FINITE LINEAR GROUPS OF DEGREE SEVEN. I

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1. This paper is the second in a series of papers discussing linear groups of prime degree, the first being (**8**). In this paper we discuss only linear groups of degree 7. Thus, G is a finite group with a faithful irreducible complex representation X of degree 7 which is unimodular and primitive. The character of X is χ. The notation of (**8**) is used except here $p = 7$. Thus P is a 7-Sylow group of G. In §§ 2 and 3 some general theorems about the 3-Sylow group and 5-Sylow group are given. In § 4 the statement of the results when G has a non-abelian 7-Sylow group is given. This corresponds to the case $|P| = 7^3$ or $|P| = 7^4$. The proof is given in §§ 5 and 6. In a subsequent paper the results when P is abelian will be given. These correspond to the case that $|P| = 7$ or $|P| = 7^2$. In § 6 characters are denoted by χ_i, their degrees by x_i. Set $|G| = g = 7^{a_7} \cdot 5^{a_5} \cdot 3^{a_3} \cdot 2^{a_2}$. By (**3**, 3E), $a_5 \leqq 7$, $a_3 \leqq 8$, $a_2 \leqq 10$. By (**8**, Theorem 2.2), $a_7 \leqq 4$.

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2. Some properties of P_5. We begin with a discussion of the 5-Sylow group P_5 of G showing first that it is abelian. This extends (**8**, Theorem 5.1).

THEOREM 2.1. *A 5-Sylow group P_5 of G is abelian.*

Proof. If P_5 is not abelian, then $X|P_5$ has a 5-dimensional constituent and two linear ones. A suitable element Q in $Z(P_5) \cap P_5'$ has five eigenvalues $\lambda = e^{2\pi i/5}$ and two eigenvalues 1.† We will show in a series of lemmas that this is impossible. We will also need the lemmas later.

LEMMA 2.2. *Suppose that H has a faithful representation Y of degree 2, not necessarily unimodular. Suppose, further, that H has two 5-elements Q_1 and Q_2 which do not commute. Then there is an element T in any commutator of H such that $Y(T)$ has eigenvalues $\{-\lambda^2, -\lambda^3\}$ where $\lambda = e^{2\pi i/5}$. Furthermore, $H' = H''$ is the unimodular subgroup of $Y(H)$. Furthermore, there is an involution J in $Z(H')$ such that*

$$Y(J) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad \textit{and} \quad H'/\langle J\rangle \cong A_5.$$

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†In (**8**, Theorem 5.1), the same method for $q \geqq 7$ gave an element contradicting Blichfeldt's theorem. However, since $360/5 > 60$, this is not true here, and other methods are needed.

Proof. Identify H with $Y(H)$. Let $c_i = \det y(Q_i)$, $i = 1, 2$. Adjoin to $Y(H)$ the matrices $\pm(\sqrt{c_i})^{-1}I$. Let K be the new group. Let S be the subgroup of K consisting of unimodular matrices. There are scalar multiples of $Y(Q_i)$, $i = 1, 2$, in S which do not commute, and hence S contains two 5-elements which do not commute. As a 5-Sylow group is abelian, it is not normal in S. The group $S/Z(S)$ must be one of the linear groups in two variables all of which are listed in (**1**). The only possibility is $S/Z(S) \cong A_5$ as this is the only one in which the 5-Sylow group is not normal. Let $S_0 = S \cap H$. Clearly, $S_0 \lhd S$ and S/S_0 is abelian. We have $S \rhd S_0 \rhd Z(S_0) \rhd e$. Since S has a composition factor A_5 and S/S_0 and $Z(S_0)$ are abelian, we have $S_0/Z(S_0) \cong A_5$. Here $Z(S_0)$ can contain only $\pm I$. Since A_5 has no representation of degree 2, we see that $Z(S_0) = \pm I$. Furthermore, $S_0' = S_0$. We have $H \rhd S_0$ with H/S_0 abelian. This shows that $H' = S_0 = H'' = H^{(n)}$. In S_0 an element T of order 5 has eigenvalues $\{\lambda^2, \lambda^3\}$ for an appropriate power of T. Multiplying by $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ we obtain an element with eigenvalues $\{-\lambda^2, -\lambda^3\}$. The proof is complete.

LEMMA 2.3. *Let H be the group generated by two non-commuting elements Q_1 and Q_2 of order* 5. *Suppose that there is a faithful representation Y of H of degree* 4 *which has a two-dimensional invariant subspace. Furthermore, $Y(Q_1)$ and $Y(Q_2)$ have eigenvalues $\{1, 1, \lambda, \bar{\lambda}\}$ or $\{1, 1, \lambda, \lambda\}$. Then there is an element T in H such that $Y(T)$ has eigenvalues $\{1, 1, -\lambda^2, -\lambda^3\}$ or*

$$\{-\lambda^2, -\lambda^2, -\lambda^3, -\lambda^3\}.$$

Here $\lambda = e^{2\pi i/5}$.

Proof. Let $Y = Y_1 \oplus Y_2$, where Y_1 and Y_2 are 2-dimensional. Identify H with $Y(H)$. Adjoin to H the matrices $\lambda^r I$, $r = 1, 2, 3, 4$. Let K be this new group. Let S be the subgroup of K such that Y_1 is unimodular. There is a scalar multiple $\lambda^r I$ of $Y(Q_1)$ in S and a scalar multiple of $Y(Q_2)$ in S. This means that S contains at least two non-commuting elements of order 5. Call these R_1 and R_2. Clearly, $Y_i(R_1)$ and $Y_i(R_2)$ cannot commute for both $i = 1$ and 2. If $Y_i(R_1)$ and $Y_i(R_2)$ do commute, the group $\langle Y_i(R_1), Y_i(R_2)\rangle$ is commutative. Applying Lemma 2.2 to a commutator of $\langle R_1, R_2\rangle$ yields an element with eigenvalues $\{1, 1, -\lambda^2, -\lambda^3\}$.

We can therefore assume that $Y_i(R_1)$ and $Y_i(R_2)$ do not commute for $i = 1, 2$. It follows easily that $S = \langle R_1, R_2\rangle$. By Lemma 2.2, $S' = S$ and thus $\det Y_2(S) = 1$. Let $Q = R_i$, $i = 1, 2$. We can assume that $Y_1(Q)$ and $Y_2(Q)$ have eigenvalues $\{\lambda^2, \lambda^3\}$. For if not, Q_1 or Q_2 cannot be written as a scalar multiple of any non-identity power of Q as the eigenvalues for such a scalar multiple are $\{c\lambda, c\lambda^2, c\lambda^3, c\lambda^4\}$. This means that $Y_1(Q)$ and $Y_2(Q)$ have eigenvalues $\{\lambda^2, \lambda^3\}$ and $Y(QJ)$ has eigenvalues $\{-\lambda^2, -\lambda^3, -\lambda^2, -\lambda^3\}$, where J is the involution in $Z(S)$. This completes the proof of the lemma.

LEMMA 2.4. *Suppose that H has a faithful representation Y of degree* 3. *Suppose also that H is generated by two non-commuting* 5*-elements Q_i, $i = 1, 2$,*

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such that $Y(Q_i)$ *has eigenvalues* $\{\alpha_i, \alpha_i, \beta_i\}$. *There is then an element* T *in any commutator of* H *such that* $Y(T)$ *has eigenvalues* $\{1, -\lambda^2, -\lambda^3\}$. *Again* $\lambda = e^{2\pi i/5}$.

Proof. $Y(Q_i)$, $i = 1, 2$, has a 2-dimensional invariant subspace U_i on which $Y(Q_i)|U_i$ is $\alpha_i I$. Any subspace of U_i is an invariant subspace of $Y(Q_i)$. Clearly, $U_1 \cap U_2$ is an invariant subspace for $Y(Q_1)$ and $Y(Q_2)$ and hence for $Y(H)$. This shows that Y is not irreducible. Since H is not abelian, there must be a 2-dimensional component. Lemma 2.2 can be applied to H' to obtain an element of the desired form.

LEMMA 2.5. *There can be no element* Q_1 *in* G *such that* $X(Q_1)$ *has eigenvalues* $\{\lambda, \lambda, \lambda, \lambda, \lambda, 1, 1\}$.

Proof. Suppose that $X(Q_1)$ has eigenvalues $\{\lambda, \lambda, \lambda, \lambda, \lambda, 1, 1\}$. Since G cannot have a normal 5-subgroup containing Q_1, there must be a conjugate Q_2 of Q_1 such that Q_1 and Q_2 do not commute. Let U_i, $i = 1, 2$, be the subspaces of the representation space V of X on which $X(Q_i) = \lambda I$. It is clear that $U_1 \cap U_2$ is an invariant subspace for $X(Q_i)$, and hence for $X(H)$, where H is the group generated by Q_i, $i = 1, 2$. Clearly, $U_1 \cap U_2$ has dimension 5, 4, or 3. Furthermore, $X(Q_i)|U_1 \cap U_2 = \lambda I$. Let W be a complementary subspace to $U_1 \cap U_2$ with respect to $X(H)$. We see that W has dimension 2, 3, or 4. If dim $W = 2$, $U_1 = U_2$, and Q_1 and Q_2 commute. This is a contradiction. If dim $W = 3$, Lemma 2.4 gives an element T in H' such that $X(T)$ has eigenvalues $\{1, 1, 1, 1, 1, -\lambda^2, -\lambda^3\}$, contradicting Blichfeldt's theorem (**1** or **8**, § 2). This follows since $X(Q_i)|W$ must have eigenvalues $\{1, 1, \lambda\}$. If dim $W = 4$, the eigenvalues of $X(Q_i)|W$ are $\{1, 1, \lambda, \lambda\}$. As in (**1**, p. 143) there is a 2-dimensional invariant subspace and Lemma 2.3 gives an element T in H such that $X(T)$ has eigenvalues contradicting Blichfeldt's theorem. This completes the proof of the lemma and the proof of Theorem 2.1.

COROLLARY 2.6. *If* $|G| = 7^a \cdot 5^b \cdot g_1$, *then* $b \leqq 6$.

Proof. This follows from (**3**, 3D) or the proof of (**3**, 3E).

We now show that there can be no element Q such that $X(Q)$ has eigenvalues $\{1, 1, 1, 1, 1, \lambda, \bar{\lambda}\}$, $\lambda = e^{2\pi i/5}$. This will be used to obtain properties of the 5-Sylow group P_5. The proof is fairly involved. Part of it will be given by means of a lemma at the end.

THEOREM 2.7. *There can be no element* Q *in* G *such that* $X(Q)$ *has eigenvalues* $\{1, 1, 1, 1, 1, \lambda, \bar{\lambda}\}$, $\lambda = e^{2\pi i/5}$.

COROLLARY 2.8. *An elementary abelian subgroup of* P_5 *has order at most* 5^4.

Proof of Corollary 2.8. Suppose that there is an elementary abelian subgroup of order 5^5. A basis $\xi_1, \ldots, \xi_5$ for P_5 can be chosen so that $X(\xi_i)$ is diagonal and $(X(\xi_i))_{ii} = \lambda$, $(X(\xi_i))_{jj} = 1$ for $1 \leqq j \leqq 5$, $j \neq i$, $(X(\xi_i))_{66} = \lambda^{r_i}$, $(X(\xi_i))_{77} = \lambda^{-(r_i+1)}$, $i = 1, 2, \ldots, 5$. Here r_i is an integer $0 \leqq r_i \leqq 4$. The last component $(X(\xi_i))_{77}$ is $\lambda^{-(r_i+1)}$ by the unimodularity. If all of the r_i are distinct for $i = 1, 2, \ldots, 5$, some r_i is 0. For this i, $X(\xi_i)$ has eigenvalues $\{1, 1, 1, 1, 1, \lambda, \bar{\lambda}\}$, contradicting Theorem 2.7. This means that two r_i are equal, say $r_i = r_k$. Now $X(\xi_i(\xi_k)^{-1})$ has eigenvalues $\{\lambda, \bar{\lambda}, 1, 1, 1, 1, 1\}$, again contradicting Theorem 2.7. This proves the corollary.

Proof of Theorem 2.7. Let Q_1 be an element in G such that $X(Q_1)$ has eigenvalues $\{1, 1, 1, 1, 1, \lambda, \bar{\lambda}\}$. Let $Q_1, Q_2, \ldots, Q_r$ be all of the conjugates of Q_1 in G. The group $H = \langle Q_1, Q_2, \ldots, Q_r\rangle$ is a normal subgroup of G not in the centre, and thus $X|H$ must be irreducible. We will obtain a contradiction by showing that this is not true.

(1) For $i = 1, 2, \ldots, r$ let U_i be the unique 5-dimensional subspace on which $X(Q_i) = I$. Any subspace of U_i is an invariant subspace for $X(Q_i)$. For any given i, $1 \leqq i \leqq r$, there is a j such that Q_i and Q_j do not commute since otherwise $Q_i \in Z(H)$ and $X|H$ cannot be irreducible. Assume that Q_i and Q_j do not commute. Let $U_{ij} = U_i \cap U_j$. We know that U_{ij} is an invariant subspace for $X(Q_i)$ and $X(Q_j)$. Let V_{ij} be a complementary invariant subspace to U_{ij} with respect to $X(\langle Q_i, Q_j\rangle)$. Let $Y_1 = X|U_{ij}$, $Y_2 = X|V_{ij}$. If V_{ij} is 2-dimensional, $Y_2(Q_i)$ and $Y_2(Q_j)$ have eigenvalues $\{\lambda, \bar{\lambda}\}$. Since Q_i and Q_j do not commute, $Y_2(Q_i)$ and $Y_2(Q_j)$ do not commute and thus Lemma 2.2 gives an element T such that $X(T)$ has eigenvalues, contradicting Blichfeldt's theorem. This means that V_{ij} has dimension 3 or 4.

Suppose now that V_{ij} has dimension 4, or, equivalently U_{ij} has dimension 3. In this case, $Y_2(Q_i)$ and $Y_2(Q_j)$ have eigenvalues $\{1, 1, \lambda, \bar{\lambda}\}$. We will show that this is impossible. To do this, the following lemma is needed. The lemma will be proved at the end of the proof of this theorem. We state the lemma here for convenience.

LEMMA 2.9. *Let $i_1, \ldots, i_s$ be a set of integers $1 \leqq i_1 < i_2 < \ldots < i_s \leqq r$ and set $H_{i_1,\ldots,i_s} = \langle Q_{i_1}, \ldots, Q_{i_s}\rangle$. Suppose that U is a 4-dimensional invariant subspace for $X(H_{i_1,\ldots,i_s})$. Let $X(H_{i_1,\ldots,i_s}) = Y \oplus Y_1$, where*

$$Y = X(H_{i_1,\ldots,i_s})|U.$$

Further, we assume that $Y(Q_{i_1}), \ldots, Y(Q_{i_s})$ has eigenvalues $\{1, 1, \lambda, \bar{\lambda}\}$ and Y_1 is the identity. Then Y must be reducible.

Proof. Lemma 2.9 will be proved after the proof of Theorem 2.7.

This lemma applies to V_{ij} since the eigenvalues of $Y_2(Q_i)$ are $\{1, 1, \lambda, \bar{\lambda}\}$. It shows that Y_2 must be reducible. If Y_2 has a 2-dimensional invariant subspace, Lemma 2.3 gives an element in $\langle Q_i, Q_j\rangle$ contradicting Blichfeldt's theorem. This means that there must be a 3-dimensional subspace. Let $Y_2 = Y_3 \oplus Y_4$, where Y_3 is 3-dimensional.

Suppose that $Y_3(Q_i)$ has eigenvalues $\{1, 1, \lambda\}$. In $\langle Q_i, Q_j\rangle$ there is a conjugate R of Q_i which does not commute with Q_i. If this were not so, Q_i would be in all 5-Sylow groups of $\langle Q_i, Q_j\rangle$, and thus Q_i and Q_j would commute since a 5-Sylow group must be abelian. Applying Lemma 2.4 to $\langle Q_i, R\rangle'$ yields an element contradicting Blichfeldt's theorem. This shows that $Y_3(Q_i)$, and similarly $Y_3(Q_j)$ have eigenvalues $\{1, \lambda, \bar{\lambda}\}$. Consequently, $Y_4(Q_i) = Y_4(Q_j) = 1$. This means that $U_i \cap U_j$ is 4-dimensional and thus V_{ij} is 3-dimensional, contrary to our assumption.

We have shown that if Q_i and Q_j do not commute, $U_i \cap U_j$ is 4-dimensional, or equivalently V_{ij} is 3-dimensional. Furthermore, Y_2 must be irreducible otherwise Lemma 2.2 yields an element contradicting Blichfeldt's theorem. Let $e_\lambda{}^i$ be an eigenvector of $X(Q_i)$ with eigenvalue λ. Similarly, let $e_{\bar{\lambda}}{}^i$, $e_\lambda{}^j$, $e_{\bar{\lambda}}{}^j$ be appropriate eigenvectors of $X(Q_i)$ and $X(Q_j)$. Since Y_2 is irreducible and V_{ij} is 3-dimensional, we see that $V_{ij} = \text{Sp}\{e_{\bar{\lambda}}{}^i, e_\lambda{}^i, e_\lambda{}^j, e_{\bar{\lambda}}{}^j\}$, where $\text{Sp}\{v_1, \ldots, v_s\}$ is the linear span of the vectors $v_1, \ldots, v_s$.

(2) Suppose that Q_k is a second element which does not commute with Q_i. We can define V_{ik} as above and show again that V_{ik} is 3-dimensional. We will show that in fact $V_{ij} = V_{ik}$. Suppose then that $V_{ij} \neq V_{ik}$.

Let $V = \text{Sp}\{V_{ij}, V_{ik}\}$. Define $e_\lambda{}^k$ and $e_{\bar{\lambda}}{}^k$ as we did $e_\lambda{}^j$ and $e_{\bar{\lambda}}{}^j$. Clearly, $e_\lambda{}^i$ and $e_{\bar{\lambda}}{}^i$ are in $V_{ij} \cap V_{ik}$ and, since $V_{ij} \neq V_{ik}$, we see that $V_{ij} \cap V_{ik}$ has dimension at most two. This means that $\langle e_\lambda{}^i, e_{\bar{\lambda}}{}^i\rangle = V_{ij} \cap V_{ik}$. Let e_j be an eigenvector of $X(Q_i)$ in V_{ij} with eigenvalue 1. A basis for V_{ij} is $\{e_\lambda{}^i, e_{\bar{\lambda}}{}^i, e_j\}$. Similarly define e_k. A basis for V is $\{e_\lambda{}^i, e_{\bar{\lambda}}{}^i, e_j, e_k\}$.

We will show now that V is an invariant subspace for $X(Q_i)$, $X(Q_j)$, and $X(Q_k)$. Certainly, V is invariant under $X(Q_i)$ as the basis is a basis of eigenvectors. For $X(Q_j)$, it is only necessary to show that e_k is mapped into V as the remaining three vectors span V_{ij}. Clearly, $e_j \notin U_i \cap U_j$ as $e_j \in V_{ij}$. This means that $U_i = \text{Sp}\{U_i \cap U_j, e_j\}$. This is so since $\dim(U_i \cap U_j) = 4$ and $e_j \in U_i$. Since $e_k \in U_i$, we see that $e_k = e + re_j$, r a scalar, $e \in U_i \cap U_j$. Furthermore, $X(Q_j)e_k = X(Q_j)e + rX(Q_j)e_j = e + rX(Q_j)e_j$. Clearly, $e \in V$ and $X(Q_j)e_j \in V$. This shows that V is invariant under $X(Q_j)$. Similarly V is invariant under $X(Q_k)$.

Let $H_{ijk} = \langle Q_i, Q_j, Q_k\rangle$, $H_{ij} = \langle Q_i, Q_j\rangle$, and $H_{ik} = \langle Q_i, Q_k\rangle$. Furthermore, define $X(H_{ijk})|V = Y$. The eigenvalues for $Y(Q_i)$, $Y(Q_j)$, and $Y(Q_k)$ are $\{1, 1, \lambda, \bar{\lambda}\}$. Lemma 2.9 shows that Y is reducible. Since $Y(H_{ij})$ has an irreducible constituent of degree 3, we see that $Y = Y_4 \oplus Y_5$, where Y_5 has degree 3 and Y_4 is linear. Clearly, $Y_5(H_{ij})$ is similar to $X(H_{ij})|V_{ij}$. We see also that $Y_5(H_{ik})$ is similar to $X(H_{ik})|V_{ik}$. If V' is the 3-dimensional subspace corresponding to Y_5, we see that $\{e_\lambda{}^i, e_{\bar{\lambda}}{}^i, e_\lambda{}^j, e_{\bar{\lambda}}{}^j, e_\lambda{}^k, e_{\bar{\lambda}}{}^k\} \in V'$. This shows that $V \subseteq V'$, and thus V cannot be 4-dimensional. We have shown that $V = V_{ij} = V_{ik}$.

(3) Now start with Q_1 and find a Q_j such that Q_1 and Q_j do not commute. Relable Q_j as Q_2. Inductively relable Q_i after Q_{i-1} has been picked so that Q_i does not commute with some Q_j, $j < i$. This will continue until a Q_s is picked such that Q_i for all $i \leqq s$ commute with all unpicked Q_t.

We know that $V = V_{12}$ is an invariant subspace for $X(H_{1,2})$. For any $i \leqq s$ there is a chain of integers $1 = j_1 < j_2 < \ldots < j_m = i$ such that Q_{j_k} and $Q_{j_{k+1}}$ do not commute. We have shown that V is an invariant subspace for $X(Q_{j_1}, Q_{j_2}), \ldots, X(Q_{j_{m-1}}, Q_{j_m})$. Clearly V is an invariant subspace for $X(\langle Q_1, \ldots, Q_s\rangle)$. Let $H_{1,2,\ldots,s} = \langle Q_1, \ldots, Q_s\rangle$.

(4) The group $\langle Q_{s+1}, \ldots, Q_r\rangle$ commutes with $H_{1,2,\ldots,s}$. Call

$$\langle Q_{s+1}, \ldots, Q_r\rangle = \tilde{H}_2 \quad \text{and} \quad H_{1,2,\ldots,s} = \tilde{H}_1.$$

We know that $X(\tilde{H}_1)$ has V as an irreducible invariant subspace. On a complementary invariant subspace, $X(\tilde{H}_1)$ is trivial. By Schur's lemma, $X(\tilde{H}_2)$ has V as an invariant subspace. This shows that V is an invariant subspace for $\langle \tilde{H}_1, \tilde{H}_2\rangle = \langle Q_1, \ldots, Q_r\rangle$, giving a contradiction.

We have completed the proof of Theorem 2.7 except for Lemma 2.9. In this lemma we can relable the elements $Q_{i_1}, \ldots, Q_{i_s}$ as $Q_1, \ldots, Q_s$.

LEMMA 2.9 (restatement). *Let $H = \langle Q_1, \ldots, Q_s\rangle$. Suppose that U is a 4-dimensional invariant subspace for $X(H)$. Let $X(H) = Y \oplus Y_1, X(H)|U = Y$. We assume that $Y(Q_i)$, $i = 1, 2, \ldots, s$, has eigenvalues $\{1, 1, \lambda, \bar{\lambda}\}$; this implies that $Y_1(H)$ is the identity. Then Y must be reducible.*

Proof. We assume that Y is irreducible. Let $|H| = h$.

(1) Clearly $5|h$. Suppose that $5^2 \nmid h$. Applying the results of (**2**) we see that Y has 5-defect 1 and $Y(Q_1)$ or a power must have eigenvalues $\{\lambda, \lambda^2, \lambda^3, \lambda^4\}$, $\{\lambda, \lambda, \bar{\lambda}, \bar{\lambda}\}$, or $\{\lambda, \lambda, \lambda, \lambda\}$. Since $\lambda(Q_1)$ has eigenvalues $\{1, 1, \lambda, \bar{\lambda}\}$, this is a contradiction and we see that $5^2|h$.

(2) Since Y is irreducible, H is not abelian and thus there must be at least one pair i, j such that Q_i and Q_j do not commute. Since the 5-Sylow group is abelian, there are at least two 5-Sylow groups in H. Let two distinct ones be S_5 and P_5. We will show that $S_5 \cap P_5 = e$.

Suppose then that $R \in S_5 \cap P_5$, $R \neq e$. The eigenvalues of $Y(R)$ cannot all be equal as det $Y(R) = 1$. This means that $Y(\langle S_5, P_5\rangle)$ must be reducible since $R \in Z(\langle S_5, P_5\rangle)$. Furthermore, $\langle S_5, P_5\rangle$ does not have a normal 5-Sylow group.

Let $\langle S_5, P_5\rangle = H_0$. Suppose that $Y(H_0)$ has two irreducible constituents of degree 2. By applying Lemma 2.2, it follows that there is an involution J in $Z(H_0)$ such that $Y(J) = -I$. The eigenvalues of $Y(R)$, or a power, are $\{\lambda, \lambda, \bar{\lambda}, \bar{\lambda}\}$ by the unimodularity. The eigenvalues of $X(RJ)$ contradict Blichfeldt's theorem. Suppose that $Y(H_0)$ has a 2-dimensional invariant subspace and two linear ones. Lemma 2.2 yields an element contradicting Blichfeldt's theorem. We see then that $Y(H_0)$ splits into two constituents of degrees 3 and 1. The eigenvalues of $Y(R)$ or a power are $\{\lambda, \lambda, \lambda, \lambda^2\}$. There is a 3-dimensional invariant subspace on which $Y(R)$ is λI. If there is a conjugate R_1 of R in H which does not commute with R, there is a 2-dimensional subspace on which $Y(R)$ and $Y(R_1)$ is λI. Lemma 2.2 yields an element contradicting Blichfeldt's theorem. This shows that all conjugates of R in H must commute, and thus the group generated by these conjugates must be an abelian normal 5-subgroup. It is therefore in all 5-Sylow groups, and hence must commute with all Q_i, $i = 1, 2, \ldots, s$. It must therefore be in the centre of H, contradicting the irreducibility of Y. This means that $S_5 \cap P_5 = e$.

(3) The group H has a faithful unimodular representation Y of degree 4. It is generated by elements Q_i of order 5 such that $Y(Q_i)$ has trace $2 + \lambda + \bar{\lambda}$. Suppose that Y is not primitive. A matrix of the form

$$\begin{bmatrix} 0 & 0 & * & * \\ 0 & 0 & * & * \\ * & * & 0 & 0 \\ * & * & 0 & 0 \end{bmatrix}$$

or a 4×4 permutation matrix cannot have order 5. This means that $Y(Q_i)$ must be diagonal or

$$Y(Q_i) = \begin{bmatrix} * & * & 0 & 0 \\ * & * & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{bmatrix}.$$

We see that in any case $Y(H)$ is reducible. This means that Y is primitive.

(4) Let $h = 7^{b_7} \cdot 5^{b_5} \cdot 3^{b_3} \cdot 2^{b_2}$. Since $H \subseteq G$, $b_7 \leqq a_7$. By (**8**, Theorem 3.1), $a_7 \leqq 4$. By (**3**, 4E) there can be no element in a 7-Sylow group with three eigenvalues 1 if $a_7 \geqq 3$. If $a_7 = 1$ or 2, this is also true by (**3**, 4A) and the fact that X is of full 7-defect. This means that $b_7 = 0$. Since $H \subseteq G$, we see that $b_2 \leqq 10$. By the primitivity of Y, $b_3 \leqq 3 + 1 = 4$, since $3 \nmid 4$ in (**3**, 3E). The number of 5-Sylow groups is congruent to 1 (mod 5^{b_5}). The only possible such number is $2^6 \cdot 3^2 = 576$. This means that $h = 5^2 \cdot 3^{b_3} \cdot 2^{b_2}$, where $2 \leqq b_3 \leqq 4$, and $b \leqq b_2 \leqq 10$. Furthermore, $|H{:}N(P_5)| = 2^6 \cdot 3^2$.

(5) Suppose that there is an element T of order 3 and an element Q of order 5 such that T and Q commute. Suppose also that T does not commute with P_5, the 5-Sylow group containing Q. Since there are no 5-Sylow intersection groups, T must normalize P_5. If P_5 is cyclic, its order must be 5 by (**3**, 3B). This is impossible and thus P_5 is elementary abelian. This means that $N(P_5)/C(P_5)$ is a subgroup of GL(2, 5). By taking a basis of P_5 containing Q, we see T must correspond to a matrix $\left(\begin{smallmatrix} 1 & a \\ 0 & b \end{smallmatrix}\right)$. No such matrix has order 3, giving a contradiction. This means that any 3-element which centralizes an element of order 5 centralizes the whole 5-Sylow group containing it.

(6) Let P_2 be a 2-Sylow subgroup of H. We know that

$$X|P_2 = 1 \oplus 1 \oplus 1 \oplus Y|P_2.$$

Any abelian subgroup of P_2 has variety at most 5 and so its order must be at most 2^4 by (**3**, 3D). We know that $|P_2| \geqq 2^6$.

If $Y(P_2)$ has two components, there is an abelian subgroup of index at most 2^2 which can be found by putting each component in monomial form and taking the diagonal matrices. This implies that $|P_2| = 2^6$. If $Y(P_2)$ is irreducible, there is an abelian subgroup of index at most 2^3. Here $|P_2| \leqq 2$. There must be a non-trivial element in $Z(P_2)$. Here $2^6 \cdot |Z(P_2)| \leqq 2^7$ and therefore $|Z(P_2)| = 2$. Also $|P_2| = 2^7$. Clearly $Z(P_2) = Z(H)$. We know that any 2-element R centralizing Q normalizes P_5. If $|P_2| = 2^6$, this is impossible. If $|P_2| = 2^7$, then $R \in Z(P_2)$ and hence $R \in C(P_5)$.

This shows that if Q is a non-trivial 5-element, $C_H(Q) = C_H(P_5)$, where P_5 is a 5-Sylow group containing Q. We can therefore apply the results of (**4**) to this case.

(7) We know that $2 \nmid |N(P_5)/C(P_5)|$ and therefore $|N(P_5)/C(P_5)|$ is 1 or 3. Since $Y|P_5$ is not rational, Y must have an exceptional character. We use the notation of (4). Let Y be in the 5-block B^α. We know that $r_\alpha s_\alpha = 1$ or 3 and therefore $s_\alpha = 1$ or 3. In either case, $b_\alpha{}^\alpha = 0$ by (**4**, Eq. 4.9) and thus $\deg Y \equiv \epsilon_\alpha r_\alpha s_\alpha \deg \theta_\alpha \pmod{25}$. This means that $4 = \epsilon_\alpha r_\alpha s_\alpha \deg \theta_\alpha$. Clearly $\epsilon_\alpha = 1$, $r_\alpha s_\alpha = 1$, and $\deg \theta_\alpha = 4$. However, θ_α is a representation of $C(P_5)$. This is impossible since $2^2 \nmid |C(P_5)|$. This contradiction establishes the lemma and completes the proof of Theorem 2.7.

Note added in proof. Theorem 2.7 can be shortened by using the linear groups of degree 4.

3. Bounds for Sylow group orders when χ is real. In this section we obtain bounds for the orders of the 3-Sylow group and the 5-Sylow group valid when χ is real on these Sylow groups. This is particularly important when discussing the case $g = 7^3 \cdot g_0$ since by (**8**, Theorem 4.6) χ is often real on these Sylow groups. It is also used later for the case $g = 7 \cdot g_0$. Here P_5 is a 5-Sylow group of G, P_3 a 3-Sylow group of G.

THEOREM 3.1. *Let $|G| = g = 7^{a_7} \cdot 5^{a_5} \cdot 3^{a_3} \cdot 2^{a_2}$. If $\chi|P_5$ is real, then $a_5 \leqq 2$. If $\chi|P_3$ is real, then $a_3 \leqq 5$.*

Proof. (1) We first treat the case of P_5. By Theorem 2.1 we know that P_5 is abelian. Let $\chi|P_5 = \sum_{i=1}^{7} \lambda_i$, where λ_i is a linear character of P_5. Since χ is real we can assume that $\chi|P_5 = \lambda_1 + \bar{\lambda}_1 + \lambda_2 + \bar{\lambda}_2 + \lambda_3 + \bar{\lambda}_3 + \lambda_4$ because the only real linear characters are trivial. This means also that $\lambda_4 = 1$. We will show that $a_5 \leqq 2$ by showing that (a) there cannot be three independent elements of order 5 and (b) there cannot be two independent elements one of order 5^2 the other of order 5. This will show that $a_5 \leqq 2$ as there cannot be an element of order 5^3 (**3**, 3B).

Suppose: (a) there are three independent elements ξ_1, ξ_2, ξ_3 of order 5 in P_5. In the usual way, they can be chosen so that $\lambda_i(\xi_i) = \lambda = e^{2\pi i/5}$ and $\lambda_j(\xi_i) = 1$ for $i \neq j$. This contradicts Theorem 2.7 for ξ_i. Here $i = 1, 2, 3$.

Suppose: (b) there is an element ξ_1 of order 5^2 and an independent element ξ_2 of order 5. We can choose ξ_1 and ξ_2 so that $\lambda_1(\xi_1) = \mu = e^{2\pi i/25}$, $\lambda_1(\xi_2) = 1$, $\lambda_2(\xi_2) = \lambda = e^{2\pi i/5}$. Suppose that $\lambda_2(\xi_1) = \eta_1$, $\lambda_3(\xi_1) = \eta_2$, and $\lambda_3(\xi_2) = \eta_3$. These specify $X(\xi_1)$ and $X(\xi_2)$ completely. We know that $\eta_3 \neq 1$ or Theorem 2.7 is contradicted. Since ξ_2 has order 5, η_3 is a fifth root of 1. Suppose that $(\eta_1)^5 = (\eta_2)^5 = 1$. Then $(\xi_1)^5$ has eigenvalues contradicting Theorem 2.7. This means that at least one of η_1, η_2 is a primitive 25th root of 1. Suppose that

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one is only a fifth root of 1, the other a 25th root of 1. By taking $\xi = \xi_1(\xi_2)^s$ for an appropriate s and rearranging if necessary we can obtain an element ξ such that $\lambda_1(\xi) = \mu$, $\lambda_2(\xi) = \eta$, $\lambda_3(\xi) = 1$. If η is a fifth root of 1, ξ^5 has eigenvalues contradicting Blichfeldt's theorem. Therefore $\eta = (\mu)^t$ for some t, $5 \nmid t$. It can be seen by inspection that there is a value $r = 1, 2, 3$, or 4 such that $rt \equiv -4, -3, -2, -1, 1, 2, 3$, or 4 (mod 25). The eigenvalues of $X(\xi^r)$ contradict Blichfeldt's theorem as they are all within an angle of $8\pi/25$ of 1 on the unit circle and 1 occurs as an eigenvalue. Also $8\pi/25 < \pi/3$.

The only case remaining is that η_1 and η_2 have primitive 25th roots of 1. In this case, either μ^5 is one of $\{(\eta_1)^5, (\bar{\eta}_1)^5, (\eta_2)^5$, or $(\bar{\eta}_2)^5\}$ or $(\eta_1)^5$ is $\{(\eta_2)^5$ or $(\bar{\eta}_2)^5\}$. In the first case after rearranging and multiplying by $(\xi_2)^r$ we can obtain an element ξ such that $\lambda_1(\xi) = \lambda_2(\xi) = \mu$. As in the paragraph above, one of ξ, ξ^2, ξ^3, ξ^4 has eigenvalues contradicting Blichfeldt's theorem. In the second case we can rearrange to find new ξ_1, ξ_2 such that $\lambda_1(\xi_1) = \eta_1$, $\lambda_2(\xi_1) = \eta_2$, $\lambda_3(\xi_1) = \mu$, $\lambda_1(\xi_2) = 1$, and $\lambda_2(\xi_2) = \lambda$. Here $(\eta_1)^5 = (\eta_2)^5$. This is the case just considered and so we have a contradiction.

We have shown that $a_5 \leqq 2$.

(2) We now treat similarly the case of P_3. There can be no element of order 3^3 by (**3**, 3B). If H is an abelian subgroup of P_3 again we have $X|P_3 = \lambda_1 + \bar{\lambda}_1 + \lambda_2 + \bar{\lambda}_2 + \lambda_3 + \bar{\lambda}_3 + 1$.

If H is an abelian subgroup with three independent elements of order 3 they can be chosen as ξ_1, ξ_2, ξ_3, where $\lambda_i(\xi_i) = \eta = e^{2\pi i/3}$, $\lambda_i(\xi_j) = 1$ for $j \neq i$. It is clear that a fourth independent element of order 3 is impossible in H.

Suppose that H is an abelian subgroup of type $(3^2, 3^2, 3)$. Again we can pick a basis ξ_1, ξ_2, ξ_3 so that $\lambda_1(\xi_1) = \mu$, $\lambda_2(\xi_1) = 1$, $\lambda_3(\xi_1) = \mu_1$; $\lambda_1(\xi_2) = 1$, $\lambda_2(\xi_2) = \mu$, $\lambda_3(\xi_2) = \mu_2$; and $\lambda_1(\xi_3) = \lambda_2(\xi_3) = 1$, $\lambda_3(\xi_3) = \eta$. Here $\mu = e^{2\pi i/9}$, $\eta = e^{2\pi i/3} = (\mu)^3$. If $(\mu_1)^3 = 1$, an element $\xi_1(\xi_3)^r$ has eigenvalues $(\mu, \bar{\mu}, 1, 1, 1, 1, 1)$, contradicting Blichfeldt's theorem. If $(\mu_1)^3 \neq 1$, an element $\xi = \xi_1(\xi_3)^r$ has $\lambda_1(\xi) = \mu$, $\lambda_2(\xi) = 1$, $\lambda_3(\xi) = \mu$ or $\bar{\mu}$. In either case, Blichfeldt's theorem is contradicted.

These arguments show that any abelian subgroup of P_3 must have order at most 3^4 or there would be a subgroup of type (3, 3, 3, 3) or $(3^2, 3^2, 3)$. If $a_3 \geqq 6$, P_3 must be non-abelian. This means that $X|P_3$ must have a non-linear component U of degree 3. Clearly U cannot be real since there is an element T in $Z(P_3/\ker U)$ for which $U(T) = \eta I$. This implies that $X|P_3 = U \oplus \bar{U} \oplus y$, where y is linear. By putting U in monomial form we obtain an abelian subgroup of order 3^5, giving a contradiction. This implies that $a_3 \leqq 5$ and proves the theorem.

4. Statement of the results.

THEOREM 4.1. *Suppose that G has a complex irreducible representation X of degree 7 which is faithful, unimodular, and primitive. If G has a non-abelian 7-Sylow group, G is one of the following groups:*

(I) *G is a uniquely determined group of order* $7^4 \cdot 48$ *which has a non-abelian*

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normal subgroup D of order 7^3 *and exponent* 1 *such that* $G/D \cong \mathrm{SL}(2, 7)$;

(II) *Certain subgroups of G in* (I) *containing D as a* 7*-Sylow group.*

Remarks. Let $|G| = g = 7^{a_7} \cdot g_0$, $(7, g_0) = 1$. If the 7-Sylow group is non-abelian, then $a_7 \geqq 3$. By (**8**, Theorem 2.2) we know that $a_7 \leqq 4$. We need only consider $a_7 = 3$ and $a_7 = 4$. The case $a_7 = 4$ is treated in § 5, the case $a_7 = 3$ is treated in § 6. The results of (**6** or **3**, 2D) show that no prime higher than 7 occurs in g.

5. The case $a_7 = 4$**.** If there are non-abelian 7-Sylow intersection groups, then (**8**, Theorem 3.1) yields case (I). We will use (**8**, Theorem 4.2) to show that there are no other possibilities. We assume then that G has no (non-abelian) 7-Sylow intersection groups.

By (**3**, 6A, 6B, 7B) and our assumption about non-abelian 7-Sylow intersection groups, we see that the only 7-Sylow intersection groups are P and Z. This means that the number of 7-Sylow groups, $|G:N(P)|$, is congruent to 1 (mod 7^3). Let this number be T. Certainly $H = O^{7'}(G)$ has T 7-Sylow groups also and $X|H$ is primitive by (**8**, Theorem 4.2). We will therefore replace G by H in our discussion and thus assume that $O^{7'}(G) = G$. Let $|G| = g = 7^4 \cdot 5^{a_5} \cdot 3^{a_3} \cdot 2^{a_2}$. We can apply (**8**, Corollaries 4.3 and 4.4) to see that $a_5 \leqq 1$.

Suppose that there is an element R in G of order 3 such that $\bar{R}$ centralizes an element of order 7 in $\bar{P}$. Since there are no non-trivial 7-Sylow intersection groups in $\bar{G}$, $\bar{R}$ must normalize $\bar{P}$ and so R must normalize P. This means that R normalizes A, the unique abelian subgroup of P of order 7^3. Let $K = \langle P, R \rangle$. Clearly $X|K$ is irreducible and can be written in monomial form. The only diagonal matrices are in A by (**3**, 4F). This means that K/A is isomorphic to a subgroup of S_7, the symmetric group on 7 elements. There is a normal 7-Sylow group. The cycle structure of $X(R)$ considered as a permutation is therefore $(abc)(def)$. Since the eigenvalues of $X(Q)$, $Q \in A$, $Q \notin Z$, have multiplicity at most 2 by (**3**, 4E), no $X(Q)$ can commute with $X(R)$. This means that there are no elements of order 21 in $\bar{G}$, and therefore by (**8**, Corollary 4.3), $a_3 \leqq 4$.

Finally, let $T = 5^{b_5} \cdot 3^{b_3} \cdot 2^{b_2}$. We have shown that $b_5 \leqq 1$, $b_3 \leqq 4$, $b_2 \leqq 10$. There are no values T of this form congruent to 1 (mod 7^3). In fact, the only such values congruent to 1 (mod 7^2) are 2304 and 540. None of these are congruent to 1 (mod 7^3) as a quick check shows. We have shown that there are no further groups with $|G| = 7^4 g_0$.

6. The case $a_7 = 3$**.** In this section we treat the case $g = 7^3 \cdot g_0$. If G has a normal 7-Sylow group, we have by (**3**, § 8) $|N(P)| = 7^3 \cdot s$ with $s | 48$. This gives case (II) of Theorem 4.1. We will show that this is the only possibility. The results of (**4**) apply to $\bar{G}$.

Suppose then that G does not have a normal 7-Sylow group. We can replace

G by $O^{7'}(G)$ and apply (**8**, Theorems 4.5 and 4.6). Here χ on 7′-elements is real and thus Theorem 3.1 applies. Let $g = 7^3 \cdot 5^{a_5} \cdot 3^{a_3} \cdot 2^{a_2}$. Theorem 3.1 shows that $a_5 \leqq 2$, $a_3 \leqq 5$. We know that $a_2 \leqq 10$. We show first that $a_2 \neq 10$.

LEMMA 6.1. *The value a_2 is not equal to* 10.

Proof. Suppose that $a_2 = 10$ and let P_2 be a 2-Sylow group of G. Suppose that $X|P_2$ has constituents at most of degree 2. There would be at most three such constituents. For each, there is an abelian subgroup of index two consisting of diagonal matrices. This means that there is an abelian subgroup of order 2^7, contradicting (**3**, 3D). This means that there must be an irreducible constituent of degree 4. Putting this constituent in monomial form yields a subgroup K of index at most 2^3 consisting of diagonal matrices on this constituent. We see that there must then also be an irreducible constituent of degree 2 or there would be an abelian subgroup of order 2^7. Let $X|P_3 = Y_1 \oplus Y_2 \oplus Y_3$. Here Y_1 is of degree 4, Y_2 of degree 2. We have seen that $Y_2(K)$ must be irreducible or there is an abelian subgroup of order 2^7. An involution J_1 in $K' \cap Z(K)$ has $Y_1(J_1) = I$, $Y_2(J_1) = -I$, $Y_3(J_1) = 1$. There is a subgroup L of order 2^9 on which Y_2 is diagonal. Again $Y_1(L)$ must be irreducible or there is an abelian subgroup of order 2^7. This means that there is an involution J_2 such that $Y_1(J_2) = -I$, $Y_2(J_2) = I$, $Y_3(J_2) = 1$. Let $J = J_1 J_2$. Clearly $J \in Z(P_2)$, $\chi(J) = -5$. We see that $\chi\bar{\chi}(J) = 25$.

Let $\chi\bar{\chi} = 1 + y$. Clearly $y(J) = 24$. If y is irreducible, this is impossible since $(g/|C(J)|)(24/48)$ is not an algebraic integer. If y is not irreducible by (**3**, § 8) we have (**3**, case II) and $y = \sum_{j=1}^{t} \chi_0{}^j$. This means that $\chi_0{}^j(J) = (24/t) \cdot 48$. Again $(g/|C(J)|)(24/48 \cdot t)$ is not an algebraic integer. This shows that $a_2 < 10$.

We now consider the different possible values $T = |G{:}N(P)|$ can have. Let $T = 5^{b_5} \cdot 3^{b_3} \cdot 2^{b_2}$. We know that $b_5 \leqq 2$, $b_3 \leqq 5$, $b_2 \leqq 9$. The possible values are $2^8 \cdot 3^2$, $2^2 \cdot 3^3 \cdot 5$, $2 \cdot 5^2$, and $2^9 \cdot 3^2 \cdot 5^2$. We treat each case separately. The results of (**4**) are used for $\bar{G}$. They are described explicitly in (**3**, § 8). We use the notation of (**3**, § 8) along with the numbering of the equations (8.4) to (8.7) of (**3**, § 8).

Suppose that $T = 2^9 \cdot 3^2 \cdot 5^2$. Let $|N(P)/P| = s$. We know that $s|48$. Also $|G| = 7^3 \cdot 5^2 \cdot 3^2 \cdot 2^9 \cdot s$. The value s is not 1 as in (**3**, § 8, last paragraph). Since $2^{10} \nmid |G|$ by Lemma 6.1, we see that $s = 3$. By Schur's theorem (**7**), χ cannot be rational on P_5. This means that there is a $\sigma \in G(K/Q[\epsilon])$, $\epsilon = e^{2\pi i/7}$ such that $\chi \neq \chi^\sigma$. By (**8**, Theorem 4.6) there is a character χ_2 of degree 48 with $\chi_2 \neq (\chi_2)^\sigma$. Equations (8.6) and (8.7) become

$$1 - 48 - 48 + x_0(16\gamma - \delta) = 0,$$

$$1 + 1 + 1 + 15(\gamma)^2 + (\gamma - \delta)^2 = 4.$$

Clearly $\gamma = 0$, $-95 - \delta x_0 = 0$, giving a contradiction. This means that $T \neq 2^9 \cdot 3^2 \cdot 5^2$.

Suppose that $T = 2^8 \cdot 3^2$. In this case $s = 2, 3, 6$. If $s = 2$, equation (8.6) becomes

$$1 - 48 + (24\gamma - \delta)x_0 = 0.$$

Again (8.7) yields $\gamma = 0$, and we have a contradiction. For the case $s = 3$, γ is again 0 and we have

$$1 - 48 + b_2 x_2 - \delta x_0 = 0.$$

One of x_2 or x_0 must be odd and therefore must be a power 3^r. No power 3^r, $r \leqq 5$ is congruent to $\pm 1 \pmod{49}$. One of x_2, x_0 must not be divisible by 3 and so must be a power 2^r. However, no power of 2^r, $r \leqq 9$, is congruent to $\pm 1 \pmod{49}$. This means that x_0 is a power of 2 and a power of 3, giving a contradiction. The case $s = 3$ is therefore impossible. If $s = 6$, we have:

$$1 - 48 + \sum_{i=3}^{l} b_i x_i + x_0(8\gamma - \delta) = 0,$$

$$1 + 1 + \sum_{i=3}^{l} (b_i)^2 + 7(\gamma)^2 + (\gamma - \delta)^2 = 7.$$

Clearly $\gamma = 0$, $b_i = \pm 1$, $l = 6$; or $\gamma = 0$, $b_3 = \pm 2$, $l = 3$. As above, if $b_i = \pm 1$, $b_i x_i$ cannot be odd. If $b_3 = \pm 2$, $b_3 x_3$ cannot be odd. This means that δx_0 must be odd. One of the x_i must then be a power of 2. This is impossible if $b_i = \pm 1$. If $b_3 = 2$, the only possibility is $x_2 = b_2 = 2$, which is impossible since P would be in the kernel of χ_2 by (8.4). It is also impossible by an examination of the linear groups in two variables. This means that $T \neq 2^8 \cdot 3^2$.

Suppose that $T = 2 \cdot 5^2 = 50$. Again by Schur's theorem (**7**), χ is not rational on P_5 and so by (**8**, Theorem 4.6) there are at least two representations of degree 48. In particular, $48|g$. Also $g = 7^3 \cdot 2 \cdot 5^2 \cdot s$. We see that $s = 24$ or 48. The possibilities for x_i and x_0 are listed in Table A.

If $s = 48$, let $b_0 = \gamma - \delta$. Equations (8.6) and (8.7) become

$$\sum_{i=0}^{l} (b_i)^2 = 49 \quad \text{and} \quad \sum_{i=0}^{l} b_i x_i = 0.$$

We know that $b_i = x_i = 1$ occurs once. It is clear from Table A that there is no other $b_i x_i$ which is odd except $b_i = x_i = 3$ and $b_i = x_i = 5$. In these cases, P is in the kernel of χ_i by (8.4), giving a contradiction. This means that $s = 48$ is impossible.

We now consider the case $s = 24$. Here we have

$$1 + \sum_{i=2}^{l} b_i x_i + (2\gamma - \delta)x_0 = 0.$$

As in the above paragraph, no $b_i x_i$ can be odd, and thus $(2\gamma - \delta)x_0$ must be odd. The only possibilities are $x_0 = 25$, $\gamma = \delta = 1$, and $x_0 = 75$, $\gamma = 2$, $\delta = 1$. The case $\delta = -1$ can be discarded by interchanging the exceptional characters.

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The case $x_0 = 75$, $\gamma = 2$ is considered first. By checking Table A, it is clear that 5^2 divides all x_i except $x_i = 48$ since $96 \nmid g$. The equations become

$$1 - 48 - 48 + \sum_{i=4}^{l} b_i x_i + 3\cdot 75 = 0,$$

$$4 + 1 + 1 + 1 + 1 + \sum_{i=4}^{l} (b_i)^2 = 25,$$

or

$$\sum_{i=4}^{l} (b_i)^2 = 17.$$

If 48 occurs u more times we have $1 - 48 - 48 - 48u \equiv 0 \pmod{5^2}$ or $2u \equiv -5 \pmod{5^2}$. The only solution is $u = 10$. Let $b_4 = b_5 = \ldots = b_{13} = -1$, $x_4 = \ldots = x_{13} = 48$. This leaves $\sum_{i=14}^{l} (b_i)^2 = 7$ and

$$1 - 12\cdot 48 + 225 + \sum_{i=14}^{l} b_i x_i = 0.$$

This last equation is $\sum_{i=14}^{l} b_i x_i = 350$. The only possible values of x_i remaining are 50 and 100. There are two possible solutions:

(a) $1 + 12\cdot 48\cdot(-1) + 50 + 50 + 50 + 2\cdot 100 + 3\cdot 75 = 0$;

(b) $1 + 12\cdot 48\cdot(-1) + 7\cdot 50 + 3\cdot 75 = 0$.

Suppose that $x_0 = 25$, $\gamma = \delta = 1$. Again there are twelve degrees 48, as 96 is again impossible. The equations become

$$\sum_{i=14}^{l} b_i x_i = 550 \quad \text{and} \quad \sum_{i=14}^{l} (b_i)^2 = 11.$$

The remaining degrees are 50, 100, and 150. Solutions are:

(c) $1 + 12\cdot 48\cdot(-1) + 3\cdot 150 + 50 + 50 + 25 = 0$;

(d) $1 + 12\cdot 48\cdot(-1) + 50 + 50 + 50 + 2\cdot 100 + 2\cdot 100 + 25 = 0$;

(e) $1 + 12\cdot 48\cdot(-1) + 2\cdot 100 + 7\cdot 50 + 25 = 0$;

(f) $1 + 12\cdot 48\cdot(-1) + 11\cdot 50 + 25 = 0$.

In each of these cases,

$$(49)^2 + \sum_{i=1}^{l} (x_i)^2 + 2(x_0)^2 = 7^2\cdot 5^2\cdot 3\cdot 2^4,$$

and thus there are no more characters in G/Z. In each case, the only possibility for the degree equation of $B_0(3)$ is $1 + 49 = 50$ as a quick inspection shows. If π is a 3-element and χ_3 is the character of degree 50 in $B_0(3)$, χ^* the character of degree 7^2, we have

$$\chi_3(\pi) = -1, \qquad \chi^*(\pi) = 1.$$

If there is an element R of order 15 or 10 in $\bar{G}$, all characters are 0 on R except 1 and χ^*. This implies that $\chi^*(R) = -1/49$, a contradiction. There are no elements of order $7\cdot 5$ in $\bar{G}$ by (**3**, 4F). We see that if π is of order 5 in G contained in the 5-Sylow group P_5, then $C(\pi) = P_5 \times Z$. The results of (**4**) can be applied to both P_5 in G and $\bar{P}_5$ in $\bar{G}$. We apply them first for $\bar{P}_5$ in $\bar{G}$. Let $s^* = N(P_5)/C(P_5)$, $t^* = 24/s^*$. By Sylow's theorem, $7^2\cdot 3\cdot 2^4 \equiv s^* \pmod 5$.

Since $s^*|24$, we have $s^* = 2$ or 12. If $s = 12$, two of the characters of degree 48 are exceptional, the rest ordinary. However, $10\cdot(2)^2 > 13$, giving a contradiction. This means that $s^* = 2$. The degree equation must be

$$1 + 48 - 49 = 0.$$

We now apply the results of (**4**) to P_5 in G. Here $C = C(P_5) = P_5 \times Z$, $H = N(P_5)$, $|H| = 5^2\cdot 7\cdot 2$, and $Z \in Z(H)$. We use the notation of (**4**). By (**4**, Theorem 3A), the characters θ_α are the seven linear characters of Z. In each case, $|F(\theta_\alpha):C| = 2 = s_\alpha$, $r_\alpha = |H:F(\theta_\alpha)| = 1$. Furthermore, $5^2 - 1 = \omega\cdot 2 = 12\cdot 2$. Here $F(\theta_\alpha)$ is the inertial group of θ_α in H. Suppose that χ is in $B(\theta_\alpha)$. This yields $\sum_i (b_i)^2 + (12-1)(b_\alpha^{(\alpha)})^2 + (b_\alpha - \epsilon_\alpha)^2 = 3$. Clearly $b_\alpha^{(\alpha)} = 0$. Furthermore, $(b_1)^2 + (b_2)^2 + (\epsilon_\alpha)^2 = 3$, $|b_i| = |\epsilon_\alpha| = 1$. This means that $\deg \chi_i^\alpha \equiv b_i \pmod{25}$ and $\deg \chi_\beta^\alpha \equiv 2\cdot\epsilon_\alpha \pmod{25}$ by (**4**, 4H). None of these can be 7 and so there is a contradiction. This eliminates the case $T = 5^2\cdot 2$.

The final case is $T = 2^2\cdot 3^3\cdot 5$. The possibilities for x_i, x_0 are listed in Table B. The cases $b_i = x_i$ are impossible since P must be in the kernel of χ_i by (8.4). Thus, in particular, $b_i = 3 = x_i$ and $b_i = 5 = x_i$ need not be considered. We distinguish the cases of different s.

(1) $s = 2$, $t = 24$. Clearly $\gamma = 0$. We have $x_0 \equiv 2\delta \pmod{7^2}$. From Table B, x_0 and x_2 are even, and therefore $1 + b_2x_2 - \delta x_0 \neq 0$.

(2) $s = 3$, $t = 16$. Again $\gamma = 0$. This time $x_0 \equiv 3\delta \pmod{7^2}$. Again x_0 and x_i are even unless $x_0 = 3$. The simple linear groups of degree 3 are known and do not have order $|\bar{G}|$.

(3) $s = 4$, $t = 12$. Here $\gamma = 0$, $x_0 \equiv 4\delta \pmod{7^2}$. The only possible odd degree is $x_0 = 45$. Here $|\bar{G}| = 7^2\cdot 5\cdot 3^3\cdot 2^4$. All $\chi_0{}^j$ are equal on all elements commuting with elements of P_3 as there are no elements of order 21 in $\bar{G}$ (**3**, § 8). This means that all $\chi_0{}^j$ are in the same 3-block. However, this is impossible since the defect group is cyclic of order 3.

(4) $s = 6$, $t = 8$. Again $\gamma = 0$. The only possibilities for non-trivial x_0, x_i are even and so this case is impossible.

(5) $s = 8$, $t = 6$. Here $|\gamma| \leqq 1$. The only possibility for x_i, $x_0{}^j$ to be odd is $x_0{}^j = 9$. Again, the $x_0{}^j$ are all in one 3-block of defect 1, giving a contradiction.

(6) $s = 12$, $t = 4$. Here $|\gamma| \leqq 1$ as $3\cdot 4 + 1 + 1 + \ldots > 13$. The only possibilities for odd x_i, x_0 are 135 and 405. Here $|G| = 7^2\cdot 5\cdot 3^4\cdot 2^4$. A character of degree 135 is of 3-defect 1. They must all be in the same 3-block, giving a contradiction. If $x_0 = 405$, $4\cdot(405)^2 > 7^2\cdot 5\cdot 3^4\cdot 2^4$, giving a contradiction.

(7) $s = 16$, $t = 3$. Here $\gamma \leqq 2$ since $9\cdot 2 + \ldots > 17$. There are no odd possibilities for b_ix_i or $(3\gamma - \delta)x_0$ and therefore this case is impossible.

(8) $s = 24$, $t = 2$. The only possibility for odd x_i, $x_0{}^j$ is $x_0{}^j = 27$. Here, two characters of degree 27 would be in the same 3-block. However, the defect group is of order 3 and implies a third character of degree 27. This third character would lie in $B_0(7)$ for $\bar{G}$ and must be non-exceptional. This is impossible.

(9) $s = 48$, $t = 1$. Since $t = 1$, there is no difference between exceptional and non-exceptional characters. None of the $b_i x_i$ are odd and therefore this case is impossible.

This completes all cases and proves the theorem.

TABLE A

$T = 50, \quad |G| = 7^3 \cdot s \cdot 50$

$x \leqq (7^2 \cdot 5^2 \cdot 2 \cdot 48)^{1/2} \leqq 7 \cdot 5 \cdot 10 = 350,\quad x \mid 5^2 \cdot 3 \cdot 2^5,$		
$x \equiv \pm 1,$	$1, 50, 48,$	
$x \equiv \pm 2,$	$2, 100, 96$	(96 is impossible if $s = 24$),
$x \equiv \pm 3,$	$3, 150,$	
$x \equiv \pm 4,$	$4, 200.$	

There are no further odd possibilities except 5, 15, 25, 75.

TABLE B

$T = 5 \cdot 3^3 \cdot 2^2, \quad |G| = 7^3 \cdot s \cdot T$

$x \leqq 7 \cdot 3^2 \cdot 2^3 \cdot \sqrt{5} \leqq 1160,\quad x \mid 5 \cdot 3^4 \cdot 2^6,$	
$x \equiv \pm 1,$	$1, 540 = 5 \cdot 3^3 \cdot 2^2, 48,$
$x \equiv \pm 2,$	$2, 1080 = 5 \cdot 3^3 \cdot 2^3, 96,$
$x \equiv \pm 3,$	$3, 144,$
$x \equiv \pm 4,$	$4, 45, 192, 2^6 \cdot 3,$
$x \equiv \pm 5,$	$5, 54, 240 = 5 \cdot 3 \cdot 2^4.$

There are no further odd possibilities except 9, 135, 405, 27, 81, 15. Here $135 \equiv -12$, $405 \equiv 13$, $27 \equiv -22$, $81 \equiv -17$, and $15 \equiv 15 \pmod{7^2}$.

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