ON NEUMANN'S FORMULA FOR THE LEGENDRE FUNCTIONS

by T. M. MACROBERT

(Received 29th November, 1948)

§ 1. *Introductory.* The formula

$$Q_n(z) = \tfrac{1}{2}\int_{-1}^{1} \frac{P_n(\mu)}{z-\mu}\,d\mu, \qquad \text{.........(1)}$$

where n is zero or a positive integer and $|z|>1$, was given by F. E. Neumann [*Crelle's Journal*, XXXVII (1848), p. 24]. In § 2 of this paper some related formulae are given ; the extension to the case when n is not integral is dealt with in § 3 ; while in § 4 the corresponding formulae for the Associated Legendre Functions when the sum of the degree and the order is a positive integer are established.

§ 2. *Related Formulae.* It will be assumed throughout that $|z|>1$. From (1), if m is a positive integer,

$$\begin{aligned} z^m Q_n(z) - \tfrac{1}{2}\int_{-1}^{1} \frac{\mu^m P_n(\mu)}{z-\mu}\,d\mu &= \tfrac{1}{2}\int_{-1}^{1} \frac{z^m-\mu^m}{z-\mu} P_n(\mu)\,d\mu \\ &= \tfrac{1}{2}\int_{-1}^{1} (z^{m-1}+z^{m-2}\mu+\ \dots\ +\mu^{m-1})P_n(\mu)\,d\mu. \qquad \text{.........(A)} \end{aligned}$$

If $m \leqq n$, this last integral vanishes. Therefore

$$z^m Q_n(z) = \tfrac{1}{2}\int_{-1}^{1} \frac{\mu^m P_n(\mu)}{z-\mu}\,d\mu,\ m \leqq n. \qquad \text{.........(2)}$$

It follows that

$$P_m(z) Q_n(z) = \tfrac{1}{2}\int_{-1}^{1} \frac{P_m(\mu) P_n(\mu)}{z-\mu}\,d\mu,\ m \leqq n. \qquad \text{.........(3)}$$

If $m=n+1$, (A) is equal to $\qquad \tfrac{1}{2}\int_{-1}^{1} \mu^n P_n(\mu)\,d\mu,$

and this is equal to $\qquad \dfrac{2^n(n!)^2}{(2n)!}\,\tfrac{1}{2}\int_{-1}^{1} \{P_n(\mu)\}^2\,d\mu$

Thus

$$z^{n+1} Q_n(z) = \tfrac{1}{2}\int_{-1}^{1} \frac{\mu^{n+1} P_n(\mu)}{z-\mu}\,d\mu + \frac{2^n(n!)^2}{(2n+1)!} \qquad \text{.........(4)}$$

From this and (2) it follows that

$$P_{n+1}(z) Q_n(z) = \tfrac{1}{2}\int_{-1}^{1} \frac{P_{n+1}(\mu) P_n(\mu)}{z-\mu}\,d\mu + \frac{1}{n+1}. \qquad \text{.........(5)}$$

This at once gives the known result

$$P_{n+1}(z) Q_n(z) - P_n(z) Q_{n+1}(z) = \frac{1}{n+1}. \qquad \text{.........(6)}$$

Other formulae of similar type are

$$z P_m(z) Q_n(z) = \tfrac{1}{2}\int_{-1}^{1} \frac{\mu P_m(\mu) P_n(\mu)}{z-\mu}\,d\mu,\ m<n, \qquad \text{.........(7)}$$

https://doi.org/10.1017/S204061850003286X Published online by Cambridge University Press

$$zP_n(z)Q_n(z)=\tfrac{1}{2}\int_{-1}^{1}\frac{\mu\{P_n(\mu)\}^2}{z-\mu}\,d\mu+\frac{1}{2n+1}, \quad \text{.........................(8)}$$

$$(z^2-1)Q_n'(z)=\tfrac{1}{2}\int_{-1}^{1}\frac{(\mu^2-1)P_n'(\mu)}{z-\mu}\,d\mu,\ n\geqq 1, \quad \text{.........................(9)}$$

$$z^m(z^2-1)Q_n'(z)=\tfrac{1}{2}\int_{-1}^{1}\frac{\mu^m(\mu^2-1)P_n'(\mu)}{z-\mu}\,d\mu,\ m<n, \quad \text{.....................(10)}$$

$$(z^2-1)P_m(z)Q_n'(z)=\tfrac{1}{2}\int_{-1}^{1}\frac{(\mu^2-1)P_m(\mu)P_n'(\mu)}{z-\mu}\,d\mu,\ m<n. \quad \text{..................(11)}$$

§ 3. *Extension to General Values of n.* If n is not integral and $-1<\mu<1$,

$$P_n(\mu)=\frac{\sin n\pi}{\pi}\sum_{p=0}^{\infty}(-1)^p\left(\frac{1}{n-p}-\frac{1}{n+p+1}\right)P_p(\mu). \quad \text{.......................(12)}$$

Thus, if $|z|>1$,

$$\tfrac{1}{2}\int_{-1}^{1}\frac{P_n(\mu)}{z-\mu}\,d\mu=\frac{\sin n\pi}{\pi}\sum_{p=0}^{\infty}(-1)^p\left(\frac{1}{n-p}-\frac{1}{n+p+1}\right)Q_p(z), \quad \text{.................(13)}$$

the series on the right being absolutely convergent * for $|z|>1$. When $n\to p$ this formula reduces to (1) with p in place of n.

The term by term integration requires justification, as the series on the right of (12) is divergent when $\mu=-1$. The proof is based on the formula

$$P_p(\cos\theta)=\frac{1}{\surd(2\pi\sin\theta)}\frac{\Gamma(p+1)}{\Gamma(p+\frac{3}{2})}$$
$$\times\sum_{i,-i}e^{\frac{1}{4}\pi i-(p+\frac{1}{2})\theta i}F\left(\tfrac{1}{2},\tfrac{1}{2};p+\tfrac{3}{2};-\frac{e^{-i\theta}}{2i\sin\theta}\right). \quad \text{...............(14)}$$

Here the hypergeometric series converges for $\frac{1}{6}\pi\leqq\theta\leqq\frac{5}{6}\pi$ and is asymptotic in p for the other values of θ in the range $0<\theta<\pi$. The series on the right of (12) therefore converges absolutely and uniformly for $0<\epsilon\leqq\theta\leqq\pi-\epsilon$.

Again, if $|\zeta|<1$,

$$F(\tfrac{1}{2},\tfrac{1}{2};p+\tfrac{3}{2};\zeta)=\frac{\Gamma(p+\frac{3}{2})}{\Gamma(\frac{1}{2})\Gamma(p+1)}\int_0^1 t^{-\frac{1}{2}}(1-t)^p(1-\zeta t)^{-\frac{1}{2}}\,dt, \quad \text{.................(15)}$$

and the expression on the right may be used to obtain the analytical continuation of the left-hand side over the ζ-plane bounded by a cross-cut along the real axis from 1 to $+\infty$.

Now, if

$$\zeta=-e^{-i\theta}/(2i\sin\theta)=\tfrac{1}{2}-\frac{1}{2i}\cot\theta,$$
$$|1-\zeta t|=|1-\tfrac{1}{2}t-\tfrac{1}{2}it\cot\theta|$$
$$=\surd(1-t+\tfrac{1}{4}t^2\operatorname{cosec}^2\theta)$$
$$=\surd\{\cos^2\theta+(\sin\theta-\tfrac{1}{2}t\operatorname{cosec}\theta)^2\}$$
$$\geqq|\cos\theta|.$$

From (15) it follows that, if $0<\theta\leqq\epsilon$ or $\pi-\epsilon\leqq\theta<\pi$,

$$\left|F\left(\tfrac{1}{2},\tfrac{1}{2};p+\tfrac{3}{2};-\frac{e^{-i\theta}}{2i\sin\theta}\right)\right|\leqq\surd(\sec\theta).$$

Therefore, from (14),

$$|\sin\theta\,P_p(\cos\theta)|\leqq\surd\left(\frac{\sin\theta}{2\pi}\right)\frac{\Gamma(p+1)}{\Gamma(p+\frac{3}{2})}2\surd(\sec\epsilon),$$

and, by continuity, this holds for $0\leqq\theta\leqq\epsilon$ and for $\pi-\epsilon\leqq\theta\leqq\pi$.

* Cf. *Phil. Mag.*, Ser. 7, XXXV (1944), p. 673–6.

https://doi.org/10.1017/S204061850003286X Published online by Cambridge University Press

It follows that the series on the right of (12), multiplied by $\sin\theta$, where $\mu = \cos\theta$, converges absolutely and uniformly for $0 \leqq \theta \leqq \pi$ and that the term by term integration is justified.

§ 4. *Legendre Functions when the sum of the Degree and the Order is a Positive Integer.* If n is zero or a positive interger,

$$P_{m+n}^{-m}(z) = \frac{2^{m+n}\,\Gamma(m+n+\frac{1}{2})}{\Gamma(\frac{1}{2})\,\Gamma(2m+n+1)}(z^2-1)^{\frac{1}{2}m} \times \left\{z^n - \frac{n(n-1)}{2(2m+2n-1)}z^{n-2} + \ldots\right\} \qquad (16)$$

and

$$T_{m+n}^{-m}(\mu) = \frac{(-1)^n(1-\mu^2)^{-\frac{1}{2}m}}{2^{m+n}\,\Gamma(m+n+1)}\frac{d^n}{d\mu^n}\{(1-\mu^2)^{m+n}\}, \qquad (17)$$

the latter formula being an extension of Rodrigues' Formula.

By using (17) and integrating by parts it can be shown that, if p is a positive integer,

$$\int_{-1}^{1}\mu^p(1-\mu^2)^{\frac{1}{2}m}\,T_{m+n}^{-m}(\mu)\,d\mu = \begin{cases} 0, & p<n, \\ \dfrac{\Gamma(\frac{1}{2})\,.\,n!}{2^{m+n}\,\Gamma(m+n+\frac{3}{2})}, & p=n. \end{cases} \qquad (18)$$

The formula *

$$(z^2-1)^{\frac{1}{2}m}\,Q_{m+n}^{-m}(z) = \tfrac{1}{2}\int_{-1}^{1}\frac{(1-\mu^2)^{\frac{1}{2}m}\,T_{m+n}^{-m}(\mu)}{z-\mu}\,d\mu \qquad (19)$$

is proved by expanding on the right in descending powers of z and using (18). It then follows that, if p is a positive integer,

$$z^p(z^2-1)^{\frac{1}{2}m}\,Q_{m+n}^{-m}(z) = \tfrac{1}{2}\int_{-1}^{1}\frac{\mu^p(1-\mu^2)^{\frac{1}{2}m}\,T_{m+n}^{-m}(\mu)}{z-\mu}\,d\mu, \quad p \leqq n, \qquad (20)$$

$$P_{m+p}^{-m}(z)\,Q_{m+n}^{-m}(z) = \tfrac{1}{2}\int_{-1}^{1}\frac{T_{m+p}^{-m}(\mu)\,T_{m+n}^{-m}(\mu)}{z-\mu}\,d\mu, \quad p \leqq n, \qquad (21)$$

$$z^{n+1}(z^2-1)^{\frac{1}{2}m}\,Q_{m+n}^{-m}(z) = \tfrac{1}{2}\int_{-1}^{1}\frac{\mu^{n+1}(1-\mu^2)^{\frac{1}{2}m}\,T_{m+n}^{-m}(\mu)}{z-\mu}\,d\mu + \frac{\Gamma(\frac{1}{2})\,.\,n!}{2^{m+n+1}\,\Gamma(m+n+\frac{3}{2})}, \qquad (22)$$

$$P_{m+n+1}^{-m}(z)\,Q_{m+n}^{-m}(z) = \tfrac{1}{2}\int_{-1}^{1}\frac{T_{m+n+1}^{-m}(\mu)\,T_{m+n}^{-m}(\mu)}{z-\mu}\,d\mu + \frac{n!}{\Gamma(2m+n+2)}, \qquad (23)$$

$$P_{m+n+1}^{-m}(z)\,Q_{m+n}^{-m}(z) - P_{m+n}^{-m}(z)\,Q_{m+n+1}^{-m}(z) = \frac{n!}{\Gamma(2m+n+2)}. \qquad (24)$$

* Cf. P. G. Gormley, *Journal of the London Math. Soc.*, IX. (1933), 149.

University of Glasgow

https://doi.org/10.1017/S204061850003286X Published online by Cambridge University Press