SOME ASPECTS OF VISUAL BINARY STAR ORBIT DETERMINATION

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The problem under consideration is that of determining the seven elements (the vector $\vec{E}$) characterizing the relative three-dimensional motion of the components of a binary star, using observations of the two-dimensional projection on the sky of the relative positions of those components as a function of time. Sooner or later a differential correction process is employed; therefore, this correction procedure will be reviewed first.

The basic approach is to derive a first estimate of the orbital parameters, form residuals from each of the observations, and assume these residuals represent a normally distributed random error, plus systematic errors due to an incorrect initial orbit. Expressing the residuals as functions of the orbital elements, these functions can be expanded in a Taylor series about the initial orbit and truncated after the first-order terms, giving the following approximate linear relation:

$$O - C \equiv R(\Delta\vec{E}) = \sum_{i=1}^{N} \left[\partial R(\vec{E}_o)/\partial\vec{E}\right]\Delta E_i . \tag{1}$$

(See da Silva 1966 for a complete presentation of all the specific forms this equation may take.) The sum over the observations of the squares of these residuals (here called $S(\Delta\vec{E})$) is formed, and the desired condition is that this sum be a minimum. This condition can be achieved by taking derivatives with respect to each element and setting each derivative equal to zero, forming equations of the following form;

$$\partial S(\Delta\vec{E}_o)/\partial\Delta E_i = 0 , \quad i = 1, \ldots, N. \tag{2}$$

One equation is generated by each element, providing N equations in N unknowns (N isusually equal to 7 in this case), and these can now be solved for the corrections to the first estimates for the orbital elements.

There are several positive features to this procedure. First, all of the observations can be used. Second, since we have error estimates for the linear corrections, we in turn have some estimate of the uncertainty of the new orbit. Third, different kinds of observations can be mixed, provided they are weighted in such a way that the observational errors come from the same parent distribution. Fourth, partial observations are permitted; that is, observations containing information on only some of the elements may be included. Fifth, by appropriate partitions, there may be solutions for subsets of the elements. For instance, in cases where there are many elongated images plus a few good separation measures, position angles of the elongations may be used to derive shape, orientation, and phase, and then the separations may be used alone to determine size.

There are also several negative features. First, the local error estimates are often optimistic and therefore misleading. Second, because the partials are evaluated on the initial orbit, they are themselves in error, which can lead to a statistically biased set of corrections. Third, the procedure should use the observations directly (rather than functions of the observations) since their

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errors are normally distributed, and this can complicate the formalism for mixed observations. It also makes it impossible to take advantage of the linear independence of some sets of elements. Fourth and finally, it is absolutely imperative to start with a very good initial estimate, in order to find the true global minimum in phase space. A poor first guess may lead into a local minimum far from the desired one, or it may even lead to divergence.

The determination of a good first estimate is therefore the crucial aspect of the problem that really deserves most of the attention. Many methods have been proposed over the last two centuries, and the books by Aitken (1964) and Heintz (1978) discuss some of these in detail. A very popular method, which will be discussed in more detail in two later papers in these proceedings, is the Thiele-Innes- van den Boss method. The difficulties of the method are the need for the simultaneous solution of three transcendental equations and the critical dependence on a very good determination of the areal constant. These are not serious if there is good orbital coverage, but they can make things extremely difficult if only a limited arc is available. Another method, known sometimes as the Kowalsky method, is extremely easy to put on a computer (since it is at least linear at the outset) but extremely risky to use, since the epochs, the only relativel accurately known data, are not employed until the last step. Again it can be of some use with good orbital coverage but disastrously misleading with limited data.

Other methods need not be mentioned, and the ones above are only illustrative, not superior. A popular method, if high precision is not required, may be called the graphical method when the formal counterpart of the Zwiers approach is used, or the educated guess method when inspection and inspiration are all that are employed. Finally, a method that can be regarded as either the first estimate or the final step is the Fourier transform method, which will be discussed in more detail by Monet in a later paper. In essence, it relates the Fourier expansions of the coordinates in elliptic motion to the Fourier coefficients in the analysis of the observed coordinates. The method has the advantage of not making any linearizing assumptions, not requiring a first estimate for the elements, and giving error estimates for the elements directly. One the other hand, for the high eccentricities found in many visual binaries, many terms are required for a satisfactory representation of the coordinates, and the inversion of these relationships to determine the elements is extremely complex. There are the problems of Fourier expansions with unequally space data. Further, the period has to be known before the expansions can be carried out, and good orbital coverage is again required. This is a very elegant method with some good applications, but it has been found wanting for visual double star orbits and is not to be recommended for these relatively long-period pairs.

The common and long-recognized limitation to all of these methods is the need to have good orbital coverage (hopefully, at least one revolution) before a reliable orbit can be obtained. However, many visual systems have only limited coverage, often with large relative errors, and these cases are, at best, very difficult. One possibility with a short arc is to express the coordinates in power series. Because of the constraint of the law of areas, five coefficients in both coordinates are necessary and sufficient to completely determine the orbit. Therefore, an examination of the statistical significance of these coefficients might indicate something of the determinancy of the orbit. However, even with power series expansions, good orbital coverage and a large number of expansion terms are required to yield reliable terms through order four, giving this approach no advantage over the ones previously discussed.

An example of this type of short arc situation appeared in the analysis of 61 Cygni discussed by Josties earlier in these proceedings. The orbital correction would not converge, and the power series had only three significant coefficients in each coordinate, indicating a lack of sufficient orbital information. A numerical experiment was then performed, in which a series of partial corrections was carried out, in each case holding the eccentricity fixed. Corrections were possible for all eccentricities above 0.4, including the parabolic one. The sum of the squares of the residuals had the same value for each correction, indicating no uniquely preferred orbit, but rather an entire family of orbits that characterize the motion. However, the total mass of the system decreased with increasing eccentricity, suggesting a possible alternative approach to the problem.

If only a short arc is available for a well observed binary, there may be several parameters of the system that are better known than is possible for the orbital elements. The areal constant is determined from the first two power series coefficients in each coordinate, and the system mass is given primarily by the third coefficients - the accelerations. In any case, the mass may be well known from indirect astrophysical considerations. It may, therefore, be advantageous to hold these parameters fixed during the orbit determination procedure, which would thus decrease the number of degrees of freedom and make the solution determinate. Apart from trial-and-error, this can be accomplished by the use of Lagrange multipliers in the least squares estimation procedure.

For the binary orbit determination problem, the variations in the areal constant and the system mass are linearized and set equal to zero, as follows:

$$dh/h = dn/n + 2da/a - 2e\,de/\sqrt{1-e^2} - \tan I\, dI = 0, \tag{3}$$

$$d\mu/\mu = 2dn/n + 3da/a = 0. \tag{4}$$

To minimize S with these constraints, the following function is formed:

$$\Phi(\Delta\vec{E}) = S(\Delta\vec{E}) + \sum_{k=1}^{L} \lambda_k \mathcal{C}_k(\Delta\vec{E}). \tag{5}$$

Here, the $\mathcal{C}_k$ are the equations of condition, and the λ_k are the Lagrange multipliers (the index k runs over the L constraints (L=2 in this case) to be applied). The following N+L equations are then solved only for the orbital corrections:

$$\partial\Phi(\Delta\vec{E})/\partial(\Delta E_i) = 0, \quad i = 1, \ldots, N \tag{6}$$

(7)

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DISCUSSION

WILSON: If the observations are sufficiently accurate, there is really no limit to the shortness of the arc to compute an orbit.

HARRINGTON: This is true, but you do not see such things in practice.

WALKER: Isn't it more important to have observations through the nodes rather than just a large arc?

HARRINGTON: Across the nodes, and across the ends of the latus rectum. Distribution is important, and, if you can go all the way around, you will have everything.

HEINTZ: The power series representation of incompletely observed long-period orbits (probably first outlined by Fletcher 50 years ago) requires a reliable fourth-order term, and in practice this is entirely beyond reach. (In isolated cases with exceptionally small relative errors, such as ADS 11632, the third order term may be marginally reliable.) When the orbit is well defined by a longer arc, the power series ceases to be a satisfactory expression of the motion, particularly the areal constant. A radial velocity difference may be a substitute quantity, but is in long-period pairs usually too small to be useful. Thus I see little application for this kind of method.

STRAND: The orbit of 61 Cygni of about 720 years which you arrived at, based upon assumed masses of the components derived from their spectral types, is about the same as I arrived at 40 years ago based upon the same assumptions, and O. Fletcher some years before me.