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ADJOINT-TRIANGLE THEOREMS FOR CONSERVATIVE FUNCTORS

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An *adjoint-triangle theorem* contemplates functors $P: C \to A$ and $T: A \to B$ where T and TP have left adjoints, and gives sufficient conditions for P also to have a left adjoint. We are concerned with the case where T is *conservative* - that is, isomorphism-reflecting; then P has a left adjoint under various combinations of completeness or cocompleteness conditions on C and A, with no explicit condition on P itself. We list systematically the strongest results we know of in this direction, augmenting those in the literature by some new ones.

Let the conservative functor $T: A \to B$ have a left adjoint S with counit $\varepsilon: ST \to 1$, and let $P: C \to A$ be such that $TP: C \to B$ has a left adjoint R. The original adjoint-triangle theorem, due independently to Dubuc [1] and to Huq [2], concerns the case where each ε_A is a coequalizer, and reads:

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THEOREM 1. *When each* ε_A *is a coequalizer,* P *has a left adjoint if* C *admits coequalizers.* □

To suppose ε_A to be a coequalizer, or even a regular epimorphism in the sense of [*4*], is strictly stronger than supposing T to be conservative; the right-adjoint functor Cat → Set sending a small category to its set of morphisms is conservative, but the corresponding ε_A is not a regular epimorphism for all A. By Proposition 4.3 of Im and Kelly [3], T is conservative when each ε_A is a strong epimorphism in the sense of [*4*], while the converse is true under various mild conditions on A.

Consider the following conditions on A:

(a) A has small limits;

(b) A has non-empty finite limits and arbitrary (even large) intersections of regular monomorphisms;

(c) A is weakly cowellpowered (that is, each object has but a small set of strongly-epimorphic quotients) and has coequalizers and all cointersections of strong epimorphisms;

(d) A has coequalizers and arbitrary cointersections of strong epimorphisms;

(e) A has pullbacks and pushouts;

and the following conditions on C:

(α) C is weakly cowellpowered and has small limits and arbitrary intersections of monomorphisms;

(β) C is weakly cowellpowered and has coequalizers and all cointersections of strong epimorphisms;

(γ) C has coequalizers and arbitrary cointersections of strong epimorphisms;

(δ) C has coequalizers and small cointersections of strong epimorphisms.

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Remark 2. (a) and (b) are completeness conditions on A , while (c) and (d) are cocompleteness conditions with (c) stronger than (d) , and (e) is a mixture of finite completeness and finite cocompleteness conditions. Again, (α) is (except for the weak cowellpowerdness) a completeness condition on C , while (β), (γ), and (δ) are cocompleteness conditions, each stronger than the next. The nasty conditions are those requiring cowellpoweredness or arbitrary cointersections of strong epimorphisms; since, while commonly satisfied by the usual categories of structures, these conditions do not pass automatically to reflective subcategories - it is not even known whether a *total* category (see [6]) satisfies such conditions. The new result (iii) of the following theorem goes beyond the results of [3] in not requiring such a condition of C , at the expense, however, of transferring it to A . The other advance on the results of [3] consists in requiring in (b) intersections, not of all monomorphisms, but only of regular ones.

THEOREM 3. *Given* $P: C \to A$ *and a conservative* $T: A \to B$ *where* T *and* TP *have left adjoints,* P *has a left adjoint under any of the following pairs of conditions on* A *and* C *:*

(i) A satisfies (b) or (d) or (e) and C satisfies (α) or (γ);

(ii) A satisfies (a) and C satisfies (α) or (β);

(iii) A satisfies (c) and C satisfies (δ).

Proof. (i) If A satisfies (b) or (d) or (e), each ε_A is a familially-strong epimorphism: this is in Theorem 4.5 of [3] for (d) and (e), but it is true for (b) too by Proposition 2.3 and 4.3 of [3]. Since (α) implies by Theorem 3.4 of [3] that every map in C factorizes as a strong epimorphism followed by a monomorphism, (i) now follows from parts (i) and (iii) of Theorem 5.4 of [3].

(ii) If A satisfies (a), each ε_A is a small-familiarly-strong epimorphism by Theorem 4.5 of [3], and (ii) now follows from parts (ii) and (iii) of Theorem 5.4 of [3].

(iii) Write A' for the full subcategory of A determined

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by the objects of the form SB for $B \in \mathcal{B}$. We need the representability for each $A \in \mathcal{A}$ of $\mathcal{A}(A,P-): \mathcal{C} \to \mathbf{Set}$, where $\mathbf{Set}$ is the category of sets in some universe containing all the hom-sets of $\mathcal{A}$; and we have it for $A \in \mathcal{A}'$, since $\mathcal{A}(SB,P-) \cong \mathcal{B}(B,TP-) \cong \mathcal{C}(RB,-)$. We therefore have it for every A , by Propositions 3.36 and 3.37 of [5], if $\mathcal{A}$ is the closure of $\mathcal{A}'$ under the class Φ of colimits consisting of the coequalizers and the small cointersections of strong epimorphisms. That this is indeed so follows from (the proof of) Proposition 3.40 of [5], since each ε_A is a strong epimorphism by Theorem 4.5 of [3]. □

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