Can. J. Math., Vol. XXIII, No. 3, 1971, pp. 461–467

BICONNECTED MULTIFUNCTIONS OF TREES WHICH HAVE AN END POINT AS FIXED POINT OR COINCIDENCE

HELGA SCHIRMER

1. Introduction. It was proved almost forty years ago that every mapping of a tree into itself has at least one fixed point, but not much is known so far about the structure of the possible fixed point sets. One topic related to this question, the study of homeomorphisms and monotone mappings of trees which leave an end point fixed, was first considered by G. E. Schweigert [**6**] and continued by L. E. Ward, Jr. [**8**] and others. One result by Schweigert and Ward is the following: any monotone mapping of a tree onto itself which leaves an end point fixed, also leaves at least one other point fixed.

It is further known that not only single-valued mappings, but also upper semi-continuous (usc) and connected-valued multifunctions of trees have a fixed point [**7**], and that two usc and biconnected multifunctions from one tree onto another have a coincidence [**5**]. The aim of this paper is to extend the result by Schweigert and Ward to these cases. This is done in the Main Theorem in § 3, which states that two usc and biconnected multifunctions φ, ψ from a tree T onto a tree T' which have an end point e of T as coincidence, also have a coincidence distinct from e if both $\varphi(e) \neq T'$ and $\psi(e) \neq T'$, and if either $\varphi(e)$ or $\psi(e)$ consists of a single point. Examples are given which show that the conditions on $\varphi(e)$ and $\psi(e)$ are necessary. They are of course always satisfied if either φ and ψ are single-valued, or if ψ is the identity so that the coincidences of φ and ψ reduce to the fixed points of φ.

Background material on trees and multifunctions is collected in § 2. The proof, in §4, of the Main Theorem uses the partial order structure of trees developed by Ward [**8**; **9**], but differs from his proof for the single-valued case [**8**, Theorem 9].

2. Biconnected multifunctions of trees. By a tree T we mean a continuum (i.e. compact connected Hausdorff space) in which every pair of distinct points is separated by a third point. Define $x \leqq y$ if and only if $x = r$, $x = y$ or x separates r and y, where r is a given point of T. It is well-known that $\leqq$ is an order dense partial order ([**8**; **9**]). We define, as usual,

$$\mathrm{L}(A) = \{y \in T | y \leqq x \text{ for some } x \in A\},$$
$$\mathrm{M}(A) = \{y \in T | x \leqq y \text{ for some } x \in A\}.$$

Received September 23, 1970. This research was partially supported by the National Research Council of Canada Grant No. A 7579.

https://doi.org/10.4153/CJM-1971-049-6 Published online by Cambridge University Press

Then $\mathrm{L}(x)$ and $\mathrm{M}(x)$ are closed for every $x \in T$, and $\mathrm{M}(x)\backslash\{x\}$ is open. If A is closed in T, then $\mathrm{L}(A)$ is closed. (The proof is completely analogous to [**10**, Lemma 2].) The set $[x, y] = \mathrm{M}(x) \cap \mathrm{L}(y)$ is a non-empty closed chain (i.e. it is linearly ordered) if $x < y$. We write $(x, y]$ for $[x, y]\backslash\{x\}$.

A point $m \in A$ is called a maximum of the subset A of T if $m \nless x$ for each $x \in A$. By a root r of A we understand a point $r \in A$ such that $A \subset \mathrm{M}(r)$. Every non-empty closed subset of T has a maximum [**8**, Theorem 1], and every non-empty closed connected subset has a root [**3**, Lemma 2]. We write $m = \max A$ if m is the only maximum of A (e.g. if A is a linearly ordered closed set).

By an arc we mean a continuum which has exactly two non-cutpoints. If $x < y$, then $[x, y]$ is an arc from x to y (see [**8**, Theorems 4, 6, and Corollary 1.1; **9**, Theorem 1]). The point e is called an end point of T if $T\backslash\{e\}$ is connected. It is shown in [**4**, p. 45] that e is an end point of T if and only if it is an end point of each subarc of T which contains it. Hence it follows that $\max\,[\mathrm{L}(x) \cap \mathrm{L}(y)] > e$ if $x \neq e$, $y \neq e$, and e is an end point.

A multifunction $\varphi : X \to X'$ from a space X into a space X' is a correspondence which assigns to each point of X a non-empty subset of X'. We say that $\varphi : X \to X'$ is upper semi-continuous (usc) if each point image $\varphi(x)$ is closed and, whenever U' is an open set containing $\varphi(x)$, then there exists an open set U containing x such that $\varphi(U) \subset U'$. It is well-known that usc of $\varphi : X \to X'$ implies that $\varphi^{-1}(A') = \{x \in X | \varphi(x) \cap A' \neq \emptyset\}$ is closed in X for every closed $A' \subset X'$.

We call a multifunction φ connected if $\varphi(A) = \bigcup(\varphi(x)| x \in A)$ is a connected subset of X' whenever A is a connected subset of X, and inverse-connected if $\varphi^{-1}(A')$ is a connected subset of X whenever A' is a connected subset of X'. φ is called biconnected if it is both connected and inverse-connected. Note that an usc multifunction $\varphi : T \to T'$ from a tree T onto a tree T' is biconnected under weaker assumptions: it follows from [**5**, Lemmas 3.1 and 3.2] that φ is connected if and only if it is connected-valued (i.e. $\varphi(x)$ is connected for every $x \in T$) and that it is inverse-connected if and only if it is monotone (i.e. $\varphi^{-1}(y)$ is connected for every $y \in T'$).

The following lemma will be frequently used in the proof of the Main Theorem.

LEMMA 1. *Let $\varphi : T \twoheadrightarrow T'$ be a biconnected multifunction from a tree T onto a tree T', and let $x_i' \in \varphi(x_i)$ for $i = 1, 2$. If $x_1 < x_2$ and $x_1' < x_2'$, then $[x_1', x_2'] \subset \varphi([x_1, x_2])$ and $[x_1, x_2] \subset \varphi^{-1}([x_1', x_2'])$.*

Proof. If there exists a $z' \in [x_1', x_2']$ such that $z' \notin \varphi([x_1, x_2])$, then $\mathrm{M}(z')\backslash\{z'\}$ and $T\backslash\mathrm{M}(z')$ would induce a separation of $\varphi([x_1, x_2])$. But $\varphi([x_1, x_2])$ is connected, hence $[x_1', x_2'] \subset \varphi([x_1, x_2])$. A similar argument shows that $[x_1, x_2] \subset \varphi^{-1}([x_1', x_2'])$.

If $\varphi : T \to T'$ is an usc biconnected multifunction from a tree T into a tree T', then $\varphi(x)$ is a non-empty, closed and connected subset of T' for every $x \in T$, and hence it has a root. Denote it by $f(x)$. Then $f : T \rightarrow T'$ is a single-valued function from T into T' which need not be continuous. But we show now that it is isotone, i.e. that $x \leqq y$ in T implies $f(x) \leqq f(y)$ in T'.

LEMMA 2. *Let $\varphi : T \rightarrow T'$ be an usc biconnected multifunction from a tree T with root r into a tree T' with root r'. If $f(r) = r'$, then $f : T \rightarrow T'$ is an isotone function.*

Proof. Take any $x, y \in T$ such that $x \leqq y$. It is sufficient to consider the case where $r < x < y$ and $f(x) \neq f(y)$. Let $m' = \max [\mathrm{L}(f(x)) \cap \mathrm{L}(f(y))]$. It follows from Lemma 1 that $m' \in [r', f(x)] \subset \varphi([r, x])$, hence there exists an $m \in [r, x]$ with $m' \in \varphi(m)$. As $[m, y] \subset \varphi^{-1}([m', f(y)])$ by Lemma 1, and as $x \in [m, y]$, we have $\varphi(x) \cap [m', f(y)] \neq \emptyset$. Therefore $f(x) \leqq f(y)$.

3. Results. We shall now state the results on fixed points and coincidences which are the purpose of this paper. They will be proved in the next paragraph.

Definition. A fixed point of a multifunction $\varphi : X \rightarrow X$ is a point $x \in X$ such that $x \in \varphi(x)$. A coincidence of two multifunctions $\varphi, \psi : X \rightarrow X'$ is a point $x \in X$ such that $\varphi(x) \cap \psi(x) \neq \emptyset$.

MAIN THEOREM. *Let $\varphi, \psi : T \twoheadrightarrow T'$ be two usc and biconnected multifunctions from a tree T onto a tree T' which have an end point e of T as coincidence. If both $\varphi(e) \neq T'$ and $\psi(e) \neq T'$, and if either $\varphi(e)$ or $\psi(e)$ consists of a single point, then φ and ψ have at least one coincidence distinct from e.*

Remark. All assumptions of the Main Theorem are clearly satisfied if φ and ψ are single-valued monotone continuous surjections which have e as a coincidence. That the restrictions on $\varphi(e)$ and $\psi(e)$ are necessary is seen from the following two examples.

Example 1. Take $T = T'$, where T is an arbitrary tree with end point e, and define $\varphi : T \twoheadrightarrow T$ by $\varphi(e) = T$, $\varphi(x) = e$ if $x \neq e$. Let $\psi : T \twoheadrightarrow T$ be the identity map $\psi(x) = x$. Then φ and ψ satisfy all the assumptions of the Main Theorem apart from the fact that $\varphi(e) = T'$, and they have e as the only coincidence.

Example 2. Let T be the segment $[e, a]$, and T' be the triod with centre m' and end points e', b', and c'. Define $\varphi : T \twoheadrightarrow T'$ by putting $\varphi(e) = [e', b']$ and extending it as a linear single-valued mapping from $(e, a]$ onto $(m', c']$. Define $\psi : T \twoheadrightarrow T'$ by putting $\psi(e) = [e', c']$ and extending it as a linear single-valued mapping from $(e, a]$ onto $(m', b']$. This time φ and ψ satisfy all assumptions of the Main Theorem apart from the fact that neither $\varphi(e)$ nor $\psi(e)$ is a single point, and again e is their only coincidence.

If in the Main Theorem $T' = T$ and ψ is the identity mapping, then we obtain as a corollary the following result on fixed points.

COROLLARY. *Let $\varphi : T \twoheadrightarrow T$ be an usc and biconnected multifunction of a tree T onto itself which leaves an end point e fixed. If $\varphi(e) \neq T$, then φ leaves at least one other point fixed.*

That the condition $\varphi(e) \neq T$ is necessary is seen from Example 1. If φ is a single-valued mapping, then the Corollary reduces to the theorem by Schweigert and Ward ([**6**; **8**]).

4. Proof of the Main Theorem. Let now $\varphi, \psi : T \twoheadrightarrow T'$ be two multifunctions which satisfy the assumptions of the Main Theorem. Take e as root of T and $e' = \varphi(e) \cap \psi(e)$ as root of T'; further let $f(x)$ denote the root of $\varphi(x)$ and $g(x)$ the root of $\psi(x)$. The following lemma will be helpful in the proof of the Main Theorem.

LEMMA 3. *If $a, b \in T$ are such that $a < b$, that $f(a)$, $f(b)$, $g(a)$, and $g(b)$ are linearly ordered, and that either $f(a) < g(a)$ and $g(b) < f(b)$, or $g(a) < f(a)$ and $f(b) < g(b)$, then φ and ψ have a coincidence on $[a, b]$.*

Proof. It is clearly sufficient to consider the case $f(a) < g(a)$ and $g(b) < f(b)$. As f and g are isotone functions (Lemma 2), we have $f(b) = \max [f(a), f(b), g(a), g(b)]$ and therefore $\{f(a), g(a), g(b)\} \in \mathrm{L}(f(b))$. For every $x \in [a, b]$, define

$$m(x) = \max [\varphi(x) \cap \mathrm{L}(f(b))],$$
$$n(x) = \max [\psi(x) \cap \mathrm{L}(f(b))],$$

and let

$$O_1 = \{x \in [a, b] | m(x) < g(x)\},$$
$$O_2 = \{x \in [a, b] | f(x) < n(x)\}.$$

We see that $a \in O_1$, $b \in O_2$ and $O_1 \cap O_2 = \emptyset$. The sets O_1 and O_2 are open in $[a, b]$: for any $x_0 \in O_1$, choose x_0' such that $m(x_0) < x_0' < g(x_0)$. Then $\psi(x_0) \subset \mathrm{M}(x_0')\backslash\{x_0'\}$, and further $\varphi(x_0) \subset T'\backslash\mathrm{M}(x_0')$ since otherwise $\mathrm{M}(x_0')\backslash\{x_0'\}$ and $T'\backslash\mathrm{M}(x_0')$ would provide a separation of the connected set $\varphi(x_0)$. As φ and ψ are usc, we can find an open set $U(x_0)$ containing x_0 such that $\varphi(x) \subset T'\backslash\mathrm{M}(x_0')$ and $\psi(x) \subset \mathrm{M}(x_0')\backslash\{x_0'\}$ for every $x \in U(x_0)$. Thus $U(x_0) \cap [a, b] \subset O_1$, so that x_0 is an interior point of O_1. An analogous argument shows that O_2 is open in $[a, b]$. If φ and ψ should have no coincidence on $[a, b]$, then $O_1 \cup O_2 = [a, b]$, and O_1 and O_2 would be a separation of the connected arc $[a, b]$. But this is impossible.

We now give the proof of the Main Theorem, which will be accomplished in four steps. The purpose of the first one is to establish the existence of a point $y \in T\backslash\{e\}$ such that $f(y)$ and $g(y)$ are comparable, say $g(y) < f(y)$. A coincidence is then obtained by an adaptation of the "dead-end method" [**2**] from fixed point theory. This is done inductively; we find a chain $x_1 < x_2 < x_3 < \ldots$ in $T\backslash\{e\}$ such that $f(x_n) \in \psi(x_{n+1})$. The second step of the proof gives the beginning of the inductive argument, the third step the

https://doi.org/10.4153/CJM-1971-049-6 Published online by Cambridge University Press

construction of x_n from $x_1, x_2, \ldots, x_{n-1}$. In the final step we deal with a possible complication which can arise in the second step.

Either $\varphi(e)$ or $\psi(e)$ consists of a single point; without loss of generality we can assume that $\psi(e)$ does. Then $\psi(e) = e'$ is the root of T'.

Step 1. There exists a point $y \in T\backslash\{e\}$ such that $f(y)$ and $g(y)$ are comparable. The proof of the existence of y depends on whether e' is an end point of T' or not.

(i) e' is an end point. Take any $a \in T\backslash\{e\}$. If $f(a)$ and $g(a)$ are not comparable, let $y' = \max [\mathrm{L}(f(a)) \cap \mathrm{L}(g(a))]$, so that $y' > e'$. As $[e', g(a)] \subset \psi[e, a]$ according to Lemma 1, there exists a $y \in (e, a)$ with $y' \in \psi(y)$. But f and g are isotone functions (Lemma 2), therefore we have $e' \leqq f(y) \leqq f(a)$ and $e' \leqq g(y) \leqq y' < f(a)$. Hence $f(y)$ and $g(y)$ are comparable.

(ii) e' is not an end point. Then we can choose points $a', b' \in T'\backslash\{e'\}$ such that $e' = \max [\mathrm{L}(a') \cap \mathrm{L}(b')]$. Select any $a \in \psi^{-1}(a')$ and $b \in \psi^{-1}(b')$. We see that $a > e, b > e$, and hence $y = \max [\mathrm{L}(a) \cap \mathrm{L}(b)] > e$. As $e < y \leqq a$ and $e < y \leqq b$, it follows from Lemma 2 that $e' \leqq g(y) \leqq g(a) \leqq a'$ and $e' \leqq g(y) \leqq g(b) \leqq b'$. Thus $g(y) = e'$, and therefore $f(y) > g(y)$.

Step 2. Inductive argument, construction of x_1 if $f(y) > g(y)$. Assume that we found in the first step a $y = y_1 \in T\backslash\{e\}$ such that $g(y_1) < f(y_1)$. We now construct x_1 and y_2 such that $e < x_1 < y_2$ and $f(x_1) \in \psi(y_2)$.

As ψ is a surjection, there exists a $z_2 \in T$ with $f(y_1) \in \psi(z_2)$. Then $z_2 \neq e$ as $\psi(e) = e'$, and $z_2 \neq y_1$ if e is the only coincidence. If $z_2 < y_1$ then $f(z_2)$ and $g(z_2)$ are both contained in $[e', f(y_1)]$ and hence comparable. But $g(z_2) \leqq f(z_2)$ implies $f(z_2) \in [g(z_2), f(y_1)] \subset \psi(z_2)$, so that z_2 is a coincidence, and $f(z_2) < g(z_2)$ implies the existence of a coincidence on $[z_2, y_1]$ by Lemma 3. So if e is the only coincidence, we must have $z_2 \nless y_1$.

In this case let $x_1 = \max [\mathrm{L}(y_1) \cap \mathrm{L}(z_2)]$. Then $e < x_1 \leqq y_1$, so that $e' \leqq f(x_1) \leqq f(y_1)$ and $e' \leqq g(x_1) \leqq g(y_1) < f(y_1)$. Therefore $f(x_1)$ and $g(x_1)$ are comparable. It follows from Lemma 3 that φ and ψ have a coincidence on $[x_1, y_1]$ if $f(x_1) \leqq g(x_1)$, so we must have $g(x_1) < f(x_1)$ if e is the only coincidence. As then $f(x_1) \in \psi((x_1, z_2])$, we can choose a $y_2 \in (x_1, z_2]$ with $f(x_1) \in \psi(y_2)$. So we see that φ and ψ either have a coincidence distinct from e, or we can find x_1 and y_2 such that $e < x_1 < y_2, f(x_1) \neq e'$ and $f(x_1) \in \psi(y_2)$.

Step 3. Inductive argument, construction of x_n if $f(y) > g(y)$. Assume now that we have obtained a chain $e < x_1 < x_2 < \ldots < x_{n-1} < y_n$ in T such that $f(x_i) \in \psi(x_{i+1})$ for $i = 1, 2, \ldots, n-2$ and $f(x_{n-1}) \in \psi(y_n)$. Then choose any $z_{n+1} \in T$ with $f(y_n) \in \psi(z_{n+1})$. As $f(x_1) \neq e'$ and $f(x_1) \leqq f(y_n)$ we see that $z_{n+1} \neq e$, and we have $z_{n+1} \neq y_n$ if e is the only coincidence. Consider first the possibility that $z_{n+1} < y_n$. Then $f(z_{n+1})$ and $g(z_{n+1})$ are both contained in $[e', f(y_n)]$ and therefore comparable. But $g(z_{n+1}) \leqq f(z_{n+1})$ implies

$f(z_{n+1}) \in \psi(z_{n+1})$, so that z_{n+1} is a coincidence, and $f(z_{n+1}) < g(z_{n+1})$ implies the existence of a coincidence on $[z_{n+1}, y_n]$ because of $g(y_n) \leqq f(x_{n-1}) \leqq f(y_n)$ and Lemma 3. So it is necessary that $z_{n+1} \not< y_n$ if e is the only coincidence.

In this case let $x_n = \max[\mathrm{L}(y_n) \cap \mathrm{L}(z_{n+1})]$. Then $e < x_n \leqq y_n$ so that $f(x_n)$ and $g(x_n)$ are both contained in $[e', f(y_n)]$ and thus comparable. If $f(x_n) \leqq g(x_n)$ then it follows from Lemma 3 that φ and ψ have a coincidence on $[x_n, y_n]$. Therefore $g(x_n) < f(x_n)$ if e is the only coincidence.

We next show that in this case it is also necessary that $x_{n-1} < x_n$. As x_{n-1} and x_n are contained in $\mathrm{L}(y_n)$ they are comparable. If $x_n \leqq x_{n-1}$, then $g(x_n) \leqq g(x_{n-1}) \leqq f(x_{n-2}) \leqq f(x_{n-1}) \leqq f(y_n)$ and $f(y_n) \in \psi(z_{n+1})$ would imply $f(x_{n-1}) \in \psi([x_n, z_{n+1}])$, or $y \in \psi^{-1}(f(x_{n-1}))$ for some $y \in [x_n, z_{n+1}]$. We have further that $y_n \in \psi^{-1}(f(x_{n-1}))$. Now $\psi^{-1}(f(x_{n-1}))$ is a non-empty, closed and connected subset of T, hence it has a root, say r_n. As in the proof of Lemma 1 we obtain $[r_n, y_n] \cup [r_n, y] \subset \psi^{-1}(f(x_{n-1}))$. If $m_n = \max[\mathrm{L}(y_n) \cap \mathrm{L}(y)]$, then $r_n \leqq m_n \leqq y_n$ and therefore $m_n \in \psi^{-1}(f(x_{n-1}))$. But

$$m_n = \max[\mathrm{L}(y_n) \cap \mathrm{L}(z_{n+1})] = x_n,$$

so that $f(x_{n-1}) \in \psi(x_n)$. From $g(x_n) < f(x_n) \leqq f(x_{n-1})$ it then follows that $f(x_n) \in \psi(x_n)$, so that x_n is a coincidence. Therefore $x_{n-1} < x_n$ if e is the only coincidence.

As $g(x_n) \leqq g(y_n) \leqq f(x_{n-1}) \leqq f(y_n)$ and $f(y_n) \in \psi(z_{n+1})$, we have $f(x_{n-1}) \in \psi([x_n, z_{n+1}])$. But we also have $f(x_{n-1}) \in \psi(y_n)$, so we see that again $f(x_{n-1}) \in \psi(x_n)$. Further $g(x_n) \leqq f(x_{n-1}) \leqq f(x_n) \leqq f(y_n)$ and $f(y_n) \in \psi(z_{n+1})$, so that Lemma 1 yields $f(x_n) \in \psi([x_n, z_{n+1}])$. Thus there exists a

$$y_{n+1} \in (x_n, z_{n+1}]$$

with $f(x_n) \in \psi(y_{n+1})$. We now have constructed a chain

$$e < x_1 < x_2 < \ldots < x_n < y_{n+1}$$

in T such that $f(x_i) \in \psi(x_{i+1})$ for $i = 1, 2, \ldots, n-1$ and $f(x_n) \in \psi(y_{n+1})$.

It follows from [**8**, p. 152] that the monotone increasing sequences $\{x_1, x_2, x_3, \ldots\}$ and $\{f(x_1), f(x_2), f(x_3), \ldots\}$ converge; let

$$x_0 = \lim_{n\to\infty} x_n \quad \text{and} \quad x_0' = \lim_{n\to\infty} f(x_n).$$

Then $x_0 \neq e$. As φ is usc, we have $x_0' \in \varphi(x_0)$ ([**1**, pp. 111 and 112]). As ψ is usc and $f(x_n) \in \psi(x_{n+1})$ for $n = 1, 2, 3, \ldots$, we also have $x_0' \in \psi(x_0)$. Therefore x_0 is a coincidence distinct from e. We have now proved the Main Theorem if there exists a $y \in T\backslash\{e\}$ with $f(y) > g(y)$.

Step 4. Inductive argument if $f(y) < g(y)$. We started our inductive argument in step 2 with a $y = y_1 \in T\backslash\{e\}$ such that $f(y_1) > g(y_1)$. But if e' is an end point, then step 1 only assures us of the existence of a $y \in T\backslash\{e\}$ such that $f(y)$ and $g(y)$ are comparable. Hence we have to investigate what happens if $f(y_1) < g(y_1)$ and e' is an end point. All arguments in step 2 and step 3 still work with the only exception of the reason given for the fact that $z_2 \neq e$, as now $\varphi(e)$ might contain $g(y_1)$. But we will prove that this cannot happen, for if $y_1 \in T \backslash \{e\}$ is such that $f(y_1) < g(y_1)$ and e is the only coincidence, then $g(y_1) \notin \varphi(e)$.

First we show that $f(y_1) < g(y_1)$ and $g(y_1) \in \varphi(e)$ would imply $\varphi((e, y_1]) = e'$. For if there exists an $x \in (e, y_1]$ such that $\varphi(x) \neq e'$, then it follows from $e' \leqq f(x) \leqq f(y_1) < g(y_1)$ and the fact that e' is an end point that $x' = \max [\varphi(x) \cap \mathrm{L}(g(y_1))] > e'$. As then $x' \in \varphi(x)$ and $x' \in \varphi(e)$, Lemma 1 shows that $[e, x] \subset \varphi^{-1}(x')$. If $g(t) < x'$ for some $t \in (e, x]$, then either $f(t) \leqq g(t) < x'$, so that $g(t) \in \varphi(t)$, or $g(t) < f(t)$, so that φ and ψ have a coincidence on $[t, y_1]$ according to Lemma 3. Hence we must have $g(t) > x' > e'$ for all $t \in (e, x]$ if e is the only coincidence. But this is impossible if $\psi(e) = e'$ and ψ is usc. So $\varphi((e, y_1]) = e'$ if $f(y_1) < g(y_1)$ and $g(y_1) \in \varphi(e)$.

Now take any $z \in T \backslash [e, y_1]$, let $z_1 = \max [\mathrm{L}(z) \cap \mathrm{L}(y_1)]$ and $w' = \max [\mathrm{L}(\varphi(z)) \cap \varphi(e)]$. If $\varphi(e) \neq e'$ and $\varphi(z) \neq e'$, then the fact that e' is an end point implies $e' < w'$. Choose any $z' \geqq w'$ with $z' \in \varphi(z)$. As $[e', z'] \subset \varphi[z_1, z]$ by Lemma 1, there exists an $s \in (z_1, z]$ such that $w' \in \varphi(s)$. But then Lemma 1 shows that $[e, s] \subset \varphi^{-1}(w')$, and in particular that $z_1 \in \varphi^{-1}(w')$ which contradicts $\varphi(z_1) = e'$. Hence $\varphi(e) = e'$ or $\varphi(z) = e'$. As $\varphi(e) = e'$ is not possible if $g(y_1) \in \varphi(e)$, we have $\varphi(z) = e'$, and as z was arbitrary, we have in fact $\varphi(T \backslash \{e\}) = e'$. But this cannot happen if $\varphi(e) \neq T'$ and φ is a surjection. Therefore $g(y_1) \notin \varphi(e)$.

This completes the proof of the Main Theorem.

References

1. C. Berge, *Topological spaces* (Oliver and Boyd, Edinburgh and London, 1963).
2. R. H. Bing, *The elusive fixed point property*, Amer. Math. Monthly *76* (1969), 119–132.
3. C. E. Capel and W. L. Strother, *Multi-valued functions and partial order*, Portugal. Math. *17* (1958), 41–47.
4. C. Eberhart, *Metrizability of trees*, Fund. Math. *65* (1969), 43–50.
5. H. Schirmer, *Coincidences and fixed points of multifunctions into trees*, Pacific J. Math. *34* (1970), 759–767.
6. G. E. Schweigert, *Fixed elements and periodic types for homeomorphisms on s.l.c. continua*, Amer. J. Math. *66* (1944), 229–244.
7. A. D. Wallace, *A fixed point theorem for trees*, Bull. Amer. Math. Soc. *47* (1941), 757–760.
8. L. E. Ward, Jr., *Partially ordered topological spaces*, Proc. Amer. Math. Soc. *5* (1954), 144–161.
9. ——— *A note on dendrites and trees*, Proc. Amer. Math. Soc. *5* (1954), 992–994.
10. ——— *A general fixed point theorem*, Colloq. Math. *15* (1966), 243–251.

Carleton University,
Ottawa, Canada

https://doi.org/10.4153/CJM-1971-049-6 Published online by Cambridge University Press