FUNDAMENTAL SOLUTIONS AND SURFACE DISTRIBUTIONS

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Introduction

When studying the solutions of elliptic boundary value problems in a bounded, smoothly bounded domain $D \subset R_n$ we often encounter the formula

$$\int_{\partial D} \left\{ u(x) \frac{\partial}{\partial n_x} \gamma(x, y) - \gamma(x, y) \frac{\partial}{\partial n_x} u(x) \right\} dS_x = \begin{cases} u(y) & \text{if } y \in D & (1a) \\ \frac{1}{2}u(y) & \text{if } y \in \partial D & (1b) \\ 0 & \text{if } y \notin D & (1c) \end{cases}$$

where $u(x) \in C^2(D) \cap C'(\bar{D})$ is a solution of the second order self-adjoint elliptic equation

$$Lu(x) \equiv (\Delta \pm k^2)u(x) = 0, \quad x \in D \tag{2}$$

and $\dfrac{\partial}{\partial n_x}$ denotes differentiation along the inward normal to ∂D at $x \in \partial D$. $\gamma(x, y)$ is a fundamental solution of (2), and as such has at $x = y$ a singularity described by

$$\gamma(x, y) \sim \begin{cases} -\dfrac{1}{2\pi} \log |x - y| & \text{if } n = 2, \\ \dfrac{1}{2n-4} \Gamma\left(\dfrac{n}{2}\right) \pi^{-n/2} |x - y|^{2-n} & \text{if } n \geqq 3. \end{cases} \tag{3}$$

The results ($1a$) and ($1c$) can be obtained in a straightforward way by applying Green's Theorem to $u(x)$ and any fundamental solution which is defined in a sufficiently large domain. However the result ($1b$) is neither as obvious nor as easily obtained as is generally claimed in textbooks, though, as we shall see, it is true *in the sense of the theory of distributions* for fundamental solutions which, apart from a singularity of type (3) at $x = y$, are regular in $\bar{D} \times \bar{D}$. For other choices of the fundamental solution (e.g. the Dirichlet Green's function) not satisfying this restriction, ($1b$) is meaningless unless a suitable definition can be given for the left hand side. In this paper we shall establish ($1b$) for fundamental solutions having the required behaviour in $\bar{D} \times \bar{D}$ and shall show that when a maximum principle is available ($L = \Delta - k^2$, $k^2 \geqq 0$) ($1b$) can be made meaningful *in the distributional sense* for the Dirichlet Green's function of L and D. It is sufficient to demonstrate this for $L = \Delta$.

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1. For convenience we restrict ourselves to the case $n = 2$. Results for $n \geqq 3$ follow in a similar manner. Fix $y \in \partial D$ and let K_ε be the disc of radius ε centred at y. Let $S_\varepsilon = \partial K_\varepsilon \cap D$. Applying Green's Theorem to $u(x)$ and $\gamma(x, y)$ in $D - K_\varepsilon$ and performing a simple residue calculation, noting that on S_ε we have

$$\gamma(x, y) \sim -\frac{1}{2\pi} \log \varepsilon, \quad \frac{\partial}{\partial n_x} = \frac{\partial}{\partial \varepsilon}$$

and $dS_x = \varepsilon d\theta \, (0 \leqq \theta \leqq \pi)$, we obtain the result

$$\lim_{\varepsilon \to 0} \int_{\partial D - K_\varepsilon} \left\{ u(x) \frac{\partial}{\partial n_x} \gamma(x, y) - \gamma(x, y) \frac{\partial}{\partial n_x} u(x) \right\} dS_x = \tfrac{1}{2} u(y).$$

Therefore to establish (1*b*) it suffices to show that

$$\lim_{\varepsilon \to 0} \int_{\partial D \cap K_\varepsilon} \left\{ \frac{\partial}{\partial n_x} \gamma(x, y) - \gamma(x, y) \frac{\partial}{\partial n_x} u(x) \right\} dS_x = 0. \tag{4}$$

Since ∂D is smooth, we have in a neighbourhood of y, $dS_x \cong dr$ where $r = |x - y|$. Since $\dfrac{\partial}{\partial n_x} u(x)$ is continuous in $\bar{D}$ it follows that

$$\lim_{\varepsilon \to 0} \int_{\partial D \cap K_\varepsilon} \gamma(x, y) \frac{\partial}{\partial n_x} u(x) dS_x = \lim_{\varepsilon \to 0} \frac{-1}{\pi} \frac{\partial}{\partial n_y} u(y) \int_0^\varepsilon \log r dr = 0. \tag{5}$$

Finally, let $r(x)$ denote the radius of the circle C_x through x and y which is tangent to ∂D at x. Since ∂D is smooth, $r(x) \to R$, the radius of the osculating circle for ∂D at y, as $x \to y$. Thus for ε small enough and $x \in \partial D \cap K_\varepsilon$ it follows that $r(x) \geqq \frac{1}{2} R$. If α is the angle between the vector from y to x and the inward normal n_x to ∂D at x we have, since $|n_x| = 1$ and $|\nabla_x |x - y|| = 1$, that

$$\left| \frac{\partial}{\partial n_x} \gamma(x, y) \right| \sim \frac{1}{2\pi} \left| n_x . \nabla_x \log |x - y| \right| = \frac{1}{2\pi |x - y|} |\cos \alpha|.$$

By the geometry of the circle C_x it follows that

$$\frac{|\cos \alpha|}{|x - y|} = \frac{1}{2r(x)} \leqq \frac{1}{R}$$

and hence we have at once that

$$\lim_{\varepsilon \to 0} \int_{\partial D \cap K_\varepsilon} u(x) \frac{\partial}{\partial n_x} \gamma(x, y) dS_x = 0.$$

This completes the proof of (4) and so of (1*b*).

2. The harmonic Green's function, $G(x, y)$, of D is defined for $x \in \bar{D}$, $y \notin \partial D$ by $G(x, y) = \gamma(x, y) + w(x, y)$, where $\gamma(x, y)$ is the function on the right hand side of (3) and

$$\begin{aligned} \Delta_x w(x, y) &= 0, & x \in D, \\ w(x, y) &= -\gamma(x, y), & x \in \partial D. \end{aligned}$$

$G(x, y)$ is positive for $x, y \in D$ (**1**, p. 262). Since $G(x, y)$ is not properly defined for $y \in \partial D$, the appropriate form of (1*b*), namely

$$\int_{\partial D} u(x) \frac{\partial}{\partial n_x} G(x, y) dS_x = \tfrac{1}{2} u(y), \quad y \in \partial D \tag{6}$$

requires interpretation.

To this end let $G_\varepsilon(x, y)$ be the harmonic Green's function for the region $D_\varepsilon = D \cap K_\varepsilon$. Applying (1*b*) over D with $\gamma(x, y) = G_\varepsilon(x, y)$, we obtain

$$\int_{\partial D} u(x) \frac{\partial}{\partial n_x} G_\varepsilon(x, y) dS_x - \int_{\partial D \cap K_\varepsilon} G_\varepsilon(x, y) \frac{\partial}{\partial n_x} u(x) dS_x = \tfrac{1}{2} u(y).$$

Fix ε_0. For $\varepsilon < \varepsilon_0$, $G_{\varepsilon_0}(x, y) - G_\varepsilon(x, y)$ is harmonic in D_ε and non-negative on ∂D_ε. By the maximum principle it is non-negative in D_ε.

Thus

$$\left| \int_{\partial D \cap K_\varepsilon} G_\varepsilon(x, y) \frac{\partial}{\partial n_x} u(x) dS_x \right| \leqq \text{const.} \int_{\partial D \cap K_\varepsilon} G_{\varepsilon_0}(x, y) dS_x.$$

As in (5) above the right hand side tends to zero with ε. Hence

$$\lim_{\varepsilon \to 0} \int_{\partial D} u(x) \frac{\partial}{\partial n_x} G_\varepsilon(x, y) dS_x = \tfrac{1}{2} u(y), \quad y \in \partial D.$$

This shows that in (6) we may interpret $\frac{\partial}{\partial n_x} G(x, y)$ as the limit in the distribution sense as $\varepsilon \to 0$ of the well defined functions $\frac{\partial}{\partial n_x} G_\varepsilon(x, y)$ (**2**, Chapter 2; **3**).

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