and P lying over p such that Q and P are in the same clique. By [**21**, Theorem 16.6] P and Q are maximal ideals of S. By 2.6 Lemma and the right injective homogeneity of S we have

$$\text{u.gr.}_R(R/p) = \text{u.gr.}_S(S/P) = \text{u.gr.}_S(S/Q) = \text{u.gr.}_R(R/q).$$

Therefore R is right inj.hom.

The same argument will give the proof of the right inj. smooth version.

2.9. It is well known that the smash product is a useful tool to translate skew group ring results to the context of group graded rings. We now demonstrate this fact in generalizing 2.8 Propositon to strongly group graded rings. We use $S\#G^*$ to denote the smash product of G over a G-graded ring S. We remind the reader that the smash product $S\#G^*$ is a free right S-module with a basis $\{p_x\}_{x\in G}$, $S\#G^* = \oplus \sum_{x\in G} p_xS$, and there exists an action of G on $S\#G^*$ defined by $(p_xs)^g = p_{(g^{-1}x)}s$, so we can form a skew group ring $(S\#G^*)*G$; see [**9**], [**22**] and [**21**, Section 2] for details.

THEOREM. *Let G be a finite group and let $S = R(G)$ be a strongly G-graded ring with coefficient ring R. Then*

(i) *R is FBN if and only if S is FBN*;

(ii) *R is a right inj.hom.* (*resp. right inj. smooth*) *FBN ring if and only if so is S.*

Proof. Suppose that G, R and S are as stated. We use $\approx$ to denote an equivalence of categories. By [**9**, Theorem 2.2] and [**10**, Theorem 2.8] we have

$$\text{Mod}(S\#G^*) \approx \text{GrMod}(S) \approx \text{Mod}(R), \tag{11}$$

where Mod( ) and GrMod( ) denote the categories of right modules and graded right modules respectively. As stated before there is a skew group ring $(S\#G^*)*G$. Using [**21**, Theorem 2.5] we have

$$(S\#G^*)*G \cong M_{|G|}(S). \tag{12}$$

(i) Since S is a finitely generated right R-module, see [**18**, Section 1], if R is FBN, then S is FBN by [**14**, Proposition 4.9].

Since $S\#G^*$ is a finitely generated S-module, if S is FBN then $S\#G^*$ is FBN, again using [**14**, Proposition 4.9]. By (11) R is FBN.

(ii) ($\Rightarrow$) Suppose that R is right inj.hom. and FBN. By 2.3 Lemma and (11) $S\#G^*$ is right inj.hom. and FBN. Then by 2.8 Proposition $(S\#G^*)*G$, a skew group ring, is also right inj.hom. and FBN, and so is S by (12).

($\Leftarrow$) Suppose that S is right inj.hom. and FBN. By (i) R is FBN. By (12) and 2.8 Proposition $S\#G^*$ is right inj.hom. Then by (11) R is right inj.hom.

The same argument will give the proof of the right inj. smooth version.

2.10. From the proof of 2.9 Theorem, we obviously have the following corollary.

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COROLLARY. *Let G be a finite group and let $S = R(G)$ be a strongly G-graded ring with coefficient ring R. Then*

(i) *S is FBN if and only if $S\#G^*$ is FBN;*

(ii) *S is right inj.hom. (resp. right inj. smooth) if and only if $S\#G^*$ is right inj.hom. (resp. right inj. smooth).*

2.11. It is well known that for a finite group G and a strongly G-graded Noetherian ring S with coefficient ring R, when R has finite global dimension, the global dimension of S may be infinite (even in the group ring case). Thus we have the following modified version of 2.9 Theorem for right hom.hom. properties, which follows easily from 2.9 Theorem and 2.2 Lemma, noting also [**28**, Lemma 2.1(ii)].

COROLLARY. *Let G be a finite group and let $S = R(G)$ be a strongly G-graded ring with coefficient ring R.*

(i) *If S is right hom.hom. (resp. right hom. smooth), then so is R.*

(ii) *If R is right hom.hom. (resp. right hom. smooth) and S has finite global dimension, then S is also right hom.hom. (resp. right hom. smooth).*

3. inj.hom. Noetherian P.I. rings are Auslander–Gorenstein. The purpose of this section is to prove that inj.hom. Noetherian P.I. rings are Auslander–Gorenstein, and to give equivalent characterizations of inj.hom. and hom.hom. Noetherian P.I. rings which have all their cliques of maximal ideals localizable. At the end of the paper we point out that a strongly graded ring by a finite group is Auslander–Gorenstein (or Macaulay) FBN if and only if so is its coefficient ring.

3.1 LEMMA. *Let R be an Auslander–Gorenstein, Macaulay, FBN ring. Then for each maximal ideal Q of R,*

$$j(R/Q) = \text{u.gr.}(R/Q) = \text{inj.dim.}(R) = \text{K.dim.}(R),$$

where R/Q and R may both be considered as right or left R-modules. In particular, R is right and left inj. smooth.

Proof. Suppose that $\text{inj.dim.}(R) = n$. Then there exists a finitely generated left (or right) R-module M such that $\text{Ext}^n_R(M, R) \neq 0$. Denote $\text{Ext}^n_R(M, R)$ by $E^n(M)$. It is easy to see that $E^n(M)$ is a finitely generated right (or left) R-module. By the Auslander-Gorenstein condition, $j(E^n(M)) \geq n$. By the Macaulay condition, we have

$$n \leq j(E^n(M)) \leq j(E^n(M)) + \text{K.dim.}(E^n(M)) = \text{K.dim.}(R). \tag{13}$$

Suppose that Q is a maximal ideal of R, so R/Q is a simple Artinian ring [**13**, Proposition 8.4], and then $\text{K.dim.}(R/Q) = 0$. Thus by the Macaulay condition and [**15**, Remark 2.2 (1)]

$$\text{K.dim.}(R) = \text{K.dim.}(R/Q) + j(R/Q) = j(R/Q) \leq n. \tag{14}$$

From (13) and (14), we obtain

$$j(R/Q) = \text{u.gr.}(R/Q) = \text{inj.dim.}(R) = \text{K.dim.}(R).$$

3.2. Since Noetherian P.I. rings are FBN, from 3.1 Lemma and [**25**, Theorem 3.10] we have the following result.

PROPOSITION. *Let R be a Noetherian P.I. ring. Then R is inj. smooth if and only if R is Auslander–Gorenstein and Macaulay.*

3.3. Suppose R is a ring, in the following we denote the Laurent series ring of R by $R((x))$, $R((x)) = \left\{ \sum_{j=n}^{\infty} r_j x^j \mid r_j \in R, n \in \mathbb{Z} \right\}$, see [**12**]. For a finitely generated right R-module M, denote $M \otimes_R R((x))$ by $M((x))$. Since it is not known at present whether the cliques of a Noetherian P.I. ring are always localizable (see 3.5 below), we are forced to pass from R to $R((x))$ in proving 3.7 Theorem. By using [**1**, 19.20 Theorem] and [**16**, 7.2.3 Proposition], we have the following lemma.

LEMMA. *Let R be a Noetherian ring. Then* $R((x))$ *is a faithfully flat R-module.*

3.4 LEMMA. *Let R be a Noetherian ring and let M be a finitely generated left R-module. Then*

(i) $\text{Ext}^i_R(M, R) \otimes_R R((x)) \cong \text{Ext}^i_{R((x))}(M((x)), R((x)))$;

(ii) $\text{Ext}^i_R(M, R) = 0$ *if and only if* $\text{Ext}^i_{R((x))}(M((x)), R((x))) = 0$;

(iii) $j_R(M) = j_{R((x))}(M((x)))$; $\text{u.gr.}_R(M) = \text{u.gr.}_{R((x))}(M((x)))$.

Proof. (i) is a direct consequence of [**8**, 1.6 Proposition]. (ii) follows from (i) since $R((x))$ is a faithfully flat R-module by 3.1 Lemma, and (iii) follows directly from (ii).

3.5. Suppose that R is a Noetherian ring and P is a prime ideal of R. We use $\mathscr{C}(P)$ to denote the set of all the elements of R which are regular modulo P. Let X be a set of prime ideals of R, we say X is *localizable* if: (i) $\cap \{\mathscr{C}(P) \mid P \in X\}$, denoted by $\mathscr{C}(X)$, is an Ore set; (ii) R_X/PR_X, for each $P \in X$, is simple Artinian and PR_X, $P \in X$, are all the primitive ideals of R_X. We call X *classically localizable* if X is localizable and has the following property: (iii) for each $P \in X$ the injective hull of the (both right and left) R_X-module R_X/PR_X is the union of its socle series; see [**13**, p. 219] and [**3**, p. 37] for details.

NOTE. It is an open question whether the cliques of a Noetherian P.I. ring are always localizable. It is also an open question whether localizable cliques of Noetherian rings are necessarily classically localizable; see [**13**, p. 289]. The following results are known: (i) if R is a Noetherian P.I. ring which is a finitely generated algebra over its centre, then every clique of R is classically localizable; (ii) if R is an FBN algebra over an uncountable field, then every clique of R is classically localizable; (iii) if R is an FBN ring, then every finite clique of R is classically localizable; see [**3**, Proposition 6.1 and Theorem 6.11] for further details.

LEMMA. *Let R be a Noetherian P.I. ring and let N be a finitely generated right R-module. If for each clique* Ω *of maximal ideals of R,* $N((x))_{\Omega((x))} = 0$, *then* $N = 0$.

Proof. If Ω is a clique of R, by [**24**] $\Omega((x)) = \{Q((x)) \mid Q \in \Omega\}$ is a clique of the Noetherian P.I. ring $R((x))$. By [**27**, Theorem 8], $\Omega((x))$ is classically localizable.

Without loss of generality, we may suppose that N_R is a non-zero simple module. Let

$P = \text{ann}_R(N)$, a maximal ideal of R. Suppose Ω is the clique of R which contains P. If $N((x))_{\Omega((x))} = 0$, then $N((x))$ is $\mathscr{C}(\Omega((x)))$-torsion, where $\mathscr{C}(\Omega((x)))$ denotes the set of elements of R which are regular modulo each prime ideal in $\Omega((x))$. Since $R/P \cong N^{(n)}$ for some positive integer n, $R((x))/P((x)) \cong N((x))^{(n)}$. Therefore $R((x))/P((x))$ is $\mathscr{C}(\Omega((x)))$-torsion. In particular there exists a $c \in \mathscr{C}(\Omega((x)))$ such that $(1 + P((x))c = 0$, that is $c \in P((x))$. This is a contradiction.

3.6. Being motivated by 3.2 Proposition, we would like to give the following definition.

Definition. Let R be an Auslander–Gorenstein Noetherian P.I. ring with all its cliques of maximal ideals localiable. If for each clique Ω of maximal ideals of R, R_Ω is Auslander–Gorenstein and Macaulay, then we call R a *locally Macaulay* ring.

Proposition. *Every commutative Noetherian ring of finite injective dimension is locally Macaulay.*

Proof. Suppose that R is a commutative Noetherian ring of finite injective dimension. Of course the cliques of R are singletons and are localizable. Let P be a prime ideal of R. Then R_P is a commutative Noetherian local ring of finite injective dimension, so it is obviously inj. smooth. By [**25**, Theorem 3.10] R_P is Macaulay. Therefore R is locally Macaulay.

Note. Let $R = k[[x]][y]$, where k is an arbitrary field and $k[[x]]$ is the power series ring. R is commutative Noetherian of finite global dimension, so it is locally Macaulay by the above Corollary; but it is not Macaulay as pointed out in [**25**, Section 2].

3.7. Theorem. *Suppose that R is a right inj.hom. Noetherian P.I. ring. Then R is Auslander–Gorenstein, and R is also left inj.hom. If each clique of maximal ideals of R is localizable, then R is locally Macaulay.*

Proof. Let Ω be a clique of maximal ideals of R. By [**24**] $\Omega((x)) = \{Q((x)) \mid Q \in \Omega\}$ is a clique of $R((x))$ (this is straightforward to prove directly) and by [**27**, Theorem 8] $\Omega((x))$ is classically localizable. We first prove that $R((x))_{\Omega((x))}$ is inj. smooth. By [**11**, Theorem 1.1]

$$\text{inj.dim}(R((x))_{\Omega((x))}) \leq \text{inj.dim.}(R((x))) \leq \text{inj.dim.}(R[[x]]).$$

But

$$\text{inj.dim.}(R[[x]]) \leq \text{inj.dim.}(R) + 1 < \infty$$

by [**23**, Corollary 11.68]. Therefore $\text{inj.dim.}(R((x))_{\Omega((x))}) < \infty$. Since $\Omega((x))$ is classically localizable, $\{P((x))_{\Omega((x))} \mid P \in \Omega\}$ are all the maximal ideals of $R((x))_{\Omega((x))}$. Suppose that $P \in \Omega$. Then

$$\text{Ext}^i_{R((x))_{\Omega((x))}}(R((x))_{\Omega((x))}/P((x))_{\Omega((x))}, R((x))_{\Omega((x))})$$

equals 0 if and only if $\text{Ext}^i_{R((x))}(R((x))/P((x)), R((x))) = 0$ by 3.5 Lemma, and if and only if $\text{Ext}^i_R(R/P, R) = 0$ by 3.4 Lemma. Thus

$$\text{u.gr.}(R((x))_{\Omega((x))}/P((x))_{\Omega((x))}) = \text{u.gr.}(R/P). \tag{15}$$

For any other element $Q \in \Omega$, since R is inj.hom, we have

$$\text{u.gr.}(R/P) = \text{u.gr.}(R/Q).$$

Then by (15) we know that

$$\text{u.gr.}(R((x))_{\Omega((x))}/Q((x))_{\Omega((x))}) = \text{u.gr.}(R((x))_{\Omega((x))}/P((x))_{\Omega((x))}),$$

and this number must be inj.dim.$(R((x))_{\Omega((x))})$ by [**25**, Lemma 3.12]. Therefore $R((x))_{\Omega((x))}$ is inj.smooth. By [**25**, Theorem 3.10] $R((x))_{\Omega((x))}$ is Auslander–Gorenstein.

Let M be a finitely generated right or left R-module, let $n \in \mathbb{N}$. Suppose that N is a submodule of $\text{Ext}^n_R(M, R)$. Then

$$N((x)) \subseteq \text{Ext}^n_{R((x))}(M((x)), R((x)))$$

by 3.4 Lemma and 3.3 Lemma. For every clique Ω of maximal ideals of R, we have

$$N((x))_{\Omega((x))} \subseteq \text{Ext}^n_{R((x))_{\Omega((x))}}(M((x))_{\Omega((x))}, R((x))_{\Omega((x))}).$$

Since $R((x))_{\Omega((x))}$ is Auslander–Gorenstein

$$\text{Ext}^m_{R((x))_{\Omega((x))}}(N((x))_{\Omega((x))}, R((x))_{\Omega((x))}) = 0, \quad \text{for all} \quad m < n.$$

By 3.4 Lemma and 3.5 Lemma we have $\text{Ext}^m_R(N, R) = 0$, for all $m < n$. Therefore R is Auslander–Gorenstein.

Let us use l.u.gr.(m) to denote the upper grade of a left module M. Suppose that R is a right inj.hom. Noetherian P.I. ring and that P is a maximal ideal of R. Let Ω be the clique of R which contains P. Since we have proved that $R((x))_{\Omega((x))}$ is (right and left) inj. smooth. By (15) and its left hand side version, we have

$$\text{l.u.gr.}(R/P) = \text{u.gr.}(R/P).$$

Therefore R is also left inj.hom.

For the final part, suppose that all the cliques of maximal ideals of R are localizable. Let Ω be a clique of maximal ideals of R. By a simpler version of the above argument, R_Ω is inj. smooth. Thus R_Ω is Macaulay by [**25**, Theorem 3.10], so R is locally Macaulay.

3.8 Corollary. *Let R be a Noetherian P.I. ring with each clique of maximal ideals localizable. Then*

(i) *R is inj.hom. if and only if R is Auslander–Gorenstein and locally Macaulay*;

(ii) *R is hom.hom. if and only if it is Auslander-regular and locally Macaulay.*

Proof. (i) ($\Rightarrow$) This is a consequence of 3.7 Theorem.

($\Leftarrow$) Suppose that R is Auslander–Gorenstein and locally Macaulay. Let P and Q be two maximal ideals of R in the same clique Ω, say. Then R_Ω is Auslander–Gorenstein and Macaulay by hypothesis. By 3.2 Proposition R_Ω is inj. smooth. As shown in the proof of 3.7 Theorem we have

$$\text{u.gr.}(R/P) = \text{u.gr.}(R_\Omega/P_\Omega) = \text{u.gr.}(R_\Omega/Q_\Omega) = \text{u.gr.}(R/Q),$$

so R is inj.hom.

The proof of (ii) follows from (i) and 3.2 Proposition.

NOTE. When R is commutative Noetherian, then as indicated in 3.6 Corollary, R Auslander–Gorenstein implies R locally Macaulay. But we don't have this implication for noncommutative Noetherian P.I. rings. The following example shows that even Auslander-regular does not imply locally Macaulay.

EXAMPLE. Let $R = \begin{bmatrix} k & k \\ 0 & k \end{bmatrix}$, where k is any field. R is Auslander-regular (as one can show by direct calculation or see also [**25**, Section 5]), but it is not Macaulay. Since R has only one clique, it is not locally Macaulay.

3.9. We studied the injective homogeneity and homological homogeneity of strongly group-graded rings in Section 2. As illustrated in the present section and [**25**], these properties are closely related to the Auslander–Gorenstein, Auslander-regular and Macaulay properties, so we would like to finish this paper with the following result.

PROPOSITION. *Let G be a finite group and let $S = R(G)$ be a strongly G-graded ring with coefficient ring R. Suppose that R is Noetherian, (but not necessarily fully bounded). Then*:

(i) *R is Auslander–Gorenstein if and only if S is Auslander–Gorenstein*;

(ii) *R is Auslander–Gorenstein and Macaulay if and only if so is S.*

Proof. Suppose that G, R and S are as stated. By [**18**, Corollary 2.7]

$$\text{inj.dim.}(R) = \text{inj.dim.}(S),$$

so R has finite injective dimension if and only if so has S.

(i) ($\Rightarrow$) Suppose that R is Auslander–Gorenstein. For each right (or left) S-module M, each integer i and each S-submodule N of $\text{Ext}^i_S(M, S)$, which is isomorphic to $\text{Ext}^i_R(M, R)$ by [**18**, Theorem 2.1], by 2.5 Lemma and the Auslander–Gorenstein property of R, we have

$$j_S(N) = j_R(N) \geq i.$$

Thus S is Auslander–Gorenstein.

($\Leftarrow$) Suppose that S is Auslander–Gorenstein. For each right (or left) R-module M, every integer i and every submodule N of $\text{Ext}^i_R(M, R)$, $S \otimes_R N$ is a submodule of $S \otimes_R \text{Ext}^i_R(M, R)$; but

$$S \otimes_R \text{Ext}^i_R(M, R) \cong \text{Ext}^i_S(M \otimes_R S, S)$$

by [**8**, 1.6 Proposition]. Since S is Auslander–Gorenstein,

$$j_S(S \otimes_R N) \geq i. \tag{16}$$

Because as R-modules $S \otimes_R N \cong \bigotimes_{g \in G} N^g$, where N^g is defined as in 2.4, by (2) we have $j_R(N) = j_R(S \otimes_R N)$. Thus by 2.5 Lemma and (16) we have

$$j_R(N) = j_R(S \otimes_R N) = j_S(S \otimes_R N) \geq i.$$

Therefore R is Auslander–Gorenstein.

(ii) For every finitely generated right (or left) S-module M, by [**18**, Theorem 1.2]

$$\text{k.dim.}_R(M) = \text{k.dim.}_S(M). \tag{17}$$

For every finitely generated right (or left) R-module N, by (16) and [**13**, Corollary 13.2] we have

$$\text{k.dim.}_R(N) = \text{k.dim.}_R(N \otimes S) = \text{k.dim.}_S(N \otimes S). \tag{18}$$

Then (ii) follows easily from (17), (18), 2.5 Lemma and the definition of the Macaulay condition.

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