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Abstract

Let $G \subseteq \widetilde{G}$ be two quasisplit connected reductive groups over a local field of characteristic zero and having the same derived group. Although the existence of L-packets is still conjectural in general, it is believed that the L-packets of G should be the restriction of those of $\widetilde{G}$. Motivated by this, we hope to construct the L-packets of $\widetilde{G}$ from those of G. The primary example in our mind is when $G = \mathrm{Sp}(2n)$, whose L-packets have been determined by Arthur [*The endoscopic classification of representations: orthogonal and symplectic groups*, Colloquium Publications, vol. 61 (American Mathematical Society, Providence, RI, 2013)], and $\widetilde{G} = \mathrm{GSp}(2n)$. As a first step, we need to consider some well-known conjectural properties of L-packets. In this paper, we show how they can be deduced from the conjectural endoscopy theory. As an application, we obtain some structural information about L-packets of $\widetilde{G}$ from those of G.

Contents

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1. Some standard notation

Suppose that F is a field; we denote its algebraic closure by $\bar{F}$. Let G be a reductive algebraic group over F and θ be an F-automorphism of G. We denote the identity component of G by G^0. If G is connected, we denote the derived group of G by G_{der} and the adjoint group of G by G_{ad}. Let G_{sc} be the simply connected cover of G_{der}. If $\widehat{G}$ is the complex reductive group dual to G, we write $\widehat{G}_{\mathrm{der}}$, $\widehat{G}_{\mathrm{ad}}$ for the derived group and adjoint group of $\widehat{G}$, respectively, and $\widehat{G}_{\mathrm{sc}}$ is the simply connected cover of $\widehat{G}_{\mathrm{der}}$. We denote the centre of G by Z_G or $Z(G)$. If G is abelian, let G^θ be the θ-invariant subgroup of G, and G_θ be the θ-coinvariant group of G, i.e., $G_\theta = G/(\theta-1)G$. For a finite group S, we denote its set of linear characters by S^*.

2. Introduction

Let F be a local field of characteristic zero and G be a quasisplit connected reductive group over F. The local Langlands conjecture asserts that the set $\Pi(G(F))$ of isomorphism classes of irreducible smooth representations of $G(F)$ can be parametrized by the set $\Phi(G)$ of local Langlands parameters. This parametrization is usually not a bijection. In fact, it is conjectured that each parameter $\phi \in \Phi(G)$ is associated with a finite set Π_ϕ of isomorphism classes of irreducible smooth representations of $G(F)$, and they give a disjoint decomposition of

$$\Pi(G(F)) = \bigsqcup_{\phi\in\Phi(G)} \Pi_\phi. \tag{2.1}$$

Such finite sets are called L-packets. This parametrization is based on the belief that there should be certain arithmetic invariants (e.g., L-factors) defined on both the representation side and the parameter side so that one could match them. From this point of view, one can think that the L-packet Π_ϕ attached to some $\phi \in \Phi(G)$ consists of all irreducible smooth representations of $G(F)$ whose arithmetic invariants match those of ϕ. However, it can be very difficult to define these arithmetic invariants on the representation side in general. On the other hand, there are some elementary properties that one would require this parametrization to always satisfy. These properties are usually given under the name 'desiderata' (see [Bor79, GGP12]). In this paper, we will mainly concern the following three desiderata.

- *Desideratum 1: central character*
  The first desideratum asserts that all irreducible smooth representations in Π_ϕ have the same central character; it can be constructed from ϕ. To see this construction, we need to give the definition of local Langlands parameters. Let $\Gamma = \mathrm{Gal}(\bar{F}/F)$ be the absolute Galois group, W_F be the Weil group and $\widehat{G}$ be the complex reductive group dual to G. The Langlands dual group is ${}^LG = \widehat{G} \rtimes W_F$, where the action of W_F factors through Γ. We define the local Langlands group to be

$$L_F := \begin{cases} W_F, & F \text{ is archimedean,} \\ W_F \times \mathrm{SL}(2,\mathbb{C}), & F \text{ is nonarchimedean.} \end{cases}$$

  Then a Langlands parameter ϕ is a $\widehat{G}$-conjugacy class of admissible homomorphisms from L_F to ${}^LG$ (see [Bor79]). In particular, it respects the projections on W_F from both L_F and ${}^LG$. We take a torus Z defined over F, containing the centre Z_G of G. For example, Z can be a maximal torus of G. Let $\widetilde{G} = (G \times Z)/Z_G$, where Z_G is included diagonally, and let $D = Z/Z_G$. Then we have an exact sequence

$$1 \longrightarrow G \longrightarrow \widetilde{G} \longrightarrow D \longrightarrow 1. \tag{2.2}$$

On the dual side, we have

$$1 \longrightarrow \widehat{D} \longrightarrow \widehat{\widetilde{G}} \longrightarrow \widehat{G} \longrightarrow 1.$$

This induces a map from $\Phi(\widetilde{G})$ to $\Phi(G)$. It follows from a result of Labesse [Lab85, Theorem 8.1] that this map is surjective. Therefore, we can lift any $\phi \in \Phi(G)$ to some $\tilde{\phi} \in \Phi(\widetilde{G})$. Note that $Z_{\widetilde{G}} = Z$ is a torus, so, dual to

$$1 \longrightarrow Z_{\widetilde{G}} \longrightarrow \widetilde{G} \longrightarrow G_{\mathrm{ad}} \longrightarrow 1,$$

we have

$$1 \longrightarrow \widehat{G}_{\mathrm{sc}} \longrightarrow \widehat{\widetilde{G}} \longrightarrow \widehat{Z}_{\widetilde{G}} \longrightarrow 1.$$

So, by composing with $\widehat{\widetilde{G}} \to \widehat{Z}_{\widetilde{G}}$, $\tilde{\phi}$ gives rise to an element $\mathbf{a}_{\tilde{\phi}} \in H^1(W_F, \widehat{Z}_{\widetilde{G}})$. Then, by the local Langlands correspondence for tori, $\mathbf{a}_{\tilde{\phi}}$ corresponds to a quasicharacter $\chi_{\tilde{\phi}}$ of $Z_{\widetilde{G}}(F)$. After we take restriction to $Z_G(F)$, we get a quasicharacter χ_ϕ of $Z_G(F)$. To see that this construction is independent of the torus Z, we need to know two things. First, if there is another torus $Z_1 \supseteq Z$, let $\widetilde{G}_1 = (G \times Z_1)/Z_G$ and $\tilde{\phi}_1 \in \Phi(\widetilde{G}_1)$ be a lift of $\tilde{\phi}$; then $\chi_{\tilde{\phi}_1}|_{Z_{\widetilde{G}}} = \chi_{\tilde{\phi}}$. Secondly, if there are two tori Z_1 and Z_2 both containing Z_G, then there exists a third torus Z_3 containing both Z_1 and Z_2. The first thing follows easily from some commutative diagrams. For the second one, we can simply take $Z_3 = (Z_1 \times Z_2)/Z_G$.

– *Desideratum 2: $G_{\mathrm{ad}}(F)$-conjugate action*

The second desideratum is more involved, and in particular it requires a different point of view towards L-packets. Roughly speaking, there are two steps in constructing the L-packets. The first one constructs the L-packets for the set $\Pi_{\mathrm{temp}}(G(F))$ of isomorphism classes of irreducible tempered representations, and then the other L-packets (nontempered) can be constructed from the tempered ones by using the theory of Langlands' quotient. Therefore, it suffices to know the tempered L-packets. The same is also true for the Langlands parametrization (2.1). That is to say, it is enough to know the parametrization of the tempered L-packets, which should correspond to the 'bounded parameters', namely the images of the Weil group part have compact closure. From the point of view of harmonic analysis, irreducible smooth representations are characterized by their 'characters', which are $G(F)$-conjugate invariant locally integrable functions over $G(F)$ and smooth over the open dense subset of strongly regular semisimple elements $G_{\mathrm{reg}}(F)$. A virtual character Θ (i.e., a finite linear combination of characters) is called *stable* if it is $G(\bar{F})$-conjugate invariant over $G_{\mathrm{reg}}(F)$, namely $\Theta(\gamma) = \Theta(\gamma')$ for any $\gamma, \gamma' \in G_{\mathrm{reg}}(F)$ such that $\gamma = g^{-1}\gamma' g$ for some $g \in G(\bar{F})$. It is conjectured that the tempered L-packets are the minimal subsets of irreducible tempered representations, within which some linear combination of the characters is stable (cf. [Sha90, Conjecture 9.2]). Therefore, the conjugate action by $G_{\mathrm{ad}}(F)$ on $\Pi(G(F))$ permutes the elements in each tempered L-packet. Moreover, there is an explicit conjectural formula for describing this action, which will be the second desideratum. To state the formula, we need to introduce a parametrization for elements inside tempered L-packets, which will be called endoscopic parametrization.

Let us denote the set of bounded Langlands parameters by $\Phi_{\mathrm{bdd}}(G)$. For $\phi \in \Phi_{\mathrm{bdd}}(G)$, we choose a representative $\underline{\phi} : L_F \to {}^L G$ and define

$$S_{\underline{\phi}} = \mathrm{Cent}(\mathrm{Im}\, \underline{\phi}, \widehat{G}),$$

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i.e., the centralizer of the image of $\underline{\phi}$ in $\widehat{G}$. Let $S^0_{\underline{\phi}}$ be the identity component of $S_{\underline{\phi}}$ and $Z(\widehat{G})^\Gamma$ be the Γ-invariant elements in the centre $Z(\widehat{G})$ of $\widehat{G}$. Then we also define $A_{\underline{\phi}} = S_{\underline{\phi}}/S^0_{\underline{\phi}}$ and $\mathcal{S}_{\underline{\phi}} = S_{\underline{\phi}}/S^0_{\underline{\phi}}Z(\widehat{G})^\Gamma$. There is an exact sequence

$$1 \longrightarrow Z_{\underline{\phi}} \longrightarrow A_{\underline{\phi}} \longrightarrow \mathcal{S}_{\underline{\phi}} \longrightarrow 1,$$

where $Z_{\underline{\phi}} = Z(\widehat{G})^\Gamma/Z(\widehat{G})^\Gamma \cap S^0_{\underline{\phi}}$. If $\underline{\phi}^g = \mathrm{Int}(g) \circ \underline{\phi}$ for $g \in \widehat{G}$, there is an isomorphism $S_{\underline{\phi}} \to S_{\underline{\phi}^g}$ unique up to $S_{\underline{\phi}}$-conjugation. This means that one cannot define a group 'S_ϕ' independent of the choice of representatives $\underline{\phi}$, but rather one can define the conjugacy classes in 'S_ϕ'.
We define a Whittaker datum to be a pair (B, Λ), where B is a Borel subgroup of G and Λ is a nondegenerate character on the unipotent radical $N(F)$ of $B(F)$. All Whittaker data can be constructed as follows. We fix an F-splitting $(B, T, \{X_\alpha\})$ of G and a nontrivial additive character $\psi_F : F \to \mathbb{C}^\times$; then we define

$$\Lambda\biggl(\exp\biggl(\sum_\alpha n_\alpha X_\alpha\biggr)\biggr) = \psi_F\biggl(\sum_\alpha n_\alpha\biggr),$$

which extends uniquely to a character of $N(F)$.

Conjecture 2.1. We fix a Whittaker datum (B, Λ) for G, and suppose that $\phi \in \Phi_{\mathrm{bdd}}(G)$.

(i) There is a unique (B, Λ)-generic representation in Π_ϕ.

(ii) There is a canonical pairing between Π_ϕ and $\mathcal{S}_{\underline{\phi}}$, which induces an inclusion from Π_ϕ to the set $\widehat{\mathcal{S}_{\underline{\phi}}}$ of characters of irreducible representations of $\mathcal{S}_{\underline{\phi}}$,

$$\Pi_\phi \longrightarrow \widehat{\mathcal{S}_{\underline{\phi}}},$$

$$\pi \longmapsto \langle \cdot, \pi \rangle_{\underline{\phi}},$$

such that it sends the (B, Λ)-generic representation to the trivial character. This becomes a bijection when F is nonarchimedean. Moreover, if $\underline{\phi}^g = \mathrm{Int}(g) \circ \underline{\phi}$ for $g \in \widehat{G}$, then

$$\langle gxg^{-1}, \pi \rangle_{\underline{\phi}^g} = \langle x, \pi \rangle_{\underline{\phi}}$$

for $\pi \in \Pi_\phi$ and $x \in \mathcal{S}_{\underline{\phi}}$.

Since $\widehat{\mathcal{S}_{\underline{\phi}}}$ are functions on conjugacy classes of $\mathcal{S}_{\underline{\phi}}$, the parametrization of elements inside Π_ϕ can be actually stated independently of the choice of representative $\underline{\phi}$ in the conjecture. Nevertheless, we would like to work with the group $\mathcal{S}_{\underline{\phi}}$ rather than its conjugacy classes, so throughout this paper we will always fix a representative $\underline{\phi}$. Let $\mathrm{Irr}(\mathcal{S}_{\underline{\phi}})$ be the set of isomorphism classes of irreducible representations of $\mathcal{S}_{\underline{\phi}}$. If $\rho \in \mathrm{Irr}(\mathcal{S}_{\underline{\phi}})$, we will denote the corresponding representation in Π_ϕ by $\pi(\rho)$. We call a parameter $\phi \in \overline{\Phi}_{\mathrm{bdd}}(G)$ *simple* if $\mathcal{S}_\phi = 1$. For simple parameters, it follows from this conjecture that their corresponding packets are singletons. Finally, we want to point out that part (i) of the conjecture is often referred to as the *generic packet conjecture*, and the pairing in part (ii) comes from the conjectural endoscopic character identity (see Conjecture 3.10), while its 'canonicity' depends on the choice of Whittaker datum.

Let $\mathcal{S}^*_{\underline{\phi}}$ be the group of linear characters of $\mathcal{S}_{\underline{\phi}}$. Then the explicit formula for describing the action of $G_{\mathrm{ad}}(F)$ on Π_ϕ can be stated in the following conjecture.

Conjecture 2.2. There exists a homomorphism

$$G_{\mathrm{ad}}(F) \longrightarrow \mathcal{S}^*_{\underline{\phi}},$$
$$g \longrightarrow \eta_g$$

such that

$$\langle \cdot, \pi^g \rangle_{\underline{\phi}} = \eta_g \langle \cdot, \pi \rangle_{\underline{\phi}}.$$

The statement of this conjecture was first given in Gan *et al.* [GGP12, § 9, item (3)], where they constructed the homomorphism $G_{\mathrm{ad}}(F) \to \mathcal{S}^*_{\underline{\phi}}$. There are three ingredients in that construction.

* (Tate local duality): there exists a perfect pairing

$$H^1(F, Z_G/Z_G^0) \times H^1(F, \pi_1(\widehat{G}_{\mathrm{der}})) \to \mathbb{C}^\times.$$

* There is a coboundary map $A_{\underline{\phi}} \to H^1(F, \pi_1(\widehat{G}_{\mathrm{der}}))$.
* There is a homomorphism $G_{\mathrm{ad}}(F) \to H^1(F, Z_G/Z_G^0)$.

Clearly, this gives a homomorphism $G_{\mathrm{ad}}(F) \to A^*_{\underline{\phi}}$, and in fact one will see that the image is in $\mathcal{S}^*_{\underline{\phi}}$ (see § 3.6).

– *Desideratum 3: twist by automorphism and quasicharacter*

Let θ be an F-automorphism of G preserving an F-splitting of G; then θ acts on $\Pi(G(F))$ by acting on $G(F)$. Let $\widehat{\theta}$ be the dual automorphism of θ on $\widehat{G}$; it gives a semidirect product $\widehat{G} \rtimes < \widehat{\theta} >$. Then θ also acts on $\Phi(G)$ through the action of $\widehat{\theta}$ on $\widehat{G}$. Let $\mathbf{a}$ be an element in $H^1(W_F, Z(\widehat{G}))$, and $\mathbf{a}$ act on $\Phi(G)$ by twisting on $Z(\widehat{G})$. One can associate a quasicharacter ω of $G(F)$ with $\mathbf{a}$ (see (3.1)). This desideratum asserts: for $\phi \in \Phi_{\mathrm{bdd}}(G)$,

$$\Pi_{\phi^\theta} = \Pi_\phi^\theta \quad \text{and} \quad \Pi_{\phi \otimes \mathbf{a}} = \Pi_\phi \otimes \omega.$$

In fact, one can refine this desideratum by making more precise the action of θ and ω on the elements in Π_ϕ. Namely, if $\pi \in \Pi_\phi$, then

$$\langle x, \pi^\theta \rangle_{\underline{\phi}^\theta} = \langle \widehat{\theta}^{-1} x \widehat{\theta}, \pi \rangle_{\underline{\phi}} \tag{2.3}$$

for $x \in \mathcal{S}_{\underline{\phi}^\theta}$, and

$$\langle x, \pi \otimes \omega \rangle_{\underline{\phi} \otimes \underline{\mathbf{a}}} = \langle x, \pi \rangle_{\underline{\phi}} \tag{2.4}$$

for $x \in \mathcal{S}_{\underline{\phi}} = \mathcal{S}_{\underline{\phi} \otimes \underline{\mathbf{a}}}$, where $\underline{\mathbf{a}}$ is a 1-cocycle of W_F in $Z(\widehat{G})$ representing $\mathbf{a}$.

The refined desideratum has the following consequence. For $\phi \in \Phi_{\mathrm{bdd}}(G)$, suppose that $\phi^\theta = \phi \otimes \mathbf{a}$, i.e., there exists $g \in \widehat{G}$ such that $(\underline{\phi}^\theta)^g = \underline{\phi} \otimes \underline{\mathbf{a}}$; then, by (2.3) and (2.4), we have for $x \in \mathcal{S}_{\underline{\phi}^\theta}$

$$\begin{aligned} \langle \widehat{\theta}^{-1} x \widehat{\theta}, \pi \rangle_{\underline{\phi}} &= \langle x, \pi^\theta \rangle_{\underline{\phi}^\theta} = \langle g x g^{-1}, \pi^\theta \rangle_{(\underline{\phi}^\theta)^g} \\ &= \langle g x g^{-1}, \pi^\theta \rangle_{\underline{\phi} \otimes \underline{\mathbf{a}}} = \langle g x g^{-1}, \pi^\theta \otimes \omega^{-1} \rangle_{\underline{\phi}}. \end{aligned}$$

By setting $s = g \rtimes \widehat{\theta}$, we have shown the following statement.

Conjecture 2.3. Suppose that $\phi \in \Phi_{\mathrm{bdd}}(G)$ and $\phi^\theta = \phi \otimes \mathbf{a}$. Let $s \in \widehat{G} \rtimes \widehat{\theta}$ satisfy $\underline{\phi}^s = \underline{\phi} \otimes \underline{\mathbf{a}}$; then

$$\langle s x s^{-1}, \pi^\theta \otimes \omega^{-1} \rangle_{\underline{\phi}} = \langle x, \pi \rangle_{\underline{\phi}}$$

for any $\pi \in \Pi_\phi$ and $x \in \mathcal{S}_{\underline{\phi}}$. In other words,

$$\pi(\rho^s)^\theta \cong \pi(\rho) \otimes \omega$$

for any $\rho \in \mathrm{Irr}(\mathcal{S}_{\underline{\phi}})$.

The first goal of this paper is to suggest a strategy towards proving the above three desiderata about L-packets. To do so, we need to assume (2.1), Conjecture 2.1 (together with its generalized form: Conjecture 2.5) and also the (twisted) endoscopic character identities (see Conjecture 3.10), which will be described in § 3. Since these conjectures can be viewed as part of the conjectural endoscopy theory, we would like to call the collection of these assumptions the *endoscopic hypothesis*. For the first desideratum, we will prove the following result under this hypothesis.

Proposition 2.4. *The desideratum about central characters of L-packets holds as long as it holds for simple parameters.*

For the second desideratum, i.e., Conjecture 2.2, we will prove a stronger result. The setup that we are going to work in is as follows. Let $G \subseteq \widetilde{G}$ be two quasisplit connected reductive groups over F such that $G_{\mathrm{der}} = \widetilde{G}_{\mathrm{der}}$. Then $\widetilde{G}/G$ is a torus, and we denote it by D. There is an exact sequence

$$1 \longrightarrow G \longrightarrow \widetilde{G} \xrightarrow{\lambda} D \longrightarrow 1. \tag{2.5}$$

Let Σ be a finite abelian group of F-automorphisms of $\widetilde{G}$ preserving a fixed F-splitting of $\widetilde{G}$; we assume that λ is Σ-*invariant*. This implies that Σ also acts on G. Let $\widetilde{G}^\Sigma = \widetilde{G} \rtimes \Sigma$ and $G^\Sigma = G \rtimes \Sigma$. Since Σ induces dual automorphisms on $\widehat{\widetilde{G}}$ and $\widehat{G}$, we denote them by $\widehat{\Sigma}$ and define $\widehat{\widetilde{G}}^\Sigma = \widehat{\widetilde{G}} \rtimes \widehat{\Sigma}$ and $\widehat{G}^\Sigma = \widehat{G} \rtimes \widehat{\Sigma}$.

Before we can state our result, we need to extend Conjecture 2.1 to the nonconnected group G^Σ. Suppose that $\phi \in \Phi_{\mathrm{bdd}}(G)$; we define $S^\Sigma_{\underline{\phi}}$, $A^\Sigma_{\underline{\phi}}$ and $\mathcal{S}^\Sigma_{\underline{\phi}}$ as before simply by taking $\widehat{G}^\Sigma$ in place of $\widehat{G}$; they are all equipped with a natural map to $\widehat{\Sigma}$. Let $S^\theta_{\underline{\phi}}$, $A^\theta_{\underline{\phi}}$ and $\mathcal{S}^\theta_{\underline{\phi}}$ be the preimages of $\widehat{\theta} \in \widehat{\Sigma}$ in $S^\Sigma_{\underline{\phi}}$, $A^\Sigma_{\underline{\phi}}$ and $\mathcal{S}^\Sigma_{\underline{\phi}}$, respectively. Note that these are not $\widehat{\theta}$-invariant elements in $S_{\underline{\phi}}$, $A_{\underline{\phi}}$ and $\mathcal{S}_{\underline{\phi}}$. Since the image in $\widehat{\Sigma}$ is the same for $S^\Sigma_{\underline{\phi}}$, $A^\Sigma_{\underline{\phi}}$ and $\mathcal{S}^\Sigma_{\underline{\phi}}$, we denote it by $\widehat{\Sigma}_{\underline{\phi}}$. Let Π^Σ_ϕ be the set of all irreducible smooth representations of $G^\Sigma(F)$, whose restriction to $G(F)$ has intersections with Π_ϕ.

A Whittaker datum (B, Λ) is called Σ-stable if Σ preserves B and Λ is Σ-invariant. In particular, if we fix a Σ-stable F-splitting of G (i.e., Σ preserves B and $\{X_\alpha\}$) and a nontrivial additive character ψ_F of F, then the associated Whittaker datum is Σ-stable. We call a representation $\pi^\Sigma \in \Pi^\Sigma_\phi$ (B, Λ)-generic if $\pi^\Sigma|_G$ is (B, Λ)-generic and the corresponding Whittaker functional is invariant under $\pi^\Sigma(\theta)$ for all $\theta \in \Sigma$.

Conjecture 2.5. We fix a Σ-stable Whittaker datum (B, Λ) for G, and suppose that $\phi \in \Phi_{\mathrm{bdd}}(G)$.

(i) There is a unique (B, Λ)-generic representation in Π^Σ_ϕ.

(ii) There is a canonical pairing between Π_{ϕ}^{Σ} and $\mathcal{S}_{\underline{\phi}}^{\Sigma}$, which induces an inclusion from Π_{ϕ}^{Σ} to the characters $\widehat{\mathcal{S}_{\underline{\phi}}}^{\Sigma}$ of irreducible representations of $\mathcal{S}_{\underline{\phi}}^{\Sigma}$,

$$\Pi_{\phi}^{\Sigma} \longrightarrow \widehat{\mathcal{S}_{\underline{\phi}}}^{\Sigma},$$

$$\pi^{\Sigma} \longmapsto \langle \cdot, \pi^{\Sigma} \rangle_{\underline{\phi}},$$

such that it sends the (B, Λ)-generic representation to the trivial character. This becomes a bijection when F is nonarchimedean. Moreover, if Σ' is a subgroup of Σ, then we have the following relation:

$$\langle \cdot, \pi^{\Sigma} \rangle_{\underline{\phi}}|_{\mathcal{S}_{\underline{\phi}}^{\Sigma'}} = \sum_{\pi^{\Sigma'} \in \Pi_{\phi}^{\Sigma'}} m(\pi^{\Sigma}, \pi^{\Sigma'}) \langle \cdot, \pi^{\Sigma'} \rangle_{\underline{\phi}}, \tag{2.6}$$

where $m(\pi^{\Sigma}, \pi^{\Sigma'})$ is the multiplicity of $\pi^{\Sigma'}$ in $\pi^{\Sigma}|_{G^{\Sigma'}}$.

Under the endoscopic hypothesis, we are able to prove the following result.

Theorem 2.6. *There exists a homomorphism*

$$\widetilde{G}(F) \longrightarrow (\mathcal{S}_{\underline{\phi}}^{\Sigma})^{*},$$

$$g \longrightarrow \varepsilon_{g}$$

such that

$$\langle \cdot, (\pi^{\Sigma})^{g} \rangle_{\underline{\phi}} = \varepsilon_{g} \langle \cdot, \pi^{\Sigma} \rangle_{\underline{\phi}}.$$

For the third desideratum, we will prove the following result under the endoscopic hypothesis.

Proposition 2.7. *The refined desideratum about L-packets under twist by automorphism and quasicharacter holds if it holds for simple parameters.*

Returning to the setup in (2.5), there is a conjectural relation between the L-packets for G and $\widetilde{G}$. That is to say, if $\tilde{\phi} \in \Phi(\widetilde{G})$ maps to $\phi \in \Phi(G)$, then the L-packet Π_{ϕ} should be the restriction of $\Pi_{\tilde{\phi}}$. The restriction multi-map $\Pi(\widetilde{G}(F)) \to \Pi(G(F))$ is surjective, in the sense that for any $\pi \in \Pi(G(F))$, there exists $\tilde{\pi} \in \Pi(\widetilde{G}(F))$, whose restriction to $G(F)$ contains π (see Corollary 6.3). Therefore, it is easy to construct the L-packets of G from those of $\widetilde{G}$. The other direction is more subtle, because for any $\pi \in \Pi(G(F))$, the preimage $\tilde{\pi} \in \Pi(\widetilde{G}(F))$ is usually not unique and they differ from each other by a twist of quasicharacters of $\widetilde{G}(F)$. So, our second goal in this paper is to make an attempt to address this problem in most generality. To be more precise, we want to establish the endoscopic hypothesis (i.e., (2.1) and Conjectures 2.5 and 3.10) for $\widetilde{G}$ by assuming it for G and the twisted endoscopic groups of G. When G is a quasisplit symplectic group or special even orthogonal group, and $\widetilde{G}$ is the corresponding similitude group, this has been essentially achieved in [Xu15].

Throughout this paper, except for § 6, we will take the endoscopic hypothesis as our working assumption. In § 3, we will describe the conjectural endoscopy theory. In particular, we will introduce Conjecture 3.10, which is part of the endoscopic hypothesis. We will prove Theorem 2.6, and deduce Conjecture 2.2 as a special case. In § 4, we will prove Proposition 2.4. In § 5, we will prove Proposition 2.7, and this implies Conjecture 2.3 for nonsimple parameters.

In § 6, we consider the problem of lifting L-packets from G to $\widetilde{G}$, where G and $\widetilde{G}$ are in the setup of (2.5). So, we will only assume the endoscopic hypothesis for G and its twisted endoscopic groups; in particular, we cannot assume Conjecture 2.3 for $\widetilde{G}$. In § 6.1, we will study the restriction multi-map $\Pi(\widetilde{G}(F)) \to \Pi(G(F))$. In §§ 6.2 and 6.3, we will discuss some special cases of Conjecture 2.3 for $\widetilde{G}$, and from there we will obtain some structural information about the L-packets of $\widetilde{G}$. In § 6.4, we will formulate a conjecture about the L-packets of $\widetilde{G}$ (see Conjecture 6.18). Finally, in § 6.5, we will take G to be a symplectic group or special even orthogonal group, and we will review various results of Arthur in [Art13], which essentially prove the endoscopic hypothesis for G. We will also take $\widetilde{G}$ to be the corresponding similitude group, and apply the previous discussion in § 6 to this case. So, the results we obtain in §§ 6.2 and 6.3 will become unconditional in this case. Moreover, we will restate Conjecture 6.18 as a theorem in this case; the proof of this theorem is included in [Xu15].

3. Endoscopy theory

3.1 Twisted endoscopic datum

Let F be a local field of characteristic zero and G be a quasisplit reductive group over F. We have an isomorphism

$$H^1(W_F, Z(\widehat{G})) \longrightarrow \mathrm{Hom}(G(F), \mathbb{C}^\times) \tag{3.1}$$

defined by Langlands (see Appendix A). Let θ be an automorphism of G and ω be a quasicharacter of $G(F)$. A twisted endoscopic datum for (G, θ, ω) is a quadruple $(H, \mathcal{H}, s, \xi)$, where H is a quasisplit reductive group over F and $\mathcal{H}$ is a split extension of W_F by $\widehat{H}$,

$$1 \longrightarrow \widehat{H} \longrightarrow \mathcal{H} \longrightarrow W_F \longrightarrow 1,$$

such that the conjugate action of W_F on $\widehat{H}$ falls into the same outer classes of automorphisms as for ${}^L H$. Note that $\mathcal{H}$ may not be isomorphic to ${}^L H$. Inside the quadruple, s is a semisimple element in $\widehat{G} \rtimes \widehat{\theta}$, ξ is an L-embedding of $\mathcal{H}$ to ${}^L G$ (i.e., it respects the projections on W_F from both $\mathcal{H}$ and ${}^L G$) and they satisfy the following conditions:

- $\mathrm{Int}(s) \circ \xi = \underline{\mathbf{a}} \cdot \xi$ for a 1-cocycle $\underline{\mathbf{a}}$ of W_F in $Z(\widehat{G})$ mapped to ω by (3.1);
- $\widehat{H} \cong \mathrm{Cent}(s, \widehat{G})^0$ through ξ.

We call H a twisted endoscopic group of G. Two twisted endoscopic data $(H, \mathcal{H}, s, \xi)$ and $(H', \mathcal{H}', s', \xi')$ are called isomorphic if there exists an element $g \in \widehat{G}$ such that $g\xi(\mathcal{H})g^{-1} = \xi'(\mathcal{H}')$ and $gsg^{-1} \in s'Z(\widehat{G})$. We denote by $\mathcal{E}(G^\theta, \omega)$ the set of isomorphism classes of twisted endoscopic data for (G, θ, ω). For abbreviation, we will use the twisted endoscopic group to denote the twisted endoscopic datum if there is no confusion.

Let $G \subseteq \widetilde{G}$ be two quasisplit connected reductive groups over F such that $G_{\mathrm{der}} = \widetilde{G}_{\mathrm{der}}$; we denote $\widetilde{G}/G$ by D. We have

$$1 \longrightarrow G \longrightarrow \widetilde{G} \xrightarrow{\lambda} D \longrightarrow 1.$$

We assume that θ is an automorphism of $\widetilde{G}$ and λ is θ-invariant. Then we have the following proposition relating the twisted endoscopic data between G and $\widetilde{G}$.

Proposition 3.1. *There is a one to one correspondence between $\mathcal{E}(G^\theta, \omega_G)$ and*

$$\bigsqcup_{\omega_{\widetilde{G}}|_G = \omega_G} \mathcal{E}(\widetilde{G}^\theta, \omega_{\widetilde{G}}).$$

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Proof. Suppose that $[(H,\mathcal{H},s,\xi)]\in\mathcal{E}(G^\theta,\omega_G)$; then $\xi(\widehat{H})=\mathrm{Cent}(s,\widehat{G})^0$. Under the projection ${}^L\widetilde{G}\longrightarrow {}^LG$, the preimage of $\mathrm{Cent}(s,\widehat{G})$ is $\{g\in\widehat{\widetilde{G}}:\widetilde{s}g\widetilde{s}^{-1}g^{-1}\in\widehat{D}\}$, where $\widetilde{s}$ is a preimage of s in $\widehat{\widetilde{G}}\rtimes\widehat{\theta}$. We claim that

$$\{g\in\widehat{\widetilde{G}}:\widetilde{s}g\widetilde{s}^{-1}g^{-1}\in\widehat{D}\}^0=\{g\in\widehat{\widetilde{G}}:\widetilde{s}g\widetilde{s}^{-1}g^{-1}=1\}^0.$$

To see this, we can consider the homomorphism defined by

$$\begin{aligned}\{g\in\widehat{\widetilde{G}}:\widetilde{s}g\widetilde{s}^{-1}g^{-1}\in\widehat{D}\}&\longrightarrow\widehat{D}\subseteq{}^L\widetilde{G}\,,\\ g&\longmapsto\widetilde{s}g\widetilde{s}^{-1}g^{-1}.\end{aligned}\tag{3.2}$$

Its composition with $A:{}^L\widetilde{G}\longrightarrow{}^L((Z^\theta_{\widetilde{G}})^0)$ is trivial. Note that A induces an isogeny

$$(Z(\widehat{\widetilde{G}})^{\widehat{\theta}})^0\to\widehat{(Z^\theta_{\widetilde{G}})^0}.$$

Since λ is θ-invariant, $\widehat{D}$ included as a subgroup of $\widehat{\widetilde{G}}$ is fixed by $\widehat{\theta}$. Therefore, $\widehat{D}\subseteq(Z(\widehat{\widetilde{G}})^{\widehat{\theta}})^0$, and we get that $\mathrm{Ker}A|_{\widehat{D}}$ is finite. This means that the homomorphism (3.2) must have finite image, so our claim becomes obvious. Since $\widehat{D}\subseteq\mathrm{Cent}(\widetilde{s},\widehat{\widetilde{G}})^0$, we can now conclude that $\mathrm{Cent}(\widetilde{s},\widehat{\widetilde{G}})^0$ is the preimage of $\xi(\widehat{H})$. Let us denote $\mathrm{Cent}(\widetilde{s},\widehat{\widetilde{G}})^0$ by $\widehat{\widetilde{H}}$.

When $\theta=\mathrm{id}$ and $s\in Z(\widehat{G})$, we have $\widehat{G}=\mathrm{Cent}(s,\widehat{G})$ and hence $\widehat{\widetilde{G}}=\{g\in\widehat{\widetilde{G}}:\widetilde{s}g\widetilde{s}^{-1}g^{-1}\in\widehat{D}\}$. Since $\widehat{\widetilde{G}}$ is connected, it follows from the above argument that $\widehat{\widetilde{G}}=\{g\in\widehat{\widetilde{G}}:\widetilde{s}g\widetilde{s}^{-1}g^{-1}=1\}^0$. Hence, $\widehat{\widetilde{G}}=\{g\in\widehat{\widetilde{G}}:\widetilde{s}g\widetilde{s}^{-1}g^{-1}=1\}$, which means that $\widetilde{s}\in Z(\widehat{\widetilde{G}})$. This shows that the preimage of $Z(\widehat{G})$ is $Z(\widehat{\widetilde{G}})$, i.e., there is an exact sequence

$$1\longrightarrow\widehat{D}\longrightarrow Z(\widehat{\widetilde{G}})\longrightarrow Z(\widehat{G})\longrightarrow 1.$$

Returning to the general situation, we can choose a splitting $c:W_F\to\mathcal{H}$, so that the composition $\underline{\phi}:\ W_F\xrightarrow{c}\mathcal{H}\xrightarrow{\xi}{}^LG$ is admissible. Then we can lift $\underline{\phi}$ to $\underline{\tilde{\phi}}:W_F\to{}^L\widetilde{G}$, which induces a Galois action on $\widehat{\widetilde{H}}$ and hence determines a quasisplit reductive group $\widetilde{H}$. We define $\widetilde{\mathcal{H}}$ to be the product $\widehat{\widetilde{H}}\cdot\mathrm{Im}\,\underline{\tilde{\phi}}$. Note that $\underline{\tilde{\phi}}$ gives a splitting of

$$1\longrightarrow\widehat{\widetilde{H}}\longrightarrow\widetilde{\mathcal{H}}\longrightarrow W_F\longrightarrow 1,$$

and we have a natural embedding $\widetilde{\xi}:\widetilde{\mathcal{H}}\to{}^L\widetilde{G}$, which is the identity on $\widehat{\widetilde{H}}$. The map

$$w\mapsto\widetilde{s}\widetilde{\xi}(\underline{\tilde{\phi}}(w))\widetilde{s}^{-1}\widetilde{\xi}(\underline{\tilde{\phi}}(w))^{-1},\quad w\in W_F$$

defines an element $\mathbf{a}\in H^1(W_F,Z(\widehat{\widetilde{G}}))$. If $\mathbf{a}$ is associated with a quasicharacter $\omega_{\widetilde{G}}$ of $\widetilde{G}(F)$, then $[(\widetilde{H},\widetilde{\mathcal{H}},\widetilde{s},\widetilde{\xi})]\in\mathcal{E}(\widetilde{G}^\theta,\omega_{\widetilde{G}})$. It is not hard to show that if we change $(H,\mathcal{H},s,\xi)$ within its isomorphism class or the splitting c or the lifting $\underline{\tilde{\phi}}$, this lifted endoscopic datum $(\widetilde{H},\widetilde{\mathcal{H}},\widetilde{s},\widetilde{\xi})$ is uniquely determined up to isomorphism. Here we need to use the fact that the preimage of $Z(\widehat{G})$ is $Z(\widehat{\widetilde{G}})$.

Finally, we have a commutative diagram

$$\begin{array}{ccc} H^1(W_F, Z(\widehat{\widetilde{G}})) & \longrightarrow & \mathrm{Hom}(\widetilde{G}(F), \mathbb{C}^\times) \\ \downarrow & & \downarrow \\ H^1(W_F, Z(\widehat{G})) & \longrightarrow & \mathrm{Hom}(G(F), \mathbb{C}^\times) \end{array}$$

which shows that $\omega_{\widetilde{G}}|_G = \omega_G$. So, we get a well-defined map from $\mathcal{E}(G^\theta, \omega_G)$ to

$$\bigsqcup_{\omega_{\widetilde{G}}|_G = \omega_G} \mathcal{E}(\widetilde{G}^\theta, \omega_{\widetilde{G}}).$$

The other direction is more straightforward, namely one can simply take the quotient by $\widehat{D}$ on the dual side. □

Remark 3.2. Following the proof, there is an exact sequence

$$1 \longrightarrow \widehat{D} \longrightarrow \widehat{\widetilde{H}} \longrightarrow \widehat{H} \longrightarrow 1\ ,$$

whose dual is

$$1 \longrightarrow H \longrightarrow \widetilde{H} \xrightarrow{\lambda_H} D \longrightarrow 1\ .$$

This suggests that the twisted endoscopic groups $\widetilde{H}$ and H also have the same derived group.

3.2 Relation with Langlands parameter

We follow the setup in the introduction. Suppose that $\phi \in \Phi(G)$ and $\tilde{\phi} \in \Phi(\widetilde{G})$ is a lift of ϕ. Let L_F act on $\widehat{D}$, $\widehat{\widetilde{G}}^\Sigma$ and $\widehat{G}^\Sigma$ by conjugation through $\underline{\phi}$. We denote the corresponding group cohomology by $H^*_{\underline{\phi}}(L_F, \cdot)$. Note that $H^0_{\underline{\phi}}(L_F, \widehat{D}) = \widehat{D}^\Gamma, H^0_{\underline{\phi}}(L_F, \widehat{G}^\Sigma) = S^\Sigma_{\underline{\phi}}$, $H^0_{\underline{\phi}}(L_F, \widehat{\widetilde{G}}^\Sigma) = S^\Sigma_{\underline{\tilde{\phi}}}$ and $H^1_{\underline{\phi}}(L_F, \widehat{D}) = H^1(W_F, \widehat{D})$. The short exact sequence

$$1 \longrightarrow \widehat{D} \longrightarrow \widehat{\widetilde{G}}^\Sigma \longrightarrow \widehat{G}^\Sigma \longrightarrow 1$$

induces a long exact sequence

$$1 \longrightarrow \widehat{D}^\Gamma \longrightarrow S^\Sigma_{\underline{\tilde{\phi}}} \longrightarrow S^\Sigma_{\underline{\phi}} \xrightarrow{\delta} H^1(W_F, \widehat{D})$$

and hence

$$1 \longrightarrow S^\Sigma_{\underline{\tilde{\phi}}}/\widehat{D}^\Gamma \xrightarrow{\iota} S^\Sigma_{\underline{\phi}} \xrightarrow{\delta} H^1(W_F, \widehat{D}). \tag{3.3}$$

To describe δ, we can write

$$S^\Sigma_{\underline{\phi}} = \{\tilde{s} \in \widehat{\widetilde{G}}^\Sigma : \tilde{s}\underline{\tilde{\phi}}(u)\tilde{s}^{-1}\underline{\tilde{\phi}}(u)^{-1} \in \widehat{D}, \text{ for all } u \in L_F\}/\widehat{D}.$$

Then $\delta(s) : u \longmapsto \tilde{s}\underline{\tilde{\phi}}(u)\tilde{s}^{-1}\underline{\tilde{\phi}}(u)^{-1}$, where $\tilde{s}$ is a preimage of s in $\widehat{\widetilde{G}}^\Sigma$, and $\delta(s)$ factors through W_F. We have the following fact about δ.

Lemma 3.3. *The image of δ consists of $\mathbf{a} \in H^1(W_F, \widehat{D})$ such that*

$$\tilde{\phi}^{\theta} = \tilde{\phi} \otimes \mathbf{a}$$

for some $\theta \in \Sigma$, and in particular it is finite.

Proof. By the definition of δ, we have $\widetilde{s}\underline{\tilde{\phi}}(u)\widetilde{s}^{-1} = \delta(s)(u) \cdot \underline{\tilde{\phi}}(u)$, where $\widetilde{s} \in \widehat{\widetilde{G}}^{\Sigma}$ is a preimage of s. Denote by $\widehat{\theta}_s$ the image of s in $\widehat{\Sigma}$. Then this means that $\tilde{\phi}^{\theta_s} = \tilde{\phi} \otimes \delta(s)$. Conversely, if $\tilde{\phi}^{\theta} = \tilde{\phi} \otimes \mathbf{a}$ for some $\mathbf{a} \in H^1(W_F, \widehat{D})$ and $\theta \in \Sigma$, then there exists $g \in \widehat{\widetilde{G}}$ such that

$$(g \rtimes \widehat{\theta})\underline{\tilde{\phi}}(u)(g \rtimes \widehat{\theta})^{-1} = \underline{\mathbf{a}}(u)\underline{\tilde{\phi}}(u)$$

for a 1-cocycle $\underline{\mathbf{a}}$ representing $\mathbf{a}$. Then it is clear that $\widetilde{s} := g \rtimes \widehat{\theta} \in \widehat{\widetilde{G}}^{\Sigma}$ maps to an element $s \in S^{\Sigma}_{\underline{\phi}}$ and $\mathbf{a} = \delta(s)$.

To see that the image of δ is finite, we consider $\delta(s)$ and let $\theta = \theta_s$. The restriction of

$$A : {}^L\widetilde{G} \longrightarrow {}^L((Z^{\theta}_{\widetilde{G}})^0)$$

to $\widehat{D}$ induces a homomorphism

$$B : H^1(W_F, \widehat{D}) \to H^1(W_F, \widehat{(Z^{\theta}_{\widetilde{G}})^0}).$$

We claim that $\delta(s)$ lies in the kernel of B. To show the claim, recall that

$$\delta(s) : u \longmapsto \widetilde{s}\underline{\tilde{\phi}}(u)\widetilde{s}^{-1}\underline{\tilde{\phi}}(u)^{-1}.$$

We can write $\widetilde{s} = g \rtimes \widehat{\theta}$ and $\underline{\tilde{\phi}}(u) = h \rtimes w_u$, where $g, h \in \widehat{\widetilde{G}}$ and $w_u \in W_F$. Then

$$\widetilde{s}\underline{\tilde{\phi}}(u)\widetilde{s}^{-1}\underline{\tilde{\phi}}(u)^{-1} := g\widehat{\theta}(h) \cdot w_u(g^{-1})h^{-1}.$$

Since

$$A(\widehat{\theta}(h)) = A(h),$$

we have

$$A(\widetilde{s}\underline{\tilde{\phi}}(u)\widetilde{s}^{-1}\underline{\tilde{\phi}}(u)^{-1}) = A(g)A(h) \cdot w_u(A(g^{-1}))A(h^{-1}) = A(g) \cdot w_u(A(g)^{-1}).$$

This proves the claim. Now the exact sequence

$$1 \longrightarrow \mathrm{Ker}\, A|_{\widehat{D}} \longrightarrow \widehat{D} \xrightarrow{A|_{\widehat{D}}} \widehat{(Z^{\theta}_{\widetilde{G}})^0}$$

induces the following exact sequence:

$$1 \longrightarrow H^1(W_F, \mathrm{Ker}\, A|_{\widehat{D}}) \longrightarrow H^1(W_F, \widehat{D}) \xrightarrow{B} H^1(W_F, \widehat{(Z^{\theta}_{\widetilde{G}})^0}) .$$

Since F is a local field and $\mathrm{Ker}\, A|_{\widehat{D}}$ is finite, it is not hard to see that $H^1(W_F, \mathrm{Ker}\, A|_{\widehat{D}})$ is finite. Then it follows from the exact sequence that the kernel of B is also finite and hence $\mathrm{Im}\, \delta$ is finite. □

We would like to modify (3.3) to have $\mathcal{S}^{\Sigma}_{\underline{\phi}}$ and $\mathcal{S}^{\Sigma}_{\underline{\tilde{\phi}}}$ in the sequence. To do so, we need to know the kernel and image of δ restricted on $Z(\widehat{G})^{\Gamma}$. Therefore, we take

$$1 \longrightarrow \widehat{D} \longrightarrow Z(\widehat{\widetilde{G}}) \longrightarrow Z(\widehat{G}) \longrightarrow 1,$$

which induces an exact sequence

$$1 \longrightarrow \widehat{D}^{\Gamma} \longrightarrow Z(\widehat{\widetilde{G}})^{\Gamma} \longrightarrow Z(\widehat{G})^{\Gamma} \xrightarrow{\ \delta\ } H^1(W_F, \widehat{D}) \longrightarrow H^1(W_F, Z(\widehat{\widetilde{G}})).$$

So, $\operatorname{Ker}\delta|_{Z(\widehat{G})^{\Gamma}} = Z(\widehat{\widetilde{G}})^{\Gamma}/\widehat{D}^{\Gamma}$. Let $\bar{H}^1(W_F, \widehat{D}) := H^1(W_F, \widehat{D})/\delta(Z(\widehat{G})^{\Gamma})$; we define $\bar{S}^{\Sigma}_{\underline{\phi}} = S^{\Sigma}_{\underline{\phi}}/Z(\widehat{G})^{\Gamma}$ and $\bar{S}^{\Sigma}_{\underline{\tilde{\phi}}} = S^{\Sigma}_{\underline{\tilde{\phi}}}/Z(\widehat{\widetilde{G}})^{\Gamma}$. By taking the quotient of (3.3) by $Z(\widehat{G})^{\Gamma}$, we get

$$1 \longrightarrow \bar{S}^{\Sigma}_{\underline{\tilde{\phi}}} \xrightarrow{\ \iota\ } \bar{S}^{\Sigma}_{\underline{\phi}} \xrightarrow{\ \bar{\delta}\ } \bar{H}^1(W_F, \widehat{D})\ . \tag{3.4}$$

Since $\operatorname{Im}\delta$ is finite, we have $(\bar{S}^{\Sigma}_{\underline{\tilde{\phi}}})^0 = (\bar{S}^{\Sigma}_{\underline{\phi}})^0$. After taking the quotient of (3.4) by the identity component, we get

$$1 \longrightarrow \mathcal{S}^{\Sigma}_{\underline{\tilde{\phi}}} \xrightarrow{\ \iota\ } \mathcal{S}^{\Sigma}_{\underline{\phi}} \xrightarrow{\ \bar{\delta}\ } \bar{H}^1(W_F, \widehat{D})\ . \tag{3.5}$$

The local Langlands correspondence for tori gives us an isomorphism

$$H^1(W_F, \widehat{D}) \cong \operatorname{Hom}(D(F), \mathbb{C}^{\times}).$$

By pulling back quasicharacters of $D(F)$ to $\widetilde{G}(F)$, we get a homomorphism

$$H^1(W_F, \widehat{D}) \to \operatorname{Hom}(\widetilde{G}(F)/G(F), \mathbb{C}^{\times}),$$

which is surjective. Note that $\delta(Z(\widehat{G})^{\Gamma})$ is trivial in $H^1(W_F, Z(\widehat{\widetilde{G}}))$, so it induces the trivial character on $\widetilde{G}(F)$. Since (3.1) is an isomorphism, we then have an isomorphism

$$r : \bar{H}^1(W_F, \widehat{D}) \to \operatorname{Hom}(\widetilde{G}(F)/G(F), \mathbb{C}^{\times}).$$

We denote the composition $r \circ \bar{\delta}$ by α. Therefore, we have the following exact sequence:

$$1 \longrightarrow \mathcal{S}^{\Sigma}_{\underline{\tilde{\phi}}} \xrightarrow{\ \iota\ } \mathcal{S}^{\Sigma}_{\underline{\phi}} \xrightarrow{\ \alpha\ } \operatorname{Hom}(\widetilde{G}(F)/G(F), \mathbb{C}^{\times}). \tag{3.6}$$

Lemma 3.4. *The image $\alpha(\mathcal{S}_{\underline{\phi}})$ is contained in* $\operatorname{Hom}(\widetilde{G}(F)/Z_{\widetilde{G}}(F)G(F), \mathbb{C}^{\times})$.

Proof. It follows from the proof of Lemma 3.3 that the image $\delta(S_{\underline{\phi}})$ is in the kernel of $H^1(W_F, \widehat{D}) \to H^1(W_F, \widehat{Z^0_{\widetilde{G}}})$. So, $\alpha(\mathcal{S}_{\underline{\phi}})$ is contained in $\operatorname{Hom}(\widetilde{G}(F)/Z^0_{\widetilde{G}}(F)G(F), \mathbb{C}^{\times})$. When $Z_{\widetilde{G}} = Z^0_{\widetilde{G}}$, this is what we want.

Suppose that $Z_{\widetilde{G}}$ is not connected; we can take an F-torus Z containing $Z_{\widetilde{G}}$, and let $\widetilde{G}' = (\widetilde{G} \times Z)/Z_{\widetilde{G}}$. Then $Z_{\widetilde{G}'} = Z$ is connected. Let $\tilde{\phi}' \in \Phi(\widetilde{G}')$ be a lift of $\tilde{\phi}$; then we have the following

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commutative diagram:

$$\begin{array}{ccccccc}
1 & \longrightarrow & \mathcal{S}_{\underline{\tilde{\phi}'}} & \xrightarrow{\iota'} & \mathcal{S}_{\underline{\phi}} & \xrightarrow{\alpha'} & \mathrm{Hom}(\widetilde{G}'(F)/G(F), \mathbb{C}^{\times}) \\
\downarrow & & \downarrow & & \| & & \downarrow \\
1 & \longrightarrow & \mathcal{S}_{\underline{\tilde{\phi}}} & \xrightarrow{\iota} & \mathcal{S}_{\underline{\phi}} & \xrightarrow{\alpha} & \mathrm{Hom}(\widetilde{G}(F)/G(F), \mathbb{C}^{\times})
\end{array}$$

Since the image of α' is trivial on $Z_{\widetilde{G}'}(F)$, the image of α is trivial on $Z_{\widetilde{G}}(F)$. This finishes the proof. □

Suppose that $\theta \in \Sigma$ for any semisimple element $s \in \bar{S}^{\theta}_{\underline{\phi}}$, let $\widehat{H} = \mathrm{Cent}(s, \widehat{G})^0$ and $\mathcal{H} = \widehat{H} \cdot \mathrm{Im}\, \underline{\phi}$; then $\mathcal{H}$ is embedded identically in ${}^L G$. The conjugate action of L_F on $\widehat{H}$ through $\underline{\phi}$ determines a Galois action on $\widehat{H}$ and hence determines a quasisplit reductive group H. Therefore, $\underline{\phi}$ factors through $\mathcal{H}$ for $[(H, \mathcal{H}, s, \xi)] \in \mathcal{E}(G^{\theta})$, where ξ is the identity embedding. For any lift $\underline{\tilde{\phi}}$ of $\underline{\phi}$, the restriction $\underline{\tilde{\phi}}|_{W_F}$ lifts $(H, \mathcal{H}, s, \xi)$ to a twisted endoscopic datum $[(\widetilde{H}, \widetilde{\mathcal{H}}, \widetilde{s}, \widetilde{\xi})] \in \mathcal{E}(\widetilde{G}^{\theta}, \omega)$ for some quasicharacter ω of $\widetilde{G}(F)/G(F)$ (cf. the proof of Proposition 3.1). By construction, we know that $\underline{\tilde{\phi}}$ factors through $\widetilde{\mathcal{H}}$. If we take a different lift $\underline{\tilde{\phi}}'$ of $\underline{\phi}$, it is easy to see that $\underline{\tilde{\phi}}'$ also factors through $\widetilde{\mathcal{H}}$. All of these can be summarized in the diagram below.

$$\begin{array}{ccccc}
L_F & \longrightarrow & \widetilde{\mathcal{H}} & \xrightarrow{\tilde{\xi}} & {}^L\widetilde{G} \\
 & \searrow & \downarrow & & \downarrow \\
 & & \mathcal{H} & \xrightarrow{\xi} & {}^L G
\end{array}$$

Then we have the following simple fact.

Lemma 3.5. $\alpha(s) = \omega$.

Proof. By definition, $\delta(s)(w) = \widetilde{s}\underline{\tilde{\phi}}(w)\widetilde{s}^{-1}\underline{\tilde{\phi}}(w)^{-1}$ for any $w \in W_F$, and $\widetilde{s}$ is a preimage of s in $\widehat{\widetilde{G}}^{\Sigma}$. Since $\underline{\tilde{\phi}}$ factors through $\widetilde{\mathcal{H}}$ and $\widehat{\widetilde{H}}$ commutes with $\widetilde{s}$, we have $\widetilde{s}\underline{\tilde{\phi}}(w)\widetilde{s}^{-1}\underline{\tilde{\phi}}(w)^{-1} = \widetilde{s}\widetilde{\xi}(w)\widetilde{s}^{-1}\widetilde{\xi}(w)^{-1}$, and this means that $\alpha(s) = \omega$. □

3.3 Endoscopic transfer

Returning to the setup in § 3.1, the reason that endoscopic data are so important is because there is a transfer map from $C_c^{\infty}(G(F))$ to $C_c^{\infty}(H(F))$ if $\mathcal{H} = {}^L H$. If $\mathcal{H} \neq {}^L H$, we have to take an extension H_1 of H by an induced torus Z_1 (called z-extension)

$$1 \longrightarrow Z_1 \longrightarrow H_1 \longrightarrow H \longrightarrow 1,$$

so that the dual homomorphism $\widehat{H} \to \widehat{H}_1$ can be extended to an L-embedding $\xi_{H_1} : \mathcal{H} \to {}^L H_1$. We call (H_1, ξ_{H_1}) a z-pair for $\mathcal{H}$. By choosing a section $c : W_F \to \mathcal{H}$, we get a quasicharacter χ_1 on Z_1 from

$$W_F \xrightarrow{c} \mathcal{H} \xrightarrow{\xi_{H_1}} {}^L H_1 \longrightarrow {}^L Z_1.$$

It is easy to see that χ_1 is independent of the choice of section c. So, the transfer map will be from $C_c^{\infty}(G(F))$ to $C_c^{\infty}(H_1(F), \chi_1)$, which is the space of χ_1^{-1}-equivariant smooth functions on $H_1(F)$ with compact support modulo Z_1.

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To define this transfer map, we need to introduce the space of twisted (stable) orbital integrals. Let ω_G be a quasicharacter of $G(F)$ and δ be a strongly θ-regular θ-semisimple element of $G(F)$, namely $\mathrm{Int}(\delta)\circ\theta$ is semisimple and the θ-twisted centralizer $G^\theta_\delta(F)$ (i.e., $\mathrm{Int}(\delta)\circ\theta$-invariant elements in $G(F)$) of δ is abelian. We assume that ω_G is trivial on $G^\theta_\delta(F)$. We fix Haar measures on $G(F)$ and $G^\theta_\delta(F)$; they induce a $G(F)$-invariant measure on $G^\theta_\delta(F)\backslash G(F)$. Then we can form the (θ,ω_G)-twisted orbital integral of $f\in C_c^\infty(G(F))$ over δ as

$$O_G^{\theta,\omega_G}(f,\delta):=\int_{G^\theta_\delta(F)\backslash G(F)}\omega_G(g)f(g^{-1}\delta\theta(g))\,dg.$$

We also form the (θ,ω_G)-twisted stable orbital integral over δ as

$$SO_G^{\theta,\omega_G}(f,\delta):=\sum_{\{\delta'\}^\theta_{G(F)}\sim_{st}\{\delta\}^\theta_{G(F)}}O_G^{\theta,\omega_G}(f,\delta'),$$

where the sum is over θ-twisted conjugacy classes $\{\delta'\}^\theta_{G(F)}$ in the θ-twisted stable conjugacy class of δ (i.e., $\delta'=g^{-1}\delta\theta(g)$ for some $g\in G(\bar F)$), and the Haar measure on $G^\theta_{\delta'}(F)$ is translated from that on $G^\theta_\delta(F)$ by conjugation. Let $\mathcal{I}(G^{\theta,\omega_G})$ ($\mathcal{SI}(G^{\theta,\omega_G})$) be the space of (θ,ω_G)-twisted (stable) orbital integrals of $C_c^\infty(G(F))$ over the set $G^\theta_{\mathrm{reg}}(F)$ of strongly θ-regular θ-semisimple elements of $G(F)$; then by definition we have projections

$$C_c^\infty(G)\twoheadrightarrow\mathcal{I}(G^{\theta,\omega_G})\twoheadrightarrow\mathcal{SI}(G^{\theta,\omega_G}).$$

Suppose that $[(H,\mathcal{H},s,\xi)]\in\mathcal{E}(G^\theta,\omega_G)$; we fix a z-pair (H_1,ξ_{H_1}) for $\mathcal{H}$. We assume that θ preserves an F-splitting of G; then there is a map from the semisimple $H_1(\bar F)$-conjugacy classes of $H_1(\bar F)$ to the θ-twisted semisimple $G(\bar F)$-conjugacy classes of $G(\bar F)$. By our assumption on θ, this map is defined over F. We call a strongly regular semisimple element $\gamma_1\in H_1(\bar F)$ strongly G-regular if its associated $H_1(\bar F)$-conjugacy class maps to a θ-twisted strongly regular semisimple $G(\bar F)$-conjugacy class of $G(\bar F)$. We denote the set of strongly G-regular semisimple elements of $H_1(F)$ by $H_{1,G\text{-reg}}(F)$. The transfer factor defined in [KS99] is a function

$$\Delta_{G,H_1}(\cdot\,,\cdot):H_{1,G\text{-reg}}(F)\times G^\theta_{\mathrm{reg}}(F)\to\mathbb{C},$$

which is nonzero only when $\gamma_1\in H_{1,G\text{-reg}}(F)$ is a norm of $\delta\in G^\theta_{\mathrm{reg}}(F)$, i.e., the $H_1(\bar F)$-conjugacy class of γ_1 maps to the θ-twisted $G(\bar F)$-conjugacy class of δ. Note that if $\delta\in G^\theta_{\mathrm{reg}}(F)$ has a norm $\gamma_1\in H_{1,G\text{-reg}}(F)$, then ω_G is trivial on $G^\theta_\delta(F)$ (see [KS99, Lemma 4.4.C]). In this paper we always normalize the transfer factor with respect to a fixed θ-stable Whittaker datum (B,Λ) for G. The transfer factor has the following basic properties (see [KS99]).

- $\Delta_{G,H_1}(\cdot\,,\cdot)$ is invariant over a stable conjugacy class in the first variable.
- There is a canonical inclusion $(Z_G)_\theta\hookrightarrow Z_H$, so that we get a homomorphism

$$Z_G\to(Z_G)_\theta\hookrightarrow Z_H. \tag{3.7}$$

  Let C be the fiber product of Z_G and Z_{H_1} over Z_H. Then there exists a quasicharacter χ_C of $C(F)$ such that

$$\Delta_{G,H_1}(z_1\gamma_1,z\delta)=\chi_C^{-1}(z_1,z)\Delta_{G,H_1}(\gamma_1,\delta),\quad z_1\in Z_{H_1}(F),z\in Z_G(F),$$

  where z_1 and z have the same image on $Z_H(F)$, and the restriction of χ_C to $Z_1(F)$ is χ_1.

– For $g \in G(F)$, $\Delta_{G,H_1}(\gamma_1, g^{-1}\delta\theta(g)) = \omega_G(g)\Delta_{G,H_1}(\gamma_1,\delta)$.

The transfer map is a correspondence from $f \in C_c^\infty(G(F))$ to $f^{H_1} \in C_c^\infty(H_1(F),\chi_1)$ such that

$$SO_{H_1}(f^{H_1},\gamma_1) = \sum_{\{\delta'\}^\theta_{G(F)} \sim_{st} \{\delta\}^\theta_{G(F)}} \Delta_{G,H_1}(\gamma_1,\delta') O_G^{\theta,\omega_G}(f,\delta'), \tag{3.8}$$

where the sum is over θ-twisted conjugacy classes $\{\delta'\}^\theta_{G(F)}$ in the θ-twisted stable conjugacy class of δ. The existence of such a correspondence has been conjectured by Langlands, Shelstad and Kottwitz. In the real case, this is now a theorem of Shelstad [She12]. In the p-adic case, Waldspurger [Wal95, Wal97, Wal06, Wal08] reduced it to the fundamental lemma for Lie algebras over the function field, and Ngô [Ngô10] proved the fundamental lemma in this form.

This transfer map can also be defined for equivariant functions. To be more precise, let Z_F be a closed subgroup of $Z_G(F)$ such that $Z_F \to Z_H(F)$ through (3.7) is injective. In particular, the preimage of Z_F in $Z_{H_1}(F)$ forms a closed subgroup of $C(F)$; we denote it by $C_{1,F}$. We fix a quasicharacter χ on Z_F; it pulls back to a quasicharacter on $C_{1,F}$. Denote the restriction of χ_C on $C_{1,F}$ by χ_{C_1}; then we claim that there is a correspondence from the space $C_c^\infty(G(F),\chi)$ of χ^{-1}-equivariant functions to the space $C_c^\infty(H_1(F),\chi\chi_{C_1})$ of $(\chi\chi_{C_1})^{-1}$-equivariant functions characterized by (3.8). This correspondence can be constructed as follows. There is a surjection from $C_c^\infty(G(F))$ to $C_c^\infty(G(F),\chi)$ defined by

$$f \mapsto \bar{f} = \int_{Z_F} f(zg)\chi(z)\,dz.$$

Similarly, we have a surjection from $C_c^\infty(H_1(F),\chi_1)$ to $C_c^\infty(H_1(F),\chi\chi_{C_1})$ defined by

$$f \mapsto \bar{f} = \int_{Z_1(F)\backslash C_{1,F}} f(zg)\chi\chi_{C_1}(z)\,dz.$$

Then it suffices to check the commutativity of the following diagram.

$$\begin{array}{ccc} C_c^\infty(G(F)) & \longrightarrow & C_c^\infty(H_1(F),\chi_1) \\ \downarrow & & \downarrow \\ C_c^\infty(G(F),\chi) & \longrightarrow & C_c^\infty(H_1(F),\chi\chi_{C_1}) \end{array}$$

Suppose that $f \in C_c^\infty(G(F))$ and γ_1 is a norm of δ,

$$O_G^{\theta,\omega_G}(\bar{f},\delta) = \int_{Z_F} O_G^{\theta,\omega_G}(f,z\delta)\chi(z)\,dz.$$

So,

$$\begin{aligned} &\sum_{\{\delta'\}^\theta_{G(F)} \sim_{st} \{\delta\}^\theta_{G(F)}} \Delta_{G,H_1}(\gamma_1,\delta') O_G^{\theta,\omega_G}(\bar{f},\delta') \\ &\quad = \sum_{\{\delta'\}^\theta_{G(F)} \sim_{st} \{\delta\}^\theta_{G(F)}} \Delta_{G,H_1}(\gamma_1,\delta') \int_{Z_F} O_G^{\theta,\omega_G}(f,z\delta')\chi(z)\,dz \\ &\quad = \int_{Z_F} \chi(z) \sum_{\{\delta'\}^\theta_{G(F)} \sim_{st} \{\delta\}^\theta_{G(F)}} \Delta_{G,H_1}(\gamma_1,\delta') O_G^{\theta,\omega_G}(f,z\delta')\,dz \end{aligned}$$

$$= \int_{Z_F} \chi(z) \sum_{\{\delta'\}^{\theta}_{G(F)} \sim_{st} \{\delta\}^{\theta}_{G(F)}} \chi_C(z_1, z) \Delta_{G,H_1}(z_1\gamma_1, z\delta') O_G^{\theta,\omega_G}(f, z\delta')\, dz$$
$$= \int_{Z_F} \chi(z)\chi_C(z_1, z) SO_{H_1}(f^{H_1}, z_1\gamma_1)\, dz$$
$$= \int_{Z_1(F)\backslash C_{1,F}} \chi\chi_{C_1}(z_1) SO_{H_1}(f^{H_1}, z_1\gamma_1)\, dz.$$

Hence, $\bar{f}$ corresponds to the image of f^{H_1} in $C_c^{\infty}(H_1(F), \chi\chi_{C_1})$.

Let $G \subseteq \widetilde{G}$ be two quasisplit connected reductive groups over F such that $G_{\mathrm{der}} = \widetilde{G}_{\mathrm{der}}$ and $\widetilde{G}/G = D$. Suppose that $[(H, \mathcal{H}, s, \xi)] \in \mathcal{E}(G^{\theta}, \omega_G)$; let $[(\widetilde{H}, \widetilde{\mathcal{H}}, \widetilde{s}, \widetilde{\xi})] \in \mathcal{E}(\widetilde{G}^{\theta}, \omega_{\widetilde{G}})$ be the corresponding lift. We also fix a z-pair $(\widetilde{H}_1, \widetilde{\xi}_{\widetilde{H}_1})$ for $\widetilde{\mathcal{H}}$ with a z-extension

$$1 \longrightarrow Z_1 \longrightarrow \widetilde{H}_1 \longrightarrow \widetilde{H} \longrightarrow 1.$$

Let H_1 be the preimage of H in $\widetilde{H}_1$; then we get a z-extension for H

$$1 \longrightarrow Z_1 \longrightarrow H_1 \longrightarrow H \longrightarrow 1.$$

Note that $\widetilde{H}_1/H_1 = D$, so on the dual side $\widetilde{\xi}_{\widetilde{H}_1}$ gives rise to an L-embedding $\xi_{H_1} : \mathcal{H} \to {}^L H_1$ by taking the quotient of $\widehat{D}$. We fix a θ-stable Whittaker datum for $\widetilde{G}$, which determines that for G. Then the relation of the transfer factors $\Delta_{\widetilde{G},\widetilde{H}_1}$ and Δ_{G,H_1} can be stated in the following lemma.

Lemma 3.6. *Suppose that δ is a strongly θ-regular θ-semisimple element in $G(F) \subseteq \widetilde{G}(F)$ and γ_1 is a strongly G-regular semisimple element in $H_1(F) \subseteq \widetilde{H}_1(F)$. Then one has*

$$\Delta_{\widetilde{G},\widetilde{H}_1}(\gamma_1, \delta) = \Delta_{G,H_1}(\gamma_1, \delta).$$

Proof. Suppose that the θ-stable Whittaker datum for $\widetilde{G}$ is constructed with respect to a θ-stable F-splitting $(\widetilde{\mathbb{B}}, \widetilde{\mathbb{T}}, \{X_\alpha\})$ of $\widetilde{G}$, and a nontrivial additive character ψ_F of F. This also determines a θ-stable F-splitting $(\mathbb{B}, \mathbb{T}, \{X_\alpha\})$ of G. Then the unnormalized transfer factor can be defined as a product

$$\Delta_0(\gamma_1, \delta) = \Delta_{\mathrm{I}}(\gamma_1, \delta)\Delta_{\mathrm{II}}(\gamma_1, \delta)\Delta_{\mathrm{III}}(\gamma_1, \delta)\Delta_{\mathrm{IV}}(\gamma_1, \delta).$$

It depends on the θ-stable F-splitting that we have fixed. First, we would like to compare the unnormalized transfer factors for $(\widetilde{G}, \widetilde{H}_1)$ and (G, H_1) term by term. To set things up, let $\widetilde{T}_{H_1}$ be the centralizer of γ_1 in $\widetilde{H}_1$ and let $T_{H_1} = \widetilde{T}_{H_1} \cap H_1$. Let $\widetilde{T}_H$ (respectively T_H) be the projection of $\widetilde{T}_{H_1}$ (respectively T_{H_1}) on $\widetilde{H}$ (respectively H). We fix an admissible embedding $\widetilde{T}_H \longrightarrow \widetilde{T}_{\theta}$; this gives an admissible embedding $T_H \longrightarrow T_{\theta}$ by restriction. Since the root system $R(\widetilde{G}, \widetilde{T})$ is isomorphic to $R(G, T)$ and the isomorphism is equivariant under the Galois action, one can assign the same a-data and χ-data [LS87] to them. They induce a-data and χ-data for $R_{\mathrm{res}}(\widetilde{G}, \widetilde{T})$ (respectively $R_{\mathrm{res}}(G, T)$), which are roots in $R(\widetilde{G}, \widetilde{T})$ (respectively $R(G, T)$) restricted to $(\widetilde{T}^{\theta})^0$ (respectively $(T^{\theta})^0$).

Let $\langle \cdot, \cdot \rangle$ denote the Tate–Nakayama pairing between $H^1(F, T_{\mathrm{sc}}^{\theta})$ and $\pi_0((\widehat{T_{\mathrm{sc}}^{\theta}})^{\Gamma})$; then the first term in the unnormalized transfer factor is defined by

$$\Delta_{I,(G,H_1)}(\gamma_1, \delta) = \langle \lambda_{a_\alpha}(T_{\mathrm{sc}}^{\theta}), s_{T,\theta} \rangle,$$

where $\lambda_{a_\alpha}(T_{\rm sc}^{\theta})$ is defined by using the a-data and the θ-stable F-splitting, and $s_{T,\theta}$ is the projection of the semisimple element $s \in \widehat{G} \rtimes \widehat{\theta}$ in the endoscopic datum $(H, \mathcal{H}, s, \xi)$ onto $(\widehat{T}_{\rm ad})_{\widehat{\theta}} = \widehat{T_{\rm sc}^{\theta}}$. Because $\widetilde{G}_{\rm sc} = G_{\rm sc}$ and we choose the a-data and the θ-stable F-splitting for $\widetilde{G}$ and G in a consistent way, $\lambda_{a_\alpha}(T_{\rm sc}^{\theta}) = \lambda_{a_\alpha}(\widetilde{T}_{\rm sc}^{\theta})$. Moreover, $\widetilde{s}$ and s have the same image in $\widehat{T_{\rm sc}^{\theta}}$; hence,

$$\Delta_{I,(G,H_1)}(\gamma_1, \delta) = \Delta_{I,(\widetilde{G},\widetilde{H}_1)}(\gamma_1, \delta).$$

For the second term, we adopt Waldspurger's modification here (see [KS12]). It is defined by the a-data and χ-data, and again because we choose them for $\widetilde{G}$ and G in a consistent way, the second term will be the same for $(\widetilde{G}, \widetilde{H}_1)$ and (G, H_1). Before discussing the third term, let us consider the fourth term first. The fourth term is defined by

$$\Delta_{\mathrm{IV},(G,H_1)}(\gamma_1, \delta) = \frac{D_{G^\theta}(\delta)}{D_{H_1}(\gamma_1)},$$

where

$$D_{G^\theta}(\delta) = |\det(Ad(\delta) \circ \theta - 1)_{\mathrm{Lie}(G)/\mathrm{Lie}\,(\mathrm{Cent}((G_\delta^\theta)^0, G)}|_F^{1/2},$$
$$D_{H_1}(\gamma_1) = |\det(Ad(\gamma_1) - 1)_{\mathrm{Lie}(H_1)/\mathrm{Lie}(T_{H_1})}|_F^{1/2}.$$

And, it is easy to see that $D_{\widetilde{G}^\theta}(\delta) = D_{G^\theta}(\delta)$ and $D_{\widetilde{H}_1}(\gamma_1) = D_{H_1}(\gamma_1)$; therefore, the fourth term is also the same for $(\widetilde{G}, \widetilde{H}_1)$ and (G, H_1).

We are now left with the third term $\Delta_{\mathrm{III},(G,H_1)}$; it is given by a pairing of hypercohomology groups $H^1(F,\, T_{\rm sc} \xrightarrow{1-\theta_1} T_1)$ and $H^1(W_F,\, \widehat{T}_1 \xrightarrow{1-\widehat{\theta}_1} \widehat{T}_{\rm ad})$, where T_1 is the fiber product of T and T_{H_1} over $T_\theta \cong T_H$, and θ_1 is a lift of θ on T_1 which fixes $Z_1 \subseteq T_1$. Similarly, we can define $\widetilde{T}_1$, and the inclusion $T_1 \to \widetilde{T}_1$ induces maps on hypercohomology groups

$$\varphi : H^1(F,\, T_{\rm sc} \xrightarrow{1-\theta_1} T_1) \longrightarrow H^1(F,\, \widetilde{T}_{\rm sc} \xrightarrow{1-\theta_1} \widetilde{T}_1),$$
$$\varphi^* : H^1(W_F,\, \widehat{\widetilde{T}}_1 \xrightarrow{1-\widehat{\theta}_1} \widehat{\widetilde{T}}_{\rm ad}) \longrightarrow H^1(W_F,\, \widehat{T}_1 \xrightarrow{1-\widehat{\theta}_1} \widehat{T}_{\rm ad}).$$

It is an easy exercise to check that they are adjoint to each other with respect to the Tate–Nakayama pairing on hypercohomology groups, i.e.,

$$\langle \varphi(\mathbf{V}), \mathbf{A} \rangle = \langle \mathbf{V}, \varphi^*(\mathbf{A}) \rangle,$$

where $\mathbf{V} \in H^1(F,\, T_{\rm sc} \xrightarrow{1-\theta_1} T_1)$ and $\mathbf{A} \in H^1(W_F,\, \widehat{\widetilde{T}}_1 \xrightarrow{1-\widehat{\theta}_1} \widehat{\widetilde{T}}_{\rm ad})$. It follows from the definition in [KS99] that there exist $\mathbf{V}_0 \in H^1(F,\, T_{\rm sc} \xrightarrow{1-\theta_1} T_1)$ and $\mathbf{A}_0 \in H^1(W_F,\, \widehat{\widetilde{T}}_1 \xrightarrow{1-\widehat{\theta}_1} \widehat{\widetilde{T}}_{\rm ad})$ such that

$$\Delta_{\mathrm{III},(\widetilde{G},\widetilde{H}_1)}(\gamma_1, \delta) = \langle \varphi(\mathbf{V}_0), \mathbf{A}_0 \rangle,$$
$$\Delta_{\mathrm{III},(G,H_1)}(\gamma_1, \delta) = \langle \mathbf{V}_0, \varphi^*(\mathbf{A}_0) \rangle.$$

Hence, $\Delta_{\mathrm{III},(\widetilde{G},\widetilde{H}_1)}(\gamma_1, \delta) = \Delta_{\mathrm{III},(G,H_1)}(\gamma_1, \delta)$.

Up to now, we have shown the equality for the unnormalized transfer factors. To define the normalizing factor, we need to choose an F-splitting $(\mathbb{B}_H, \widetilde{\mathbb{T}}_H, \{X_{\alpha_H}\})$ of $\widetilde{H}$; it determines an F-splitting $(\mathbb{B}_H, \mathbb{T}_H, \{X_{\alpha_H}\})$ of H. Let $V_{\widetilde{G}}$ be the representation of Γ on $X^*(\widetilde{\mathbb{T}})^\theta \otimes \mathbb{C}$ and $V_{\widetilde{H}}$

be the representation of Γ on $X^*(\widetilde{\mathbb{T}}_H)\otimes\mathbb{C}$. Let $\widetilde{V}=V_{\widetilde{G}}-V_{\widetilde{H}}$; then the normalizing factor for $(\widetilde{G},\widetilde{H}_1)$ is given by the local ϵ-factor

$$\epsilon_L(\widetilde{V},\psi_F)=\epsilon_L(V_{\widetilde{G}},\psi_F)/\epsilon_L(V_{\widetilde{H}},\psi_F)$$

(see [Tat79, § 3.6]). Similarly, we can define V_G, V_H and $V=V_G-V_H$. Then it is enough to show that $\epsilon_L(\widetilde{V},\psi_F)=\epsilon_L(V,\psi_F)$. Note

$$\epsilon_L(\widetilde{V},\psi_F)/\epsilon(V,\psi_F)=\epsilon_L(V_{\widetilde{G}},\psi_F)/\epsilon_L(V_G,\psi_F)\cdot\epsilon_L(V_H,\psi_F)/\epsilon_L(V_{\widetilde{H}},\psi_F).$$

By the following exact sequences:

$$1\longrightarrow\mathbb{T}\longrightarrow\widetilde{\mathbb{T}}\longrightarrow D\longrightarrow 1,$$

$$1\longrightarrow\mathbb{T}_H\longrightarrow\widetilde{\mathbb{T}}_H\longrightarrow D\longrightarrow 1,$$

we have

$$1\longrightarrow V_D\longrightarrow V_{\widetilde{G}}\longrightarrow V_G\longrightarrow 1,$$

$$1\longrightarrow V_D\longrightarrow V_{\widetilde{H}}\longrightarrow V_H\longrightarrow 1,$$

where $V_D=X^*(D)\otimes\mathbb{C}$ is θ-invariant. Therefore,

$$\epsilon_L(V_{\widetilde{G}},\psi_F)/\epsilon_L(V_G,\psi_F)=\epsilon_L(V_{\widetilde{H}},\psi_F)/\epsilon_L(V_H,\psi_F)=\epsilon_L(V_D,\psi_F).$$

This finishes the proof. □

Following the notation in this lemma, note that $G(F)$ is θ-twisted conjugate invariant under $\widetilde{G}(F)$, so we have the following corollary.

Corollary 3.7. *Suppose that δ is a strongly θ-regular θ-semisimple element in $G(F)$ and γ_1 is a strongly G-regular semisimple element in $H_1(F)$. Then one has*

$$\Delta_{G,H_1}(\gamma_1,g^{-1}\delta\theta(g))=\omega_{\widetilde{G}}(g)\Delta_{G,H_1}(\gamma_1,\delta).$$

Proof. From the previous lemma, we know that

$$\Delta_{G,H_1}(\gamma_1,\delta)=\Delta_{\widetilde{G},\widetilde{H}_1}(\gamma_1,\delta)\quad\text{and}\quad\Delta_{\widetilde{G},\widetilde{H}_1}(\gamma_1,g^{-1}\delta\theta(g))=\Delta_{G,H_1}(\gamma_1,g^{-1}\delta\theta(g)).$$

It follows from the property of the transfer factor that

$$\Delta_{\widetilde{G},\widetilde{H}_1}(\gamma_1,g^{-1}\delta\theta(g))=\omega_{\widetilde{G}}(g)\Delta_{\widetilde{G},\widetilde{H}_1}(\gamma_1,\delta).$$

Then the corollary is clear. □

Remark 3.8. An equivalent way of stating this corollary is as follows. Let $f\in C_c^\infty(G(F)\rtimes\theta)$; we can view it as in $C_c^\infty(G(F))$ by sending g to $g\rtimes\theta$, and define its transfer as before. For $g\in\widetilde{G}(F), h\in G(F)\rtimes\theta$, we denote $f^g(h)=f(ghg^{-1})$. Then this corollary says that

$$(f^g)^{H_1}=\omega_{\widetilde{G}}(g)f^{H_1}. \tag{3.9}$$

Let $\widetilde{Z}_F$ be a closed subgroup of $Z_{\widetilde{G}}(F)$ such that $\widetilde{Z}_F \to (Z_{\widetilde{G}})_\theta(F)$ is injective and $D(F)/\lambda(\widetilde{Z}_F)$ is finite (this is possible because we assume that λ is θ-invariant). Let $Z_F = \widetilde{Z}_F \cap G(F)$. We choose Haar measures on $\widetilde{Z}_F$ and Z_F such that the measure on $Z_F \backslash G(F)$ is the restriction of that on $\widetilde{Z}_F \backslash \widetilde{G}(F)$. We fix a quasicharacter $\widetilde{\chi}$ of $\widetilde{Z}_F$ and denote its restriction to Z_F by χ. Note that $\widetilde{Z}_F G(F) \backslash \widetilde{G}(F)$ is finite, so we get an inclusion map

$$\begin{aligned} C_c^\infty(G(F), \chi) &\hookrightarrow C_c^\infty(\widetilde{G}(F), \widetilde{\chi}), \\ f &\longmapsto \widetilde{f}, \end{aligned} \tag{3.10}$$

where $\widetilde{f}$ is the extension of f by zero outside $\widetilde{Z}_F G(F)$. We can identify $C_c^\infty(G(F), \chi)$ with its image in $C_c^\infty(\widetilde{G}(F), \widetilde{\chi})$. Because $\widetilde{Z}_F G(F)$ is θ-twisted conjugate invariant under $\widetilde{G}(F)$, the map (3.10) induces a map from $\mathcal{I}(G^{\theta, \omega_G}, \chi)$ to $\mathcal{I}(\widetilde{G}^{\theta, \omega_{\widetilde{G}}}, \widetilde{\chi})$, where $\omega_{\widetilde{G}}|_G = \omega_G$. Moreover, $\widetilde{Z}_F G(\bar{F})$ is θ-twisted conjugate invariant under $\widetilde{G}(\bar{F})$, so it also induces a map from $\mathcal{SI}(G^{\theta, \omega_G}, \chi)$ to $\mathcal{SI}(\widetilde{G}^{\theta, \omega_{\widetilde{G}}}, \widetilde{\chi})$.

Let $\omega_{\widetilde{G}}$ be a quasicharacter of $\widetilde{G}(F)$ and $\omega_G = \omega_{\widetilde{G}}|_G$. Let δ be a strongly θ-regular θ-semisimple element of $G(F) \subseteq \widetilde{G}(F)$ such that $\omega_{\widetilde{G}}$ is trivial on the θ-twisted centralizer $\widetilde{G}^\theta_\delta(F)$ of δ. We choose Haar measures on $\widetilde{G}^\theta_\delta(F)$ and $G^\theta_\delta(F)$ such that the measure on $G^\theta_\delta(F) \backslash G(F)$ is the restriction of that on $\widetilde{G}^\theta_\delta(F) \backslash \widetilde{G}(F)$. Then

$$O^{\theta, \omega_{\widetilde{G}}}_{\widetilde{G}}(\tilde{f}, \delta) = \sum_{\{\delta'\}^\theta_{G(F)} \sim_{\widetilde{G}(F)} \{\delta\}^\theta_{G(F)}} O^{\theta, \omega_G}_G(f, \delta') \omega_{\widetilde{G}}(g),$$

where the sum is over θ-twisted $G(F)$-conjugacy classes $\{\delta'\}^\theta_{G(F)}$ in the θ-twisted $\widetilde{G}(F)$-conjugacy classes $\{\delta\}^\theta_{\widetilde{G}(F)}$ with $\delta' = g^{-1}\delta g$ for $g \in \widetilde{G}(F)$, and the Haar measure on $G^\theta_{\delta'}(F)$ is translated from that on G^θ_δ by conjugation.

Suppose that $[(\widetilde{H}, \widetilde{\mathcal{H}}, \widetilde{s}, \widetilde{\xi})] \in \mathcal{E}(\widetilde{G}^\theta, \omega_{\widetilde{G}})$ and $[(H, \mathcal{H}, s, \xi)] \in \mathcal{E}(G^\theta, \omega_G)$ correspond to each other according to Proposition 3.1. We also fix a z-pair $(\widetilde{H}_1, \widetilde{\xi}_{\widetilde{H}_1})$ for $\widetilde{\mathcal{H}}$, which induces a z-pair (H_1, ξ_{H_1}) for $\mathcal{H}$. Let $\widetilde{C}_{1,F}$ be the preimage of $\widetilde{Z}_F$ in $Z_{\widetilde{H}_1}(F)$ and $C_{1,F}$ be the preimage of Z_F in $Z_{H_1}(F)$. Then

$$C_{1,F} \hookrightarrow \widetilde{C}_{1,F} \xrightarrow{\lambda_1} D(F)$$

with $\lambda_1(\widetilde{C}_{1,F}) = \lambda(\widetilde{Z}_F)$. It is easy to check that the restriction of $\chi_{\widetilde{C}}$ to $C(F)$ is χ_C. Note that $\widetilde{\chi}$ and χ pull back to quasicharacters of $\widetilde{C}_{1,F}$ and $C_{1,F}$, respectively. So, let $\widetilde{\chi}' = \widetilde{\chi}\chi_{\widetilde{C}_1}$ and $\chi' = \chi\chi_{C_1}$; then we have an inclusion map analogous to (3.10)

$$\begin{aligned} C_c^\infty(H_1(F), \chi') &\hookrightarrow C_c^\infty(\widetilde{H}_1(F), \widetilde{\chi}'), \\ f &\longmapsto \widetilde{f}\,. \end{aligned}$$

The next lemma shows that these inclusion maps are compatible with twisted endoscopic transfers.

Lemma 3.9. *Suppose that $f \in C_c^\infty(G(F), \chi)$; then the $(\theta, \omega_{\widetilde{G}})$-twisted endoscopic transfer of the extension $\tilde{f}$ of f is equal to the extension of the (θ, ω_G)-twisted endoscopic transfer f^{H_1} of f as elements in $\mathcal{SI}(\widetilde{H}_1, \widetilde{\chi}')$, i.e.,*

$$\tilde{f}^{\widetilde{H}_1} = \widetilde{(f^{H_1})}.$$

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Proof. Let us assume that δ is a strongly θ-regular θ-semisimple element in $G(F) \subseteq \widetilde{G}(F)$ and γ_1 is a strongly G-regular semisimple element in $H_1(F) \subseteq \widetilde{H}_1(F)$, and γ_1 is a norm of δ. By the definition of twisted endoscopic transfer,

$$SO_{\widetilde{H}_1}(\tilde{f}^{\widetilde{H}_1}, \gamma_1) = \sum_{\{\delta'\}^{\theta}_{\widetilde{G}(F)} \sim_{st} \{\delta\}^{\theta}_{\widetilde{G}(F)}} \Delta_{\widetilde{G},\widetilde{H}_1}(\gamma_1, \delta') O^{\theta,\omega_{\widetilde{G}}}_{\widetilde{G}}(\tilde{f}, \delta'),$$

where the sum is over θ-twisted $\widetilde{G}(F)$-conjugacy classes $\{\delta'\}^{\theta}_{\widetilde{G}(F)}$ in the θ-twisted stable conjugacy class of δ. Meanwhile,

$$O^{\theta,\omega_{\widetilde{G}}}_{\widetilde{G}}(\tilde{f}, \delta') = \sum_{\{\delta''\}^{\theta}_{G(F)} \sim_{\widetilde{G}(F)} \{\delta'\}^{\theta}_{G(F)}} O^{\theta,\omega_G}_{G}(f, \delta'') \omega_{\widetilde{G}}(g),$$

where the sum is over θ-twisted $G(F)$-conjugacy classes $\{\delta''\}^{\theta}_{G(F)}$ in the θ-twisted $\widetilde{G}(F)$-conjugacy class $\{\delta'\}^{\theta}_{\widetilde{G}(F)}$, and $\delta'' = g^{-1}\delta'\theta(g)$ for $g \in \widetilde{G}(F)$. By the property of twisted transfer factor, one has

$$\Delta_{\widetilde{G},\widetilde{H}_1}(\gamma_1, g^{-1}\delta'\theta(g)) = \Delta_{\widetilde{G},\widetilde{H}_1}(\gamma_1, \delta')\omega_{\widetilde{G}}(g).$$

Therefore,

$$\begin{aligned} SO_{\widetilde{H}_1}(\tilde{f}^{\widetilde{H}_1}, \gamma_1) &= \sum_{\{\delta'\}^{\theta}_{\widetilde{G}(F)} \sim_{st} \{\delta\}^{\theta}_{\widetilde{G}(F)}} \Delta_{\widetilde{G},\widetilde{H}_1}(\gamma_1, \delta') \Bigg(\sum_{\{\delta''\}^{\theta}_{G(F)} \sim_{\widetilde{G}(F)} \{\delta'\}^{\theta}_{G(F)}} O^{\theta,\omega_G}_{G}(f, \delta'')\omega_{\widetilde{G}}(g) \Bigg) \\ &= \sum_{\{\delta''\}^{\theta}_{G(F)} \sim_{st} \{\delta\}^{\theta}_{G(F)}} \Delta_{\widetilde{G},\widetilde{H}_1}(\gamma_1, \delta'') O^{\theta,\omega_G}_{G}(f, \delta''). \end{aligned}$$

On the other hand,

$$SO_{\widetilde{H}_1}(\widetilde{f^{H_1}}, \gamma_1) = SO_{H_1}(f^{H_1}, \gamma_1) = \sum_{\{\delta''\}^{\theta}_{G(F)} \sim_{st} \{\delta\}^{\theta}_{G(F)}} \Delta_{G,H_1}(\gamma_1, \delta'') O^{\theta,\omega_G}_{G}(f, \delta'').$$

It follows from Lemma 3.6 that

$$\Delta_{\widetilde{G},\widetilde{H}_1}(\gamma_1, \delta'') = \Delta_{G,H_1}(\gamma_1, \delta''),$$

where δ'' is in the θ-twisted stable $G(F)$-conjugacy class of δ. So, we have shown that

$$SO_{\widetilde{H}_1}(\tilde{f}^{\widetilde{H}_1}, \gamma_1) = SO_{\widetilde{H}_1}(\widetilde{f^{H_1}}, \gamma_1) \tag{3.11}$$

for $\gamma_1 \in H_1(F)$ being a norm.

If γ_1 is not a norm, it follows from the property of the transfer factor that both sides of (3.11) are zero. By the equivariance property, we can extend (3.11) to $\widetilde{C}_{1,F}H_1(F)$. It is also easy to see that $SO_{\widetilde{H}_1}(\widetilde{f^{H_1}}, \gamma_1) \neq 0$ only when $\gamma_1 \in \widetilde{C}_{1,F}H_1(F)$. Finally, one can show that $SO_{\widetilde{H}_1}(\tilde{f}^{\widetilde{H}_1}, \gamma_1) \neq 0$ only when $\gamma_1 \in \widetilde{C}_{1,F}H_1(F)$ by using the condition on the support of $\tilde{f}$. This finishes the proof. □

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3.4 Character identity

Let π be an irreducible smooth representation of $G(F)$ and χ_π be the central character of π. Let C_F be a closed subgroup of $Z_G(F)$, and $\zeta = \chi_\pi|_{Z_F}$. Suppose that $\pi^\theta \cong \pi \otimes \omega_G$, let $A_\pi(\theta, \omega_G)$ be an intertwining operator between $\pi \otimes \omega_G$ and π^θ (this is uniquely determined up to a scalar); we then define the (θ, ω_G)-twisted character of π to be the distribution

$$f_{G^\theta}(\pi, \omega_G) := \operatorname{trace} \int_{C_F\backslash G(F)} f(g)\pi(g)\, dg \circ A_\pi(\theta, \omega_G) \tag{3.12}$$

for $f \in C_c^\infty(G(F), \zeta)$. In particular, we can define the distribution for $f \in C_c^\infty(G(F))$ by taking C_F to be trivial. By results of Harish-Chandra [Har63, Har99] in the nontwisted case and Bouaziz [Bou87], Lemaire [Lem13] and Clozel [Clo87] in the twisted case, there exists a locally integrable function $\Theta_\pi^{G^\theta,\omega_G}$ on $G(F)$ such that for $x \in G^\theta_{\mathrm{reg}}(F), g \in G(F)$,

$$\Theta_\pi^{G^\theta,\omega_G}(g^{-1}x\theta(g)) = \omega_G(g)\Theta_\pi^{G^\theta,\omega_G}(x)$$

and

$$f_{G^\theta}(\pi, \omega_G) = \int_{C_F\backslash G(F)} f(g)\Theta_\pi^{G^\theta,\omega_G}(g)\, dg.$$

By the twisted Weyl integration formula (cf. [Lem13, § 7.3] and [Mez13, § 5.4.1]), one can show that this character defines a linear functional on $\mathcal{I}(G^{\theta,\omega_G}, \chi)$. A linear functional on $\mathcal{I}(G^{\theta,\omega_G}, \chi)$ is called *stable* if it factors through $\mathcal{SI}(G^{\theta,\omega_G}, \chi)$. This notion of stability is equivalent to the one we gave in the introduction.

We assume that θ preserves an F-splitting of G. For $\phi \in \Phi(G)$, suppose that $\underline{\phi}$ factors through $\mathcal{H}$ for a twisted endoscopic datum $[(H, \mathcal{H}, s, \xi)] \in \mathcal{E}(G^\theta)$; let us write $\underline{\phi} = \xi \circ \underline{\phi}_{\mathcal{H}}$. Clearly, $sZ(\widehat{G}) \cap S^\theta_{\underline{\phi}} \neq \emptyset$; we denote its image in $\bar{S}^\theta_{\underline{\phi}}$ again by s. Let us fix a z-pair (H_1, ξ_{H_1}) for $\mathcal{H}$ and define $\underline{\phi}_{H_1} = \xi_{H_1} \circ \underline{\phi}_{\mathcal{H}}$. We say that $(H_1, \underline{\phi}_{H_1})$ corresponds to $(\underline{\phi}, s)$ for $s \in \bar{S}^\theta_{\underline{\phi}}$. It is easy to see that for any semisimple $s \in \bar{S}^\theta_{\underline{\phi}}$, such a pair $(H_1, \underline{\phi}_{H_1})$ always exists (see § 3.2). For abbreviation, we write $(H_1, \underline{\phi}_{H_1}) \to (\underline{\phi}, s)$. We always assume that the Haar measure is preserved for any admissible embedding $T_H \xrightarrow{\simeq} T_\theta$ for a maximal torus $T_H \subseteq H$ and a θ-stable maximal torus $T \subseteq G$.

Now we can state the conjectural twisted endoscopic character identity.

Conjecture 3.10. Suppose that $\phi \in \Phi_{\mathrm{bdd}}(G)$.

(i)

$$f(\underline{\phi}) := \sum_{\pi\in\Pi_\phi} \langle 1, \pi\rangle_{\underline{\phi}} f_G(\pi), \quad f \in C_c^\infty(G(F)) \tag{3.13}$$

is stable.

(ii) Suppose that s is a semisimple element in $\bar{S}^\theta_{\underline{\phi}}$, and $(H_1, \underline{\phi}_{H_1}) \to (\underline{\phi}, s)$. Then

$$f^{H_1}(\underline{\phi}_{H_1}) = \sum_{\substack{\pi\in\Pi_\phi \\ \pi\cong\pi^\theta}} \langle x, \pi^+\rangle_{\underline{\phi}} f_{G^\theta}(\pi) \tag{3.14}$$

for $f \in C_c^\infty(G(F))$, where x is the image of s in $\mathcal{S}^\theta_{\underline{\phi}}$, and π^+ is an extension of π to $G^+(F) := G(F) \rtimes \langle\theta\rangle$ with $\pi^+(\theta) = A_\pi(\theta)$.

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Remark 3.11. In the statement of this conjecture, the character $\langle\cdot,\pi\rangle_{\underline{\phi}}$ is given in Conjecture 2.1 and $\langle\cdot,\pi^+\rangle_{\underline{\phi}}$ is given in Conjecture 2.5, where one takes $\Sigma=\langle\theta\rangle, G^{\Sigma}=G^+$ and $\pi^{\Sigma}=\pi^+$.

In the setup of this conjecture, we can let

$$\Theta_{\underline{\phi},x}=\sum_{\pi\in\Pi_{\phi}}\langle x,\pi^+\rangle_{\underline{\phi}}\Theta_{\pi}^{G^{\theta}}$$

and

$$\Theta_{\underline{\phi}_{H_1}}=\sum_{\pi\in\Pi_{\phi_{H_1}}}\langle 1,\pi\rangle_{\underline{\phi}_{H_1}}\Theta_{\pi}^{H_1}.$$

Then, by expanding (3.14) using the twisted Weyl integration formula, we get

$$\Theta_{\underline{\phi},x}(\delta)=\sum_{\gamma_1\to\delta}\frac{D_{H_1}(\gamma_1)^2}{D_{G^{\theta}}(\delta)^2}\Delta_{G,H_1}(\gamma_1,\delta)\Theta_{\underline{\phi}_{H_1}}(\gamma_1),$$

where the sum is over stable conjugacy classes of norms $\gamma_1\in H_{1,G\text{-reg}}(F)$ of $\delta\in G^{\theta}_{\mathrm{reg}}(F)$.

Let Z_F be a closed subgroup of $Z_G(F)$ such that $Z_F\to Z_H(F)$ through (3.7) is injective. If the elements in $\Pi_{\phi_{H_1}}$ all have the same central character, let us denote its restriction to $C_{1,F}$ by χ'. Then, for $z\in Z_F$ and z_1 in its preimage in $C_{1,F}$, we have

$$\begin{aligned}\Theta_{\underline{\phi},x}(z\delta)&=\sum_{\gamma_1\to\delta}\frac{D_{H_1}(z_1\gamma_1)^2}{D_{G^{\theta}}(z\delta)^2}\Delta_{G,H_1}(z_1\gamma_1,z\delta)\Theta_{\underline{\phi}_{H_1}}(z_1\gamma_1)\\&=\sum_{\gamma_1\to\delta}\chi_{C_1}(z_1)^{-1}\frac{D_{H_1}(\gamma_1)^2}{D_{G^{\theta}}(\delta)^2}\Delta_{G,H_1}(\gamma_1,\delta)\chi'(z_1)\Theta_{\underline{\phi}_{H_1}}(\gamma_1)\\&=\chi_{C_1}(z_1)^{-1}\chi'(z_1)\Theta_{\underline{\phi},x}(\delta).\end{aligned}$$

Note that $\chi_{C_1}^{-1}\chi'$ is trivial on $Z_1(F)$ and hence descends to a quasicharacter on Z_F, which we denote by χ. By the linear independence of twisted characters of irreducible smooth representations, one must have

$$\Theta_{\pi}^{G^{\theta}}(zg)=\chi(z)\Theta_{\pi}^{G^{\theta}}(g)$$

for $z\in Z_F$ and $\pi\in\Pi_{\phi}$. In particular, we can let $\theta=\mathrm{id}$ and $Z_F=Z_G(F)$. Then the central character of elements in Π_{ϕ} is χ. This suggests that if we want to show for any $\phi\in\Phi_{\mathrm{bdd}}(G)$ that the elements in Π_{ϕ} have the same central character, we can reduce to the case of simple parameters. Since the L-packet for a simple parameter consists of only one element, there is nothing to show in that case. So, we have the following proposition as a consequence of Conjecture 3.10.

PROPOSITION 3.12. *Suppose that $\phi\in\Phi_{\mathrm{bdd}}(G)$; then the elements in Π_{ϕ} all have the same central character.*

This proposition can be extended to all L-packets by the theory of Langlands quotient.

3.5 Proof of Theorem 2.6

Let $G \subseteq \widetilde{G}$ be two quasisplit connected reductive groups over F such that $G_{\mathrm{der}} = \widetilde{G}_{\mathrm{der}}$; we denote $\widetilde{G}/G$ by D. We have

$$1 \longrightarrow G \longrightarrow \widetilde{G} \xrightarrow{\lambda} D \longrightarrow 1.$$

We assume that θ is an automorphism of $\widetilde{G}$ preserving an F-splitting of $\widetilde{G}$, and λ is θ-invariant. Let $G^+ = G \rtimes \langle\theta\rangle$.

Lemma 3.13. *Suppose that $\phi \in \Phi_{\mathrm{bdd}}(G)$ and $\pi \in \Pi_\phi$; then*

$$\langle x, (\pi^+)^g\rangle_{\underline{\phi}} = \omega_x(g)\langle x, \pi^+\rangle_{\underline{\phi}} \tag{3.15}$$

for any $g \in \widetilde{G}(F)$ and $x \in \mathcal{S}^\theta_{\underline{\phi}}$, where $\omega_x = \alpha(x)$ and π^+ is an irreducible representation of $G^+(F)$ containing π in its restriction.

Proof. Let $\pi = \pi(\rho)$ for $\rho \in \mathrm{Irr}(\mathcal{S}_\phi)$. If $\pi \ncong \pi^\theta$, then $\rho^x \ncong \rho$ for $x \in \mathcal{S}^\theta_{\underline{\phi}}$ (cf. Lemma 5.1 and Conjecture 2.3). Therefore, $\langle x, \pi^+\rangle_{\underline{\phi}} = 0$ for $x \in \mathcal{S}^\theta_{\underline{\phi}}$ and hence (3.15) is clear. Now we will only concern the case $\pi \cong \pi^\theta$. Suppose that $s \in \bar{S}^\theta_{\underline{\phi}}$ and $(H_1, \underline{\phi}_{H_1}) \to (\underline{\phi}, s)$; then, by (3.14), we have

$$f^{H_1}(\underline{\phi}_{H_1}) = \sum_{\substack{\pi\in\Pi_\phi\\ \pi\cong\pi^\theta}} \langle x, \pi^+\rangle_{\underline{\phi}} f_{G^\theta}(\pi)$$

for $f \in C_c^\infty(G(F))$, where x is the image of s in $\mathcal{S}^\theta_{\underline{\phi}}$. We can also reformulate this identity by taking $f \in C_c^\infty(G(F) \rtimes \theta)$ and view it as in $C_c^\infty(G(F))$ by sending g to $g \rtimes \theta$, so that we can define its transfer as before. The resulting identity is

$$f^{H_1}(\underline{\phi}_{H_1}) = \sum_{\pi\in\Pi_\phi} \langle x, \pi^+\rangle_{\underline{\phi}} f_{G^+}(\pi^+).$$

For $g \in \widetilde{G}(F), h \in G(F) \rtimes \theta$, we denote $f^g(h) = f(ghg^{-1})$. Then, by Lemma 3.5 and (3.9), we have

$$(f^g)^{H_1} = \omega_x(g) f^{H_1}$$

and hence

$$(f^g)^{H_1}(\underline{\phi}_{H_1}) = \omega_x(g) f^{H_1}(\underline{\phi}_{H_1}).$$

Using the character identity to expand each side, we get

$$\sum_{\pi\in\Pi_\phi} \langle x, \pi^+\rangle_{\underline{\phi}} f^g_{G^+}(\pi^+) = \sum_{\pi\in\Pi_\phi} \omega_x(g)\langle x, \pi^+\rangle_{\underline{\phi}} f_{G^+}(\pi^+). \tag{3.16}$$

The left-hand side of (3.16) is equal to

$$\sum_{\pi\in\Pi_\phi} \langle x, \pi^+\rangle_{\underline{\phi}} f_{G^+}((\pi^+)^{g^{-1}}) = \sum_{\pi\in\Pi_\phi} \langle x, (\pi^+)^g\rangle_{\underline{\phi}} f_{G^+}(\pi^+), \tag{3.17}$$

where we substitute π^+ for $(\pi^+)^{g^{-1}}$. Compared with the right-hand side of (3.16), this may possibly change the extension of π by some twist of characters of $G^+(F)/G(F)$. Nevertheless, the right-hand side of (3.16) is independent of the extensions, so we can certainly choose the same extension as the right-hand side of (3.17). So, after these changes, we get

$$\sum_{\pi\in\Pi_\phi} \langle x, (\pi^+)^g\rangle_{\underline{\phi}} f_{G^+}(\pi^+) = \sum_{\pi\in\Pi_\phi} \omega_x(g)\langle x, \pi^+\rangle_{\underline{\phi}} f_{G^+}(\pi^+)$$

and hence

$$\langle x, (\pi^+)^g\rangle_{\underline{\phi}} = \omega_x(g)\langle x, \pi^+\rangle_{\underline{\phi}}$$

by the linear independence of twisted characters. □

Now we are going to prove Theorem 2.6. For $\phi \in \Phi_{\mathrm{bdd}}(G)$, recall that there is a homomorphism

$$\alpha : \mathcal{S}^\Sigma_{\underline{\phi}} \to \mathrm{Hom}(\widetilde{G}(F)/G(F), \mathbb{C}^\times),$$

so we can define the homomorphism $\widetilde{G}(F) \to (\mathcal{S}^\Sigma_{\underline{\phi}})^*$ in the theorem by letting $\varepsilon_g(x) = \alpha(x)(g) = \omega_x(g)$. Fix $\pi \in \Pi_\phi$ and $x \in \mathcal{S}^\Sigma_{\underline{\phi}}$, we denote the image of x in $\widehat{\Sigma}$ by $\widehat{\theta}$; then $x \in \mathcal{S}^\theta_{\underline{\phi}}$. Let $\Sigma' = \langle\theta\rangle$ and $\pi^{\Sigma'} = \pi^+$; it follows from Lemma 3.13 that for any $g \in \widetilde{G}(F)$,

$$\langle x, (\pi^{\Sigma'})^g\rangle_{\underline{\phi}} = \varepsilon_g(x)\langle x, \pi^{\Sigma'}\rangle_{\underline{\phi}}.$$

On the other hand, we have from (2.6)

$$\langle \cdot, \pi^\Sigma\rangle_{\underline{\phi}}|_{\mathcal{S}^{\Sigma'}_{\underline{\phi}}} = \sum_{\pi^{\Sigma'}\in\Pi^{\Sigma'}_\phi} m(\pi^\Sigma, \pi^{\Sigma'})\langle \cdot, \pi^{\Sigma'}\rangle_{\underline{\phi}}.$$

Since $m((\pi^\Sigma)^g, (\pi^{\Sigma'})^g) = m(\pi^\Sigma, \pi^{\Sigma'})$,

$$\begin{aligned}\langle x, (\pi^\Sigma)^g\rangle_{\underline{\phi}} &= \sum_{\pi^{\Sigma'}\in\Pi^{\Sigma'}_\phi} m((\pi^\Sigma)^g, (\pi^{\Sigma'})^g)\langle x, (\pi^{\Sigma'})^g\rangle_{\underline{\phi}} \\ &= \sum_{\pi^{\Sigma'}\in\Pi^{\Sigma'}_\phi} m(\pi^\Sigma, \pi^{\Sigma'})\varepsilon_g(x)\langle x, \pi^{\Sigma'}\rangle_{\underline{\phi}} \\ &= \varepsilon_g(x)\langle x, \pi^\Sigma\rangle_{\underline{\phi}}.\end{aligned}$$

As we vary $\pi \in \Pi_\phi$ and $x \in \mathcal{S}^\Sigma_{\underline{\phi}}$, this equality still holds. Therefore, we have proved the theorem.

3.6 Proof of Conjecture 2.2

In this section, we want to show that Conjecture 2.2 is a special case of Theorem 2.6. The main step is to clarify the three ingredients in defining the homomorphism $G_{\mathrm{ad}}(F) \to \mathcal{S}^*_{\underline{\phi}}$ in the statement of the conjecture. First, we need to recall the construction of z-extension. It is a consequence of the following more general construction.

PROPOSITION 3.14. *Suppose that F is a field of characteristic zero and G and G' are reductive groups over F. If G' is semisimple and there is a covering $G' \to G_{\mathrm{der}}$, then there exists a central extension of G*

$$1 \longrightarrow Z \longrightarrow \widetilde{G}' \longrightarrow G \longrightarrow 1$$

such that:

- $\widetilde{G}'_{\mathrm{der}} = G'$;
- *the projection $\widetilde{G}'_{\mathrm{der}} \to G_{\mathrm{der}}$ coincides with $G' \to G_{\mathrm{der}}$;*
- *Z is an induced torus; in particular, $H^1(F, Z) = 1$.*

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Remark 3.15. When $G' = G_{\mathrm{sc}}$, $\widetilde{G}'$ is the usual z-extension of G. For the proof of this proposition, we refer the reader to [MS82, Proposition 3.1] and [Lan79].

Now we want to construct the Tate local duality for nonabelian reductive groups. Let F be a local field of characteristic zero and G be a connected reductive group over F. Let $G' = G/Z_G^0$; then $Z_{G'} = Z_G/Z_G^0$. We apply Proposition 3.14 to the natural projection $G' \to G_{\mathrm{ad}}$, and we get an extension $\widetilde{G}'$ of G_{ad}

$$1 \longrightarrow Z \longrightarrow \widetilde{G}' \longrightarrow G_{\mathrm{ad}} \longrightarrow 1$$

such that $\widetilde{G}'_{\mathrm{der}} = G'$ and $H^1(F, Z) = 1$. Moreover, $Z = Z_{\widetilde{G}'}$; we denote $\widetilde{G}'/G'$ by D. Consider the exact sequence

$$1 \longrightarrow G' \longrightarrow \widetilde{G}' \xrightarrow{\lambda} D \longrightarrow 1.$$

The restriction to the centres gives

$$1 \longrightarrow Z_{G'} \longrightarrow Z_{\widetilde{G}'} \xrightarrow{\lambda} D \longrightarrow 1$$

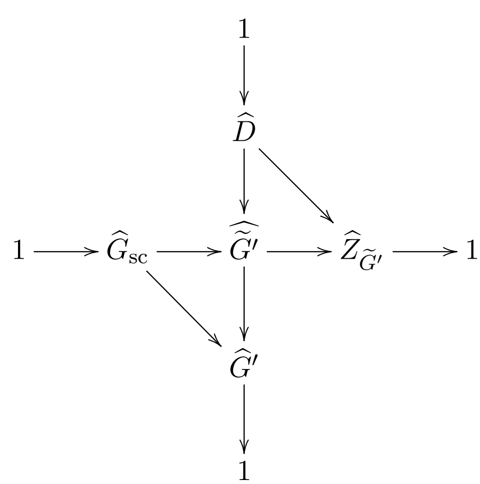
and it induces the following exact sequence:

$$Z_{\widetilde{G}'}(F) \xrightarrow{\lambda} D(F) \longrightarrow H^1(F, Z_{G'}) \longrightarrow H^1(F, Z_{\widetilde{G}'}) = 1.$$

Therefore,

$$H^1(F, Z_{G'}) = D(F)/\mathrm{Im}(Z_{\widetilde{G}'}(F) \xrightarrow{\lambda} D(F)).$$

On the other hand, one considers the following diagram.



Note that $\pi_1(\widehat{G'}) = \widehat{G}_{\mathrm{sc}} \cap \widehat{D}$ and $\widehat{G'} \cong \widehat{G}_{\mathrm{der}}$, so we get a short exact sequence

$$1 \longrightarrow \pi_1(\widehat{G}_{\mathrm{der}}) \longrightarrow \widehat{D} \longrightarrow \widehat{Z}_{\widetilde{G}'} \longrightarrow 1.$$

This induces the following exact sequence:

$$\pi_0(\widehat{Z}_{\widetilde{G}'}^{\Gamma}) \longrightarrow H^1(F, \pi_1(\widehat{G}_{\mathrm{der}})) \longrightarrow H^1(F, \widehat{D}) \longrightarrow H^1(F, \widehat{Z}_{\widetilde{G}'}).$$

By the Tate–Nakayama duality for tori (see [Kot84, § 3.3, equation (3.3.1)]), we have $\pi_0(\widehat{Z}^{\Gamma}_{\widetilde{G}'})^* = H^1(F, Z_{\widetilde{G}'}) = 1$ and hence $\pi_0(\widehat{Z}^{\Gamma}_{\widetilde{G}'}) = 1$. Therefore,

$$H^1(F, \pi_1(\widehat{G}_{\mathrm{der}})) = \mathrm{Ker}(H^1(F, \widehat{D}) \to H^1(F, \widehat{Z}_{\widetilde{G}'})).$$

It also follows from the Tate–Nakayama duality for tori that $H^1(F, \widehat{D})$ (respectively $H^1(F, \widehat{Z}_{\widetilde{G}'})$) is canonically isomorphic to the group of continuous characters of finite orders on $D(F)$ (respectively $Z_{\widetilde{G}'}(F)$) (see [Kot84, § 3,3, equation (3.3.2)]). Since $\mathrm{Im}(Z_{\widetilde{G}'}(F) \xrightarrow{\lambda} D(F))$ has finite index in $D(F)$, $\mathrm{Ker}(H^1(F, \widehat{D}) \to H^1(F, \widehat{Z}_{\widetilde{G}'}))$ is isomorphic to characters of $D(F)$ that are trivial on $\mathrm{Im}(Z_{\widetilde{G}'}(F) \xrightarrow{\lambda} D(F))$, and this is exactly the dual of $D(F)/\mathrm{Im}(Z_{\widetilde{G}'}(F) \xrightarrow{\lambda} D(F))$. Hence, we get a perfect pairing

$$H^1(F, Z_{G'}) \times H^1(F, \pi_1(\widehat{G}_{\mathrm{der}})) \to \mathbb{C}^\times. \tag{3.18}$$

The fact that this pairing is independent of the choice of extension with respect to $G' \to G_{\mathrm{ad}}$ is because of the following proposition.

Proposition 3.16. (i) *If there is another extension*

$$1 \longrightarrow Z_1 \longrightarrow \widetilde{G}'_1 \longrightarrow G_{\mathrm{ad}} \longrightarrow 1$$

dominating the original extension, i.e., there is a surjection $\widetilde{G}'_1 \to \widetilde{G}'$ *such that the following diagram commutes:*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & Z_1 & \longrightarrow & \widetilde{G}'_1 & \longrightarrow & G_{\mathrm{ad}} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & Z & \longrightarrow & \widetilde{G}' & \longrightarrow & G_{\mathrm{ad}} & \longrightarrow & 1 \end{array}$$

then the pairing (3.18) obtained from this extension is the same as the original one.

(ii) *If there are two extensions*

$$1 \longrightarrow Z_i \longrightarrow \widetilde{G}'_i \longrightarrow G_{\mathrm{ad}} \longrightarrow 1 \quad (i = 1, 2),$$

then one can find a third extension which dominates both of them.

The proof of part (i) is straightforward and we leave it to the reader. The proof of part (ii) can be found in [Kot82, Lemma 1.1].

Since $H^1(F, Z_{\widetilde{G}'}) = 1$, $G_{\mathrm{ad}}(F) = \widetilde{G}'(F)/Z_{\widetilde{G}'}(F)$ and

$$G_{\mathrm{ad}}(F) = \widetilde{G}'(F)/Z_{\widetilde{G}'}(F) \xrightarrow{\ \lambda\ } D(F)/\mathrm{Im}(Z_{\widetilde{G}'}(F) \xrightarrow{\lambda} D(F)) = H^1(F, Z_{G'})$$

defines the homomorphism $G_{\mathrm{ad}}(F) \to H^1(F, Z_{G'})$ in the introduction. Just like the Tate local duality pairing, one can show that this homomorphism is independent of the choice of extension with respect to $G' \to G_{\mathrm{ad}}$.

Finally, if $\phi \in \Phi_{\mathrm{bdd}}(G)$, we have defined a homomorphism $\delta : S_{\underline{\phi}} \to H^1(W_F, \widehat{D})$ (see (3.3)). By Lemma 3.3, the image of δ is finite. So, δ factors through $A_{\underline{\phi}}$ and the image lies in $H^1(F, \widehat{D})$. Moreover, we claim that the image of δ lies in $\mathrm{Ker}(H^1(F, \widehat{D}) \to H^1(F, \widehat{Z}_{\widetilde{G}'}))$. In fact, this follows

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from the proof of Lemma 3.3. For the convenience of the reader, we repeat that argument here. For $s \in S_{\underline{\phi}}$, let $\widetilde{s}$ be a preimage of s in $\widehat{\widetilde{G}'}$. Recall that

$$\delta(s) : u \mapsto \widetilde{s}\tilde{\phi}(u)\widetilde{s}^{-1}\tilde{\phi}(u)^{-1} = \widetilde{s}\sigma_u(\widetilde{s}^{-1}) \cdot \underbrace{\sigma_u(\widetilde{s})\tilde{\phi}(u)\widetilde{s}^{-1}\tilde{\phi}(u)^{-1}}_{\in \widehat{G}_{\mathrm{sc}}}$$

for $u \in L_F$ and σ_u is the image of u in Γ. Note that the decomposition of $\delta(s)(u)$ factors through Γ. Then our claim follows from the following diagram:

$$\begin{array}{ccccc} & & H^1(F, \widehat{D}) & & \\ & & \downarrow & \searrow & \\ H^1(F, \widehat{G}_{\mathrm{sc}}) & \longrightarrow & H^1(F, \widehat{\widetilde{G}'})) & \longrightarrow & H^1(F, \widehat{Z}_{\widetilde{G}'}) \end{array}$$

So, we obtain a homomorphism $\delta : A_{\underline{\phi}} \to H^1(F, \pi_1(\widehat{G}_{\mathrm{der}}))$.

From the construction above, we obtain a homomorphism $G_{\mathrm{ad}}(F) \to A^*_{\underline{\phi}}$, which sends g to η_g. It is easy to check that

$$\eta_g(s) = \alpha(x)(\widetilde{g})$$

for $s \in A_{\underline{\phi}}$ with image $x \in \mathcal{S}_\phi$, and $\widetilde{g} \in \widetilde{G}'(F)$ with image $g \in G_{\mathrm{ad}}(F)$. As a consequence, $\eta_g \in \mathcal{S}^*_{\underline{\phi}}$ and Conjecture 2.2 follows from Theorem 2.6 immediately.

4. Central character

For $\phi \in \Phi(G)$, one can associate a character χ_ϕ of $Z_G(F)$ as in the introduction. By Proposition 3.12, we see that the central characters of elements in Π_ϕ are the same. So, we can talk about the central character of an L-packet, and we would like to show that it is equal to χ_ϕ. By the construction of χ_ϕ and also nontempered L-packets, we see that it suffices to prove this for $\phi \in \Phi_{\mathrm{bdd}}(G)$. Note that if ϕ is simple, Π_ϕ contains only one element and we would like to assume that the central character of Π_ϕ is χ_ϕ. Then it is enough to check how χ_ϕ and the central character of representations change with respect to the endoscopic transfer.

Lemma 4.1. *Let $\phi \in \Phi_{\mathrm{bdd}}(G)$ and $s \in \bar{S}_{\underline{\phi}}$. Suppose that, for any $(H_1, \underline{\phi}_{H_1}) \to (\underline{\phi}, s)$, the central character of $\Pi_{\phi_{H_1}}$ is $\chi_{\phi_{H_1}}$; then the central character of Π_ϕ is χ_ϕ.*

Proof. First, we want to reduce to the case $\mathcal{H} = {}^L H$. To do so, we can simply take a z-extension of G

$$1 \longrightarrow Z_1 \longrightarrow G_1 \longrightarrow G \longrightarrow 1$$

and denote the image of ϕ in $\Phi_{\mathrm{bdd}}(G_1)$ by ϕ_1. Note that $(G_1)_{\mathrm{der}} = G_{\mathrm{sc}}$; then, by a result of Langlands (see [Lan79, Proposition 1]), $\underline{\phi}_1$ factors through an endoscopic datum $(H_1, {}^L H_1, s, \xi_1)$. This gives a natural embedding $\xi_{H_1} : \mathcal{H} \to {}^L H_1$ and a z-extension

$$1 \longrightarrow Z_1 \longrightarrow H_1 \longrightarrow H \longrightarrow 1.$$

Therefore, (H_1, ξ_{H_1}) is a z-pair for $\mathcal{H}$. By our assumption, $\chi_{\phi_{H_1}}$ is the central character of $\Pi_{\phi_{H_1}}$. If we can show that χ_{ϕ_1} is the central character of Π_{ϕ_1}, then the same is true for χ_ϕ.

From now on, we assume that $\mathcal{H} = {}^L H$ and we take $H_1 = H$. By the definition of χ_ϕ, we need to take a torus Z containing the centre Z_G of G and form $\widetilde{G} = (G \times Z)/Z_G$. Then H can be lifted to a twisted endoscopic group $\widetilde{H}$ of $\widetilde{G}$. Let $\underline{\tilde{\phi}}_H$ be a lift of $\underline{\phi}_H$; it gives a lift $\underline{\tilde{\phi}}$ of $\underline{\phi}$. Then $\chi_{\phi_H} = \chi_{\tilde{\phi}_H}|_{Z_H}$ and $\chi_\phi = \chi_{\tilde{\phi}}|_{Z_G}$. Note that $\chi_{\tilde{\phi}} = \chi_{\tilde{\xi}} \cdot (\chi_{\tilde{\phi}_H}|_{Z_{\widetilde{G}}})$, where $\chi_{\tilde{\xi}}$ is dual to

$$W_F \xrightarrow{\tilde{\xi}|_{W_F}} {}^L\widetilde{G} \to {}^L Z_{\widetilde{G}}.$$

On the other hand, the central character of Π_ϕ only differs from the restriction of that of Π_{ϕ_H} to Z_G by χ_C^{-1}. This follows from our proof of Proposition 3.12 by taking $C = C_1 = Z_G$. Since $\chi_C = \chi_{\widetilde{C}}|_{Z_G}$, it is enough to show that $\chi_{\tilde{\xi}} = \chi_{\widetilde{C}}^{-1}$. To give the definition of $\chi_{\widetilde{C}}$, we need to fix Γ-splittings $(\widetilde{\mathcal{B}}_{\widetilde{H}}, \widetilde{\mathcal{T}}_{\widetilde{H}}, \{\mathcal{X}_{\alpha_{\widetilde{H}}}\})$ and $(\widetilde{\mathcal{B}}, \widetilde{\mathcal{T}}, \{\mathcal{X}_\alpha\})$ for both $\widehat{\widetilde{H}}$ and $\widehat{\widetilde{G}}$. By taking a certain $\widehat{\widetilde{G}}$-conjugate of $\tilde{\xi}$, we can assume that $\tilde{\xi}(\widetilde{\mathcal{T}}_{\widetilde{H}}) = \widetilde{\mathcal{T}}$ and $\widetilde{\mathcal{B}}_{\widetilde{H}} \subseteq \widetilde{\mathcal{B}}$. We also choose a maximal torus $\widetilde{T}_{\widetilde{H}}$ of $\widetilde{H}$ defined over F, and choose an admissible embedding $\widetilde{T}_{\widetilde{H}} \to \widetilde{T}$ together with χ-data on $R(\widetilde{G}, \widetilde{T})$. The admissible embedding identifies ${}^L\widetilde{T}_{\widetilde{H}}$ with ${}^L\widetilde{T}$, and transports χ-data from $R(\widetilde{G}, \widetilde{T})$ to $R(\widetilde{H}, \widetilde{T}_{\widetilde{H}})$. The χ-data give embeddings $\xi_{\widetilde{T}_{\widetilde{H}}} : {}^L\widetilde{T}_{\widetilde{H}} \to {}^L\widetilde{H}$ and $\xi_{\widetilde{T}} : {}^L\widetilde{T} \to {}^L\widetilde{G}$. Then there exists a 1-cocycle $a_{\widetilde{T}}$ of W_F in $\widehat{\widetilde{\mathcal{T}}}$ with transported Galois action from $\widetilde{T}$ such that

$$\tilde{\xi} \circ \xi_{\widetilde{T}_{\widetilde{H}}} = a_{\widetilde{T}} \cdot \xi_{\widetilde{T}}, \tag{4.1}$$

and $\chi_{\widetilde{C}}^{-1}$ is dual to

$$W_F \xrightarrow{a_{\widetilde{T}}} \widehat{\widetilde{\mathcal{T}}} \to {}^L\widetilde{G} \to {}^L Z_{\widetilde{G}}.$$

By the constructions of $\xi_{\widetilde{T}_{\widetilde{H}}}$ and $\xi_{\widetilde{T}}$ (see [LS87, § 2.5]), one can see that $\xi_{\widetilde{T}_{\widetilde{H}}}(W_F) \subseteq \widehat{\widetilde{H}}_{\mathrm{der}} \rtimes W_F$ and $\xi_{\widetilde{T}}(W_F) \subseteq \widehat{\widetilde{G}}_{\mathrm{der}} \rtimes W_F$, so if we restrict both sides of (4.1) to W_F and compose them with ${}^L\widetilde{G} \to {}^L Z_{\widetilde{G}}$, we get an equality for the duals of $\chi_{\tilde{\xi}}$ and $\chi_{\widetilde{C}}^{-1}$. Therefore, $\chi_{\tilde{\xi}} = \chi_{\widetilde{C}}^{-1}$. □

It is clear that this lemma implies Proposition 2.4.

5. Twist by automorphism and quasicharacter

Let θ be an automorphism of G preserving an F-splitting. Let $\mathbf{a}$ be an element in $H^1(W_F, Z(\widehat{G}))$, which is associated with a quasicharacter ω of $G(F)$. In this section, we want to prove Proposition 2.7.

Lemma 5.1. *Suppose that $\phi \in \Phi_{\mathrm{bdd}}(G)$ is not simple; then $\Pi_{\phi^\theta} = \Pi_\phi^\theta$. Moreover,*

$$\langle x, \pi^\theta \rangle_{\underline{\phi}^\theta} = \langle \widehat{\theta}^{-1} x \widehat{\theta}, \pi \rangle_{\underline{\phi}}$$

for any $\pi \in \Pi_\phi$ and $x \in \mathcal{S}_{\underline{\phi}^\theta}$.

Proof. For $s \in \bar{S}_{\underline{\phi}}$, we assume that $(H_1, \underline{\phi}_{H_1}) \to (\underline{\phi}, s)$ with respect to $(H, \mathcal{H}, s, \xi)$ and the z-pair (H_1, ξ_{H_1}). Then we have $(H_1, \underline{\phi}_{H_1}) \to (\underline{\phi}^\theta, \widehat{\theta} s \widehat{\theta}^{-1})$ with respect to $(H, \mathcal{H}, \widehat{\theta} s \widehat{\theta}^{-1}, \xi^\theta)$ and the same z-pair (H_1, ξ_{H_1}). To make a distinction, we denote the transfer factor with respect to $(H, \mathcal{H}, \widehat{\theta} s \widehat{\theta}^{-1}, \xi^\theta)$ by $\Delta_{G, H_1'}$, and the transfer by $f^{H_1'}$ for $f \in C_c^\infty(G(F))$. Note that $f^{H_1'}$ is defined on $H_1(F)$.

If $f \in C_c^\infty(G(F))$, we can choose the transfers so that they satisfy

$$(f^\theta)^{H'_1} = f^{H_1}. \tag{5.1}$$

To see this, let γ_1 be a semisimple strongly G-regular element of $H_1(F)$ and let T_{H_1} be the centralizer of γ_1. Let T_H be the projection of T_{H_1} on H and $\gamma \in H(F)$ be the image of γ_1. We fix an admissible embedding $T_H \to T$ with respect to $(H, \mathcal{H}, s, \xi)$ and denote the image of γ by δ. Then the admissible embedding of T_H with respect to $(H, \mathcal{H}, \widehat{\theta} s \widehat{\theta}^{-1}, \xi^\theta)$ becomes the composition of

$$T_H \to T \xrightarrow{\theta^{-1}} \theta^{-1}(T).$$

This is because the endoscopic embedding ξ changes to ξ^θ. Note that γ maps to $\theta^{-1}(\delta)$ under this admissible embedding. Then

$$\begin{aligned} SO_{H_1}((f^\theta)^{H'_1}, \gamma_1) &= \sum_{\{\delta'\}_{G(F)} \sim_{st} \{\delta\}_{G(F)}} \Delta_{G,H'_1}(\gamma_1, \theta^{-1}(\delta')) O_G(f^\theta, \theta^{-1}(\delta')) \\ &= \sum_{\{\delta'\}_{G(F)} \sim_{st} \{\delta\}_{G(F)}} \Delta_{G,H'_1}(\gamma_1, \theta^{-1}(\delta')) O_G(f, \delta'). \end{aligned}$$

By the definition of transfer factors, one can check that

$$\Delta_{G,H'_1}(\gamma_1, \theta^{-1}(\delta')) = \Delta_{G,H_1}(\gamma_1, \delta'),$$

so we have

$$SO_{H_1}((f^\theta)^{H'_1}, \gamma_1) = SO_{H_1}(f^{H_1}, \gamma_1).$$

It follows from (5.1) that

$$f^{H_1}(\underline{\phi}_{H_1}) = (f^\theta)^{H'_1}(\underline{\phi}_{H_1}).$$

Now we can expand both sides by the endoscopic character identities:

$$f^{H_1}(\underline{\phi}_{H_1}) = \sum_{\pi \in \Pi_\phi} \langle x, \pi \rangle_{\underline{\phi}} f_G(\pi)$$

and

$$\begin{aligned} (f^\theta)^{H'_1}(\underline{\phi}_{H_1}) &= \sum_{\pi \in \Pi_{\phi^\theta}} \langle \widehat{\theta} x \widehat{\theta}^{-1}, \pi \rangle_{\underline{\phi}^\theta} f^\theta_G(\pi) \\ &= \sum_{\pi \in \Pi_{\phi^\theta}} \langle \widehat{\theta} x \widehat{\theta}^{-1}, \pi \rangle_{\underline{\phi}^\theta} f_G(\pi^{\theta^{-1}}), \end{aligned}$$

where x is the image of s in $\mathcal{S}_{\underline{\phi}}$. By linear independence of characters, for any $\pi' \in \Pi_{\phi^\theta}$, there exists $\pi \in \Pi_\phi$ such that $\pi^\theta \cong \pi'$. This shows that $\Pi^\theta_\phi = \Pi_{\phi^\theta}$. Moreover,

$$\langle x, \pi \rangle_{\underline{\phi}} = \langle \widehat{\theta} x \widehat{\theta}^{-1}, \pi' \rangle_{\underline{\phi}^\theta} = \langle \widehat{\theta} x \widehat{\theta}^{-1}, \pi^\theta \rangle_{\underline{\phi}^\theta}.$$

Let $x' = \widehat{\theta} x \widehat{\theta}^{-1} \in \mathcal{S}_{\underline{\phi}^\theta}$; then we get $\langle \widehat{\theta}^{-1} x' \widehat{\theta}, \pi \rangle_{\underline{\phi}} = \langle x', \pi^\theta \rangle_{\underline{\phi}^\theta}$. □

Lemma 5.2. *Suppose that $\phi \in \Phi_{\mathrm{bdd}}(G)$ is not simple; then $\Pi_{\phi \otimes \mathbf{a}} = \Pi_\phi \otimes \omega$. Moreover,*

$$\langle x, \pi \otimes \omega \rangle_{\underline{\phi} \otimes \underline{\mathbf{a}}} = \langle x, \pi \rangle_{\underline{\phi}}$$

for any $\pi \in \Pi_\phi$ and $x \in \mathcal{S}_{\underline{\phi}} = \mathcal{S}_{\underline{\phi} \otimes \underline{\mathbf{a}}}$, where $\underline{\mathbf{a}}$ is a 1-cocycle of W_F in $Z(\widehat{G})$ representing $\mathbf{a}$.

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Proof. For $s \in \bar{S}_{\phi}$, we assume that $(H_1, \underline{\phi}_{H_1}) \to (\underline{\phi}, s)$ with respect to $(H, \mathcal{H}, s, \xi)$ and the z-pair (H_1, ξ_{H_1}). Then we have $(H'_1, \underline{\phi}_{H_1}) \to (\underline{\phi} \otimes \underline{\mathbf{a}}, s)$ with respect to $(H, \mathcal{H}, s, \xi \otimes \underline{\mathbf{a}})$ and the same z-pair (H_1, ξ_{H_1}). To make a distinction, we denote the transfer factor with respect to $(H, \mathcal{H}, s, \xi \otimes \underline{\mathbf{a}})$ by Δ_{G,H'_1}, and the transfer by $f^{H'_1}$ for $f \in C_c^\infty(G(F))$. Note that $f^{H'_1}$ is defined on $H_1(F)$.

If $f \in C_c^\infty(G(F))$, one can choose the transfers so that they satisfy

$$(f \otimes \omega^{-1})^{H'_1} = f^{H_1}. \tag{5.2}$$

To see this, let γ_1 be a semisimple strongly G-regular element of $H_1(F)$ and let T_{H_1} be the centralizer of γ_1. Let T_H be the projection of T_{H_1} on H and $\gamma \in H(F)$ be the image of γ_1. We fix an admissible embedding $T_H \to T$ with respect to $(H, \mathcal{H}, s, \xi)$ and denote the image of γ by δ. Then the admissible embedding of T_H with respect to $(H, \mathcal{H}, s, \xi \otimes \underline{\mathbf{a}})$ is the same. We have

$$\begin{aligned} SO_{H_1}((f \otimes \omega^{-1})^{H'_1}, \gamma_1) &= \sum_{\{\delta'\}_{G(F)} \sim_{st} \{\delta\}_{G(F)}} \Delta_{G,H'_1}(\gamma_1, \delta') O_G(f \otimes \omega^{-1}, \delta') \\ &= \sum_{\{\delta'\}_{G(F)} \sim_{st} \{\delta\}_{G(F)}} \Delta_{G,H'_1}(\gamma_1, \delta') \omega^{-1}(\delta') O_G(f, \delta'). \end{aligned}$$

Moreover, we have

$$\Delta_{G,H'}(\gamma, \delta') = \omega(\delta') \Delta_{G,H}(\gamma, \delta').$$

In fact, this difference between transfer factors only comes from Δ_{III} or more precisely Δ_2 (see [LS87, § 3.5] for its definition). Therefore,

$$SO_{H_1}((f \otimes \omega^{-1})^{H'_1}, \gamma_1) = SO_{H_1}(f^{H_1}, \gamma_1).$$

It follows from (5.2) that

$$f^{H_1}(\underline{\phi}_{H_1}) = (f \otimes \omega^{-1})^{H'_1}(\underline{\phi}_{H_1}).$$

Now we can expand both sides by the endoscopic character identities:

$$f^{H_1}(\underline{\phi}_{H_1}) = \sum_{\pi \in \Pi_\phi} \langle x, \pi \rangle_{\underline{\phi}} f_G(\pi)$$

and

$$\begin{aligned} (f \otimes \omega^{-1})^{H'_1}(\underline{\phi}_{H_1}) &= \sum_{\pi \in \Pi_{\phi \otimes \mathbf{a}}} \langle x, \pi \rangle_{\underline{\phi} \otimes \underline{\mathbf{a}}} (f \otimes \omega^{-1})_G(\pi) \\ &= \sum_{\pi \in \Pi_{\phi \otimes \mathbf{a}}} \langle x, \pi \rangle_{\underline{\phi} \otimes \underline{\mathbf{a}}} f_G(\pi \otimes \omega^{-1}), \end{aligned}$$

where x is the image of s in $\mathcal{S}_{\underline{\phi}} = \mathcal{S}_{\underline{\phi} \otimes \underline{\mathbf{a}}}$. By linear independence of characters, for any $\pi' \in \Pi_{\phi \otimes \mathbf{a}}$, there exists $\pi \in \Pi_\phi$ such that $\pi' \otimes \omega^{-1} \cong \pi$. This implies that $\Pi_{\phi \otimes \mathbf{a}} = \Pi_\phi \otimes \omega$. Furthermore,

$$\langle x, \pi \otimes \omega \rangle_{\underline{\phi} \otimes \underline{\mathbf{a}}} = \langle x, \pi' \rangle_{\underline{\phi} \otimes \underline{\mathbf{a}}} = \langle x, \pi \rangle_{\underline{\phi}}.$$

This finishes the proof. □

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6. Lifting L-packet

Let $G \subseteq \widetilde{G}$ be two quasisplit connected reductive groups over F such that $G_{\mathrm{der}} = \widetilde{G}_{\mathrm{der}}$; we denote $\widetilde{G}/G$ by D. Suppose that $\tilde{\phi} \in \Phi_{\mathrm{bdd}}(\widetilde{G})$ and ϕ is the image of $\tilde{\phi}$ under $\Phi_{\mathrm{bdd}}(\widetilde{G}) \to \Phi_{\mathrm{bdd}}(G)$; then it is conjectured that $\Pi_{\tilde{\phi}}|_G = \Pi_{\phi}$. The problem we want to study is to what extent one can understand the L-packet of $\widetilde{G}$ from that of G. Therefore, we will only assume the endoscopic hypothesis (i.e., (2.1) and Conjectures 2.5 and 3.10) for G and all its twisted endoscopic groups. To be more precise, this will be our working assumption in §§ 6.2–6.4. It follows that the previous results that we have proved about the desiderata of L-packets are valid for G.

6.1 Representation-theoretic preparation

We start by investigating the restriction multi-map $\Pi(\widetilde{G}(F)) \to \Pi(G(F))$. Similar discussions of this restriction multi-map can also be found in [LL79], [HS12] and [GK82].

Lemma 6.1. *If $\tilde{\pi}$ is an irreducible smooth representation of $\widetilde{G}(F)$, then the restriction of $\tilde{\pi}$ to $G(F)$ is a direct sum of finitely many irreducible smooth representations.*

Proof. Since $\tilde{\pi}$ has a central character $\chi_{\tilde{\pi}}$, it is enough to show that the restriction of $\tilde{\pi}$ to $Z_{\widetilde{G}}(F)G(F)$ is a direct sum of finitely many irreducible smooth representations. Note that $|D(F) : \lambda(Z_{\widetilde{G}}(F))|$ is finite, so the index $|\widetilde{G}(F) : Z_{\widetilde{G}}(F)G(F)| = |\lambda(\widetilde{G}(F)) : \lambda(Z_{\widetilde{G}}(F))| < |D(F) : \lambda(Z_{\widetilde{G}}(F))|$ is also finite. Then this lemma follows from the following algebraic result. □

Lemma 6.2. *Let G and H be two groups such that H is a normal subgroup of G and G/H is finite.*

(i) *If $\tilde{\pi}$ is an irreducible representation of G, then the restriction of $\tilde{\pi}$ to H is a direct sum of finitely many irreducible representations.*

(ii) *If π is an irreducible representation of H, then there exists an irreducible representation $\tilde{\pi}$ of G which contains π in its restriction to H.*

Proof. (i) Let $g_1, g_2, \ldots, g_r$ be the representatives of G/H and $g_1 = 1$. Let us assume that the restriction of $\tilde{\pi}$ to H is reducible. We first need to show that there exists a direct sum decomposition of the representation space $V = V(\tilde{\pi}|_H) = \bigoplus_{i=1}^{l} \tilde{\pi}(g_{v_i})W$ for some proper H-invariant subspace W and $1 \leqslant v_i \leqslant r$. Suppose that there exists a direct sum $0 \neq \bigoplus_{i=1}^{l} \tilde{\pi}(g_{v_i}) W \subsetneq V$; then $\bigcap_{k=1}^{r} \bigoplus_{i=1}^{l} \tilde{\pi}(g_k g_{v_i})W = 0$ for it is invariant under G, but not equal to V. Hence, we can choose $\{k_1, k_2, \ldots, k_m\} \subseteq \{1, 2, \ldots, r\}$ so that $W \bigcap_{j=1}^{m} \bigoplus_{i=1}^{l} \tilde{\pi}(g_{k_j} g_{v_i})W = 0$, but $W' = W \bigcap_{j=1}^{m-1} \bigoplus_{i}^{l} \tilde{\pi}(g_{k_j} g_{v_i})W \neq 0$. Here we let $W' = W$ if $m = 1$. Since $W' \cap \bigoplus_{i=1}^{l} \tilde{\pi}(g_{k_m} g_{v_i}) W = 0$, $W' + \bigoplus_{i=1}^{l} \tilde{\pi}(g_{k_m} g_{v_i})W'$ is again a direct sum. Note that we have increased the number of direct summands by 1. By repeating this argument, we will end up with a direct sum which is either the whole space V or equal to $\bigoplus_{i=1}^{r} \tilde{\pi}(g_i)W''$ with respect to some H-invariant subspace $0 \neq W'' \subseteq W$. In the latter case, it is again equal to V for it is invariant under G.

Now we can assume that there is a direct sum decomposition of $V = \bigoplus_{i=1}^{l} \tilde{\pi}(g_{v_i})W$ with respect to some W. Suppose that W is reducible; then there exists an H-invariant subspace W' in W and $\bigoplus_{i=1}^{l} \tilde{\pi}(g_{v_i})W' \neq V$. This implies that $l < r$. Hence, W must be irreducible if $l = r$. In case $l < r$, we can apply the argument in the previous paragraph and find that W'' in W', so that $V = \bigoplus_{i=1}^{m} \tilde{\pi}(g_{v_i})W''$ and $m > l$. If W'' is reducible, we can repeat this argument until either we get an irreducible subrepresentation in which case the proof is done, or we decompose V into a direct sum of r subspaces. In the latter case, it is clear now that each subspace has to be irreducible. Therefore, $\tilde{\pi}$ can be decomposed into a finite direct sum of irreducible H-representations. Moreover, it is easy to see that the direct summands run over all the isomorphism classes of G-conjugates of any irreducible representation π contained in $\tilde{\pi}|_H$.

(ii) Let $\tilde{\pi}$ be any irreducible representation of G; from Frobenius reciprocity, we have

$$\mathrm{Hom}_H(\,\mathrm{Res}_H^G\tilde{\pi},\pi\,) \cong \mathrm{Hom}_G(\tilde{\pi},\mathrm{Ind}_H^G\pi).$$

Then it is easy to see from part (i) of the lemma that $\tilde{\pi}$ contains π in its restriction to H if and only if $\tilde{\pi}$ is a subrepresentation of $\sigma = \mathrm{Ind}_H^G\pi$. So, it is enough to show that σ has an irreducible subrepresentation. Note that $\sigma|_H = \bigoplus_{i=1}^r \pi^{g_i}$, so we have projections $p_i : V(\sigma) \to V(\pi^{g_i})$. If W is a G-invariant subspace of $V(\sigma)$, we are going to define a sequence of subspaces as follows. Let $W_1 = W, W_2 = \mathrm{Ker}\, p_1|_{W_1}, W_3 = \mathrm{Ker}\, p_2|_{W_2}, \ldots, W_r = \mathrm{Ker}\, p_{r-1}|_{W_{r-1}}$ and $W_{r+1} = 0$. Then we have

$$0 = W_{r+1} \subseteq W_r \subseteq W_{r-1} \subseteq \cdots \subseteq W_1 = W,$$

where $W_i/W_{i+1} \simeq \pi^{g_i}$ or 0 for $1 \leqslant i \leqslant r$. In particular, there exists a unique sequence of integers $r \geqslant s_m > s_{m-1} > \cdots > s_1 \geqslant 1$ such that

$$0 \subsetneq W_{s_m} \subsetneq W_{s_{m-1}} \subsetneq \cdots \subsetneq W_{s_1} = W$$

with $W_{s_i}/W_{s_{i+1}} \simeq \pi^{g_{s_i}}$ for $1 \leqslant i \leqslant m-1$ and $W_{s_m} = \pi^{g_m}$. We call $m = m(W)$ the length of W. Now let us take a proper G-invariant subspace W of minimal length; then W has to be irreducible. Otherwise, there exists another G-invariant subspace $W' \subsetneq W$ and, if $\{s_1, s_2, \ldots, s_m\}$ is associated with W, then $W'_{s_i}/W'_{s_{i+1}} \subseteq W_{s_i}/W_{s_{i+1}} \simeq \pi^{g_{s_i}}$. From here, we see that $m(W') \leqslant m(W)$ and hence $m(W') = m(W)$. This means that $W'_{s_m} = W_{s_m}$ and $W'_{s_i}/W'_{s_{i+1}} = W_{s_i}/W_{s_{i+1}}$. Therefore, $W' = W$. □

As an immediate consequence of part (ii) of this lemma, we have the following corollary.

Corollary 6.3. *If π is an irreducible smooth representation of $G(F)$, then there exists an irreducible smooth representation $\tilde{\pi}$ of $\widetilde{G}(F)$ which contains π in its restriction to $G(F)$. In particular, the central character χ_π can be extended to a character of $Z_{\widetilde{G}}(F)$.*

Proof. Let $\widetilde{Z}_F$ be a closed subgroup of $Z_{\widetilde{G}}(F)$ such that $\widetilde{Z}_F \cap Z_G(F) = 1$ and $D(F)/\lambda(\widetilde{Z}_F)$ is finite. Then we can extend π to $\widetilde{Z}_F G(F)$ through the trivial character on $\widetilde{Z}_F$. Since $\widetilde{G}(F)/\widetilde{Z}_F G(F)$ is finite, the existence of $\tilde{\pi}$ follows from Lemma 6.2 directly, and moreover its central character $\chi_{\tilde{\pi}}$ extends χ_π.

The closed subgroup $\widetilde{Z}_F$ can be constructed as follows. We first choose an F-subtorus C of $Z^0_{\widetilde{G}}$ such that $Z^0_{\widetilde{G}} = CZ^0_G$ and $C \cap Z^0_G$ is finite. It is easy to see that $\lambda(C(F))$ has finite index in $D(F)$ and $|C(F) \cap Z_G(F)|$ is finite. Next, we choose an integer m such that $\widetilde{Z}_F := \{x^m : x \in C(F)\}$ has no torsion points. Then $\widetilde{Z}_F \cap Z_G(F) = 1$ and $\lambda(\widetilde{Z}_F)$ also has finite index in $D(F)$. □

Reviewing part (ii) of Lemma 6.2, we see that the irreducible subrepresentations of $\mathrm{Ind}_H^G\pi$ give all the irreducible representations of G whose restriction to H contains π. So, it is interesting to determine the structure of $\mathrm{Ind}_H^G\pi$. This may not be easy in general, but when G/H is abelian and the irreducible representations of H satisfy Schur's lemma, one can actually compute the induction very explicitly. Especially, note that if $\widetilde{Z}_F$ is a closed subgroup of $Z_{\widetilde{G}}(F)$ such that

$D(F)/\lambda(\widetilde{Z}_F)$ is finite, $\widetilde{G}(F)/\widetilde{Z}_F G(F)$ is also abelian. So, now we are going to calculate $\mathrm{Ind}_H^G\pi$ under the assumption that G/H is abelian. In fact, we can take any sequence of normal subgroups

$$H = H_0 \subseteq H_1 \subseteq \cdots \subseteq H_i \subseteq \cdots \subseteq H_r = G$$

such that H_{i+1}/H_i is cyclic and of prime order. Then

$$\mathrm{Ind}_H^G\pi = \mathrm{Ind}_{H_{r-1}}^G \cdots \mathrm{Ind}_{H_i}^{H_{i+1}} \cdots \mathrm{Ind}_H^{H_1}\pi.$$

As we will see, for any irreducible representation σ of H_i, the induction $\mathrm{Ind}_{H_i}^{H_{i+1}}\sigma$ is always semisimple, so it is enough for us to consider the case when G/H is a cyclic group of prime order p. Let $g \in G$ be a generator of the cyclic group G/H and let us assume that $\pi^g \cong \pi$; then there exists an intertwining operator A of $V(\pi)$ such that for all $h \in H$, we have

$$A \circ \pi(h) = \pi(ghg^{-1}) \circ A.$$

So,

$$A^p \circ \pi(h) = \pi(g^p h g^{-p}) \circ A^p = \pi(g^p) \circ \pi(h) \circ \pi(g^p)^{-1} \circ A^p$$

and

$$(\pi(g^p)^{-1} \circ A^p) \circ \pi(h) = \pi(h) \circ (\pi(g^p)^{-1} \circ A^p).$$

Since π is irreducible, $\pi(g^p)^{-1} \circ A^p = cI$ for some constant c. After rescaling A, we can assume that $c = 1$ and hence $A^p = \pi(g^p)$. This shows that we can extend π to an irreducible representation $\tilde{\pi}$ of G by defining $\tilde{\pi}(g) = A$. In fact, if we change the scaling of A by a pth root of unity, we can get another extension $\tilde{\pi} \otimes \omega$ for some character ω of G/H. Let $\{\omega_i\}_{i=1}^p$ be all the characters of G/H; then it is easy to see that $\tilde{\pi} \otimes \omega_i$ are distinct for all $1 \leqslant i \leqslant p$. Our claim is that

$$\mathrm{Ind}_H^G\pi \cong \bigoplus_{i=1}^{p} \tilde{\pi} \otimes \omega_i. \tag{6.1}$$

To see this, we first get inclusions from $\tilde{\pi} \otimes \omega_i$ to $\mathrm{Ind}_H^G\pi$ for all $1 \leqslant i \leqslant p$ by Frobenius reciprocity. Then this gives a G-invariant homomorphism from $\bigoplus_{i=1}^p \tilde{\pi} \otimes \omega_i$ to $\mathrm{Ind}_H^G\pi$. Since $\tilde{\pi} \otimes \omega_i$ are distinct, this homomorphism must be injective. Otherwise, the image of some $\tilde{\pi} \otimes \omega_k$ will be contained in the image of $\bigoplus_{i \neq k} \tilde{\pi} \otimes \omega_i$, but that is impossible. The surjectivity will follow from a simple argument on the lengths of representations as defined in the proof of part (ii) of Lemma 6.2, when we restrict to H. Finally, if $\pi^g \ncong \pi$, then $\mathrm{Ind}_H^G\pi$ is irreducible because any irreducible subrepresentation of $\mathrm{Ind}_H^G\pi$ contains π^{g^i} for $1 \leqslant i \leqslant p$ in its restriction to H.

Next, we will give a formula for $\mathrm{Ind}_{\widetilde{Z}_F G(F)}^{\widetilde{G}(F)}\pi$, where π is an irreducible smooth representation of $G(F)$, which can be extended to $\widetilde{Z}_F G(F)$ through some quasicharacter $\widetilde{\chi}$ of $\widetilde{Z}_F$. Let us denote

$$\widetilde{G}(\pi) = \{g \in \widetilde{G}(F) : \pi^g \cong \pi\}.$$

Suppose that G_F^1 is a maximal subgroup of $\widetilde{G}(F)$, to which one can extend π. Note that such G_F^1 may not be unique. If we denote such an extension by π^1, then by (6.1) we have

$$\mathrm{Ind}_{\widetilde{Z}_F G(F)}^{G_F^1}\pi \cong \bigoplus_{\omega \in (G_F^1/\widetilde{Z}_F G(F))^*} \pi^1 \otimes \omega$$

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and

$$\mathrm{Ind}_{\widetilde{Z}_F G(F)}^{\widetilde{G}(F)} \pi \cong \mathrm{Ind}_{G_F^1}^{\widetilde{G}(F)} \mathrm{Ind}_{\widetilde{Z}_F G(F)}^{G_F^1} \pi \cong \bigoplus_{\omega \in (G_F^1/\widetilde{Z}_F G(F))^*} \mathrm{Ind}_{G_F^1}^{\widetilde{G}(F)} (\pi^1 \otimes \omega).$$

Note that $\mathrm{Ind}_{G_F^1}^{\widetilde{G}(F)}(\pi^1 \otimes \omega)$ is irreducible, so we can assume that $\tilde{\pi} \cong \mathrm{Ind}_{G_F^1}^{\widetilde{G}(F)} \pi^1$ by making a good choice of π^1. Now we want to count the multiplicities in the decomposition of $\mathrm{Ind}_{\widetilde{Z}_F G(F)}^{\widetilde{G}(F)} \pi$ above. Observe that $\mathrm{Ind}_{G_F^1}^{\widetilde{G}(F)} \pi^1 \cong \mathrm{Ind}_{G_F^1}^{\widetilde{G}(F)} (\pi^1 \otimes \omega)$ if and only if $(\pi^1)^g \cong \pi^1 \otimes \omega$ for some $g \in \widetilde{G}(F)$. In fact, such g must be in $\widetilde{G}(\pi)$. So, we consider the homomorphism

$$\widetilde{G}(\pi) \longrightarrow (G_F^1/\widetilde{Z}_F G(F))^*,$$

$$g \longmapsto \omega : (\pi^1)^g \cong \pi^1 \otimes \omega$$

and the kernel is G_F^1 by maximality. If we denote the image of this homomorphism by $c(\pi)$, then

$$\mathrm{Ind}_{\widetilde{Z}_F G(F)}^{\widetilde{G}(F)} \pi \cong |c(\pi)| \bigoplus_{\omega \in (G_F^1/\widetilde{Z}_F G(F))^*/c(\pi)} \mathrm{Ind}_{G_F^1}^{\widetilde{G}(F)} (\pi^1 \otimes \omega). \tag{6.2}$$

As a consequence of this formula, we have the following corollaries.

COROLLARY 6.4. *If π is an irreducible smooth representation of $G(F)$, then the irreducible smooth representation $\tilde{\pi}$ of $\widetilde{G}(F)$, which contains π in its restriction to $G(F)$, is unique up to twisting by* $\mathrm{Hom}(\widetilde{G}(F)/G(F), \mathbb{C}^\times)$.

Proof. As in Corollary 6.3, we can let $\widetilde{Z}_F$ be a closed subgroup of $Z_{\widetilde{G}}(F)$ such that $\widetilde{Z}_F \cap Z_G(F) = 1$ and $D(F)/\lambda(\widetilde{Z}_F)$ is finite. Then, for any two irreducible smooth representations $\tilde{\pi}_1, \tilde{\pi}_2$, which contain π in their restrictions to $G(F)$, one can choose $\omega \in \mathrm{Hom}(\widetilde{G}(F)/G(F), \mathbb{C}^\times)$ such that the restrictions of $\tilde{\pi}_1 \otimes \omega$ and $\tilde{\pi}_2$ to $\widetilde{Z}_F G(F)$ all contain the same representation which extends π. By Frobenius reciprocity and (6.2), $\tilde{\pi}_1 \otimes \omega \cong \tilde{\pi}_2 \otimes \omega'$ for some $\omega' \in (\widetilde{G}(F)/\widetilde{Z}_F G(F))^*$. Therefore, $\tilde{\pi}_1 \cong \tilde{\pi}_2 \otimes \omega' \omega^{-1}$. □

COROLLARY 6.5. *If $\tilde{\pi}$ is an irreducible smooth representation of $\widetilde{G}(F)$, then the irreducible smooth representations π of $G(F)$ in the restriction of $\tilde{\pi}$ all have the same multiplicity and it is equal to $|c(\pi)|$.*

Proof. It follows from the proof of Lemma 6.2 that $\mathrm{Res}_{G(F)}^{\widetilde{G}(F)} \tilde{\pi}$ consists of isomorphism classes of π^g for $g \in \widetilde{G}(F)$. By (6.2) and Frobenius reciprocity, the multiplicity of π^g is $|c(\pi^g)| = |c(\pi)|$. This finishes the proof. □

LEMMA 6.6. *Suppose that $\tilde{\pi}$ is an irreducible smooth generic representation of $\widetilde{G}(F)$; then the multiplicity of the irreducible smooth representation π of $G(F)$ in the restriction of $\tilde{\pi}$ is equal to one.*

Proof. Since $\tilde{\pi}$ is generic, there exists a generic representation π of $G(F)$ in the restriction of $\tilde{\pi}$. For $g \in \widetilde{G}(\pi)$, the intertwining operator $A_g : \pi \to \pi^g$ will preserve the Whittaker functional up to a scalar. Here we are using the uniqueness of the Whittaker model. As a consequence, we can normalize A_g for all $g \in \widetilde{G}(\pi)$ so that they all preserve the Whittaker functional. Then one can check easily that π can be extended by these intertwining operators to $\widetilde{G}(\pi)$. This means that $G_F^1 = \widetilde{G}(\pi)$ and hence $|c(\pi)| = 1$. Now this lemma will follow from Corollary 6.5. □

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If $\tilde{\pi}$ is an irreducible smooth representation of $\widetilde{G}(F)$, let us denote

$$X(\tilde{\pi}) = \{\omega \in (\widetilde{G}(F)/Z_{\widetilde{G}}(F)G(F))^* : \tilde{\pi} \otimes \omega \cong \tilde{\pi}\}.$$

We denote the multiplicity of an irreducible smooth representation π of $G(F)$ in the restriction of $\tilde{\pi}$ by $m(\tilde{\pi}, \pi)$. Next, we want to give a formula for $m(\tilde{\pi}, \pi)$ in terms of $X(\tilde{\pi})$ and $\widetilde{G}(\pi)$.

Corollary 6.7. *If $\tilde{\pi}$ is an irreducible smooth representation of $\widetilde{G}(F)$ and π is contained in its restriction to $G(F)$, then*

$$m(\tilde{\pi}, \pi)^2 = \frac{|X(\tilde{\pi})|}{|\widetilde{G}(F)/\widetilde{G}(\pi)|}. \tag{6.3}$$

Proof. It follows from Corollary 6.5 that $m(\tilde{\pi}, \pi) = |c(\pi)|$. By definition, $|c(\pi)| = |\widetilde{G}(\pi)/G^1_F|$. On the other hand, it follows from (6.2) that $X(\tilde{\pi})$ is the preimage of $c(\pi)$ under

$$(\widetilde{G}(F)/Z_{\widetilde{G}}(F)G(F))^* \longrightarrow (G^1_F/Z_{\widetilde{G}}(F)G(F))^*.$$

Note that the kernel of this map is $(\widetilde{G}(F)/G^1_F)^*$, so $|X(\tilde{\pi})| = |c(\pi)| \cdot |\widetilde{G}(F)/G^1_F|$. Cancelling G^1_F from these two identities, we get

$$|c(\pi)|^2 = \frac{|X(\tilde{\pi})|}{|\widetilde{G}(F)/\widetilde{G}(\pi)|}.$$

This finishes the proof. □

Remark 6.8. In the next section, we will consider the situation that both G and H as in Lemma 6.2 are finite groups. It is not hard to see that the corollaries above can also be stated for such pairs, and the proofs are the same.

Finally, we show that the restriction multi-map $\Pi(\widetilde{G}(F)) \to \Pi(G(F))$ preserves temperedness.

Lemma 6.9. *Suppose that $\tilde{\pi}$ is an irreducible smooth unitary representation of $\widetilde{G}(F)$; then $\tilde{\pi}$ is an essential discrete series representation of $\widetilde{G}(F)$ if and only if its restriction to $G(F)$ consists of essential discrete series representations. The same is true of the tempered representations.*

Proof. If $\tilde{\pi}$ is an essential discrete series representation, then the matrix coefficient $\langle \tilde{\pi}(g)v, w^\vee \rangle$ for $v \in V(\tilde{\pi})$ and $w^\vee \in V(\tilde{\pi})^\vee$ is a square integrable function modulo the centre. In particular, its restriction to $G(F)$ is square integrable modulo the centre; hence, the restriction of $\tilde{\pi}$ consists of essential discrete series representations. Conversely, we can write the matrix coefficient of $\tilde{\pi}$ as a piecewise-defined function on the components of $\widetilde{G}(F)/Z_{\widetilde{G}}(F)G(F)$, where on each component it is defined as

$$\langle \tilde{\pi}(hg)v, w^\vee \rangle = \langle \tilde{\pi}(h)(\tilde{\pi}(g)v), w^\vee \rangle$$

for some fixed representatives $g \in \widetilde{G}(F)$ of $\widetilde{G}(F)/Z_{\widetilde{G}}(F)G(F)$ and $h \in Z_{\widetilde{G}}(F)G(F)$, which is a matrix coefficient of the restriction of $\tilde{\pi}$. So, the restriction of $\tilde{\pi}$ consisting of essential discrete series representations implies that $\tilde{\pi}$ is an essential discrete series representation. The same kind of argument also applies to tempered representations when we replace the condition of square integrability by $L^{2+\epsilon}$. □

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6.2 Coarse L-packet

In this section, we want to describe the preimage of L-packets of G under $\Pi(\widetilde{G}(F)) \to \Pi(G(F))$. To do so, we need the following hypothesis.

Hypothesis 1. Suppose that $\phi \in \Phi_{\mathrm{bdd}}(G)$ and let $\rho \in \mathrm{Irr}(\mathcal{S}_{\underline{\phi}})$ and $\tau \in \mathrm{Irr}(\mathcal{S}_{\underline{\tilde{\phi}}})$ be in the restriction $\rho|_{\mathcal{S}_{\underline{\tilde{\phi}}}}$. Let $\tilde{\pi}$ be an irreducible smooth representation of $\widetilde{G}(F)$, whose restriction to $G(F)$ contains $\pi = \pi(\rho)$; then, for any $x \in \mathcal{S}_{\underline{\phi}}$,

$$\tau^x \cong \tau \Longleftrightarrow \tilde{\pi} \cong \tilde{\pi} \otimes \omega_x.$$

Moreover,

$$X(\tilde{\pi}) = \alpha(\mathcal{S}_{\underline{\phi}}(\tau)),$$

where $\mathcal{S}_{\underline{\phi}}(\tau) = \{x \in \mathcal{S}_{\underline{\phi}} : \tau^x \cong \tau\}$.

It is clear that this hypothesis is a consequence of Conjecture 2.3 for $\widetilde{G}$, which is not assumed in § 6. Since this hypothesis will be used on top of our working assumption for this section, we will point it out whenever we assume this hypothesis. The next proposition is kind of dual to this hypothesis.

Proposition 6.10. *Suppose that* $\phi \in \Phi_{\mathrm{bdd}}(G)$ *and let* $\rho \in \mathrm{Irr}(\mathcal{S}_{\underline{\phi}})$ *and* $\pi = \pi(\rho)$. *Let*

$$X(\rho) = \{\varepsilon \in (\mathcal{S}_{\underline{\phi}}/\mathcal{S}_{\underline{\tilde{\phi}}})^* : \rho \otimes \varepsilon \cong \rho\};$$

then $\{\varepsilon_g : g \in \widetilde{G}(\pi)\} = X(\rho)$, *where* $\varepsilon_g(x) = \omega_x(g) = \alpha(x)(g)$ *for* $x \in \mathcal{S}_{\underline{\phi}}$.

Proof. For $g \in \widetilde{G}(\pi)$, by Lemma 3.13,

$$\langle x, \pi \rangle_{\underline{\phi}} = \langle x, \pi^g \rangle_{\underline{\phi}} = \varepsilon_g(x) \langle x, \pi \rangle_{\underline{\phi}}$$

and hence $\varepsilon_g \in X(\rho)$. This shows that $\{\varepsilon_g : g \in \widetilde{G}(\pi)\} \subseteq X(\rho)$.

For the other direction, note that the map $\alpha : x \mapsto \omega_x$ embeds $\mathcal{S}_{\underline{\phi}}/\mathcal{S}_{\underline{\tilde{\phi}}}$ into $\mathrm{Hom}(\widetilde{G}(F)/G(F), \mathbb{C}^\times)$ (see (3.6)), so the map $g \mapsto \varepsilon_g$ from $\widetilde{G}(F)$ to $(\mathcal{S}_{\underline{\phi}}/\mathcal{S}_{\underline{\tilde{\phi}}})^*$ is surjective. Hence, for any $\varepsilon \in X(\rho)$, we can assume that $\varepsilon = \varepsilon_g$ for some $g \in \widetilde{G}(F)$. Then

$$\langle x, \pi^g \rangle_{\underline{\phi}} = \varepsilon_g(x) \langle x, \pi \rangle_{\underline{\phi}} = \langle x, \pi \rangle_{\underline{\phi}}.$$

By injectivity of the map $\pi \to \langle \cdot, \pi \rangle_{\underline{\phi}}$, one must have $\pi^g \cong \pi$, i.e., $g \in \widetilde{G}(\pi)$. □

Corollary 6.11. *Suppose that* $\phi \in \Phi_{\mathrm{bdd}}(G)$ *and let* $\rho \in \mathrm{Irr}(\mathcal{S}_{\underline{\phi}})$ *and* $\pi = \pi(\rho)$. *Let*

$$\mathrm{Ker}(X(\rho)) = \{x \in \mathcal{S}_{\underline{\phi}} : \varepsilon(x) = 1 \text{ for all } \varepsilon \in X(\rho)\};$$

then $\alpha(\mathrm{Ker}(X(\rho))) = (\widetilde{G}(F)/\widetilde{G}(\pi))^*$.

Proof. Consider the pairing $\widetilde{G}(F) \times \mathcal{S}_{\underline{\phi}} \to \mathbb{C}^\times$ which sends (g, x) to $\varepsilon_g(x) = \alpha(x)(g)$. It becomes a perfect pairing of abelian groups after taking quotients by $\mathcal{S}_{\underline{\tilde{\phi}}} \subseteq \mathcal{S}_{\underline{\phi}}$, and $U \subseteq \widetilde{G}(F)$, which is annihilated by $\mathcal{S}_{\underline{\phi}}$. We claim that $U \subseteq \widetilde{G}(\pi)$. This is because if $\varepsilon_g = 1$, then

$$\langle x, \pi^g \rangle_{\underline{\phi}} = \varepsilon_g(x) \langle x, \pi \rangle_{\underline{\phi}} = \langle x, \pi \rangle_{\underline{\phi}}.$$

By injectivity of the map $\pi \to \langle \cdot, \pi \rangle_{\underline{\phi}}$, one must have $\pi^g \cong \pi$, i.e., $g \in \widetilde{G}(\pi)$. By Proposition 6.10 and the Pontryagin duality applied to the perfect pairing $\widetilde{G}(F)/U \times \mathcal{S}_{\underline{\phi}}/\mathcal{S}_{\underline{\tilde{\phi}}} \to \mathbb{C}^\times$, we have a perfect pairing $(\widetilde{G}(F)/U)/(\widetilde{G}(\pi)/U) \times \mathrm{Ker}(X(\rho))/\mathcal{S}_{\underline{\tilde{\phi}}} \to \mathbb{C}^\times$. Therefore, $(\widetilde{G}(F)/\widetilde{G}(\pi))^* = ((\widetilde{G}(F)/U)/(\widetilde{G}(\pi)/U))^* = \alpha(\mathrm{Ker}(X(\rho)))$. □

Proposition 6.12. *Suppose that $\phi \in \Phi_{\mathrm{bdd}}(G)$ and let $\rho \in \mathrm{Irr}(\mathcal{S}_{\underline{\phi}})$ and $\tau \in \mathrm{Irr}(\mathcal{S}_{\underline{\tilde{\phi}}})$ be in the restriction $\rho|_{\mathcal{S}_{\underline{\tilde{\phi}}}}$ with multiplicity $m(\rho,\tau)$. Let $\tilde{\pi}$ be an irreducible smooth representation of $\widetilde{G}(F)$ whose restriction to $G(F)$ contains $\pi = \pi(\rho)$. Under Hypothesis 1, we have $m(\tilde{\pi},\pi) = m(\rho,\tau)$.*

Proof. By Corollary 6.7, we have

$$m(\tilde{\pi},\pi)^2 = \frac{|X(\tilde{\pi})|}{|\widetilde{G}(F)/\widetilde{G}(\pi)|}.$$

Similarly, one can show that

$$m(\rho,\tau)^2 = \frac{|X(\rho)|}{|\mathcal{S}_{\underline{\phi}}/\mathcal{S}_{\underline{\phi}}(\tau)|}$$

(see Remark 6.8). To relate these two expressions, we take Hypothesis 1 and apply Corollary 6.11 to the formula of $m(\tilde{\pi},\pi)^2$, and we get

$$\begin{aligned} m(\tilde{\pi},\pi)^2 &= \frac{|\alpha(\mathcal{S}_{\underline{\phi}}(\tau))|}{|\alpha(\mathrm{Ker}(X(\rho)))|} = \frac{|\mathcal{S}_{\underline{\phi}}(\tau)/\mathcal{S}_{\underline{\tilde{\phi}}}|}{|\mathrm{Ker}(X(\rho))/\mathcal{S}_{\underline{\tilde{\phi}}}|} \\ &= \frac{|\mathcal{S}_{\underline{\phi}}(\tau)/\mathcal{S}_{\underline{\tilde{\phi}}}|}{|(\mathcal{S}_{\underline{\phi}}/\mathcal{S}_{\underline{\tilde{\phi}}})^*/X(\rho)|} = \frac{|\mathcal{S}_{\underline{\phi}}(\tau)/\mathcal{S}_{\underline{\tilde{\phi}}}||X(\rho)|}{|(\mathcal{S}_{\underline{\phi}}/\mathcal{S}_{\underline{\tilde{\phi}}})|} = \frac{|X(\rho)|}{|\mathcal{S}_{\underline{\phi}}/\mathcal{S}_{\underline{\phi}}(\tau)|} \\ &= m(\rho,\tau)^2. \end{aligned}$$

Hence, $m(\tilde{\pi},\pi) = m(\rho,\tau)$. □

This proposition suggests that $m(\tilde{\pi},\pi) = 1$ if $\mathcal{S}_{\underline{\phi}}$ is abelian. For classical groups, it has been shown that $\mathcal{S}_{\underline{\phi}}$ is always abelian (see [Art13, Mok14]). On the other hand, when G is a symplectic group or special even orthogonal group, and $\widetilde{G}$ is its similitude group, it has been proved that $m(\tilde{\pi},\pi) = 1$ (see [AP06, Theorem 1.4]). In fact, one can prove Hypothesis 1 under the assumption that $m(\tilde{\pi},\pi) = m(\rho,\tau) = 1$.

Proposition 6.13. *Suppose that $\phi \in \Phi_{\mathrm{bdd}}(G)$ and let $\rho \in \mathrm{Irr}(\mathcal{S}_{\underline{\phi}})$ and $\tau \in \mathrm{Irr}(\mathcal{S}_{\underline{\tilde{\phi}}})$ be in the restriction $\rho|_{\mathcal{S}_{\underline{\tilde{\phi}}}}$. Let $\tilde{\pi}$ be an irreducible smooth representation of $\widetilde{G}(F)$, whose restriction to $G(F)$ contains $\pi = \pi(\rho)$. If $m(\tilde{\pi},\pi) = m(\rho,\tau) = 1$, then, for any $x \in \mathcal{S}_{\underline{\phi}}$,*

$$\tau^x \cong \tau \Longleftrightarrow \tilde{\pi} \cong \tilde{\pi} \otimes \omega_x.$$

Moreover,

$$X(\tilde{\pi}) = \alpha(\mathcal{S}_{\underline{\phi}}(\tau)),$$

where $\mathcal{S}_{\underline{\phi}}(\tau) = \{x \in \mathcal{S}_{\underline{\phi}} : \tau^x \cong \tau\}$.

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Proof. If $m(\tilde{\pi},\pi)=m(\rho,\tau)=1$, then $X(\tilde{\pi})=(\widetilde{G}(F)/\widetilde{G}(\pi))^*$ and $X(\rho)=(\mathcal{S}_{\underline{\phi}}/\mathcal{S}_{\underline{\phi}}(\tau))^*$. It follows that $\mathrm{Ker}(X(\rho))=\mathcal{S}_{\underline{\phi}}(\tau)$. By Corollary 6.11, $X(\tilde{\pi})=\alpha(\mathrm{Ker}(X(\rho)))=\alpha(\mathcal{S}_{\underline{\phi}}(\tau))$. This implies the direction '⇒'. For the other direction, one can always choose $x_0\in\mathcal{S}_{\underline{\phi}}(\tau)$ such that $\omega_x=\omega_{x_0}$, which implies that $xx_0^{-1}\in\mathcal{S}_{\underline{\tilde{\phi}}}$. Hence, $x\in\mathcal{S}_{\underline{\phi}}(\tau)$. □

For $\phi\in\Phi_{\mathrm{bdd}}(G)$, we assume that the central character of Π_ϕ is χ_ϕ. Let us fix a character $\widetilde{\chi}_\phi$ of $Z_{\widetilde{G}}(F)$ such that $\widetilde{\chi}_\phi|_{Z_G(F)}=\chi_\phi$. Then we define $\tilde{\Pi}_{\phi,\widetilde{\chi}_\phi}$ to be the subset of $\Pi(\widetilde{G}(F))$ with central character $\widetilde{\chi}_\phi$, whose restriction to $G(F)$ is contained in Π_ϕ. Let $X=\mathrm{Hom}(\widetilde{G}(F)/Z_{\widetilde{G}}(F)G(F),\mathbb{C}^\times)$; then X acts on $\tilde{\Pi}_{\phi,\widetilde{\chi}_\phi}$ by twisting. We call $\tilde{\Pi}_{\phi,\widetilde{\chi}_\phi}$ a coarse L-packet for $\widetilde{G}$; its structure is described in the following proposition.

Proposition 6.14. *Suppose that $\phi\in\Phi_{\mathrm{bdd}}(G)$ and $\widetilde{\chi}_\phi$ is chosen as above. We assume Hypothesis 1.*

(i) *If $\rho\in\mathrm{Irr}(\mathcal{S}_{\underline{\phi}})$, then the $\widetilde{G}(F)$-conjugate orbit of $\pi(\rho)$ has size $|\alpha(\mathrm{Ker}(X(\rho)))|$.*

(ii) *There is a pairing (not necessarily unique)*

$$\tilde{\pi}\longrightarrow\langle\cdot,\tilde{\pi}\rangle_{\underline{\phi}} \tag{6.4}$$

from $\tilde{\Pi}_{\phi,\widetilde{\chi}_\phi}$ to $\widehat{\mathcal{S}_{\underline{\tilde{\phi}}}}$ such that:

(a)

$$\langle\cdot,\tilde{\pi}\otimes\omega_x\rangle_{\underline{\phi}}=\langle x(\cdot)x^{-1},\tilde{\pi}\rangle_{\underline{\phi}}$$

for $x\in\mathcal{S}_{\underline{\phi}}$;

(b)

$$\langle\cdot,\pi\rangle_{\underline{\phi}}|_{\mathcal{S}_{\underline{\tilde{\phi}}}}=m(\tilde{\pi},\pi)\sum_{x\in\mathcal{S}_{\underline{\phi}}/\mathcal{S}_{\underline{\phi}}(\tau)}\langle\cdot,\tilde{\pi}\otimes\omega_x\rangle_{\underline{\phi}}$$

for any $\pi\in\Pi_\phi$ in the restriction of $\tilde{\pi}$.

Moreover, it sends the generic representation to the trivial character of $\mathcal{S}_{\underline{\tilde{\phi}}}$.

Proof. Suppose that $\pi\in\Pi_\phi$; then the orbit of π under the conjugate action of $\widetilde{G}(F)$ has size $|\widetilde{G}(F)/\widetilde{G}(\pi)|$. By Corollary 6.11, we know that $\alpha(\mathrm{Ker}(X(\rho)))=(\widetilde{G}(F)/\widetilde{G}(\pi))^*$. Hence, $|(\widetilde{G}(F)/\widetilde{G}(\pi))|=|(\widetilde{G}(F)/\widetilde{G}(\pi))^*|=|\alpha(\mathrm{Ker}(X(\rho)))|$.

For the second part, we can choose any $\pi(\rho)$ in the restriction of $\tilde{\pi}\in\tilde{\Pi}_{\phi,\widetilde{\chi}_\phi}$ and choose any irreducible subrepresentation τ in $\rho|_{\mathcal{S}_{\underline{\tilde{\phi}}}}$. We also fix a set of representatives $\{\omega_i\}$ in X of $X/\alpha(\mathcal{S}_{\underline{\phi}})$. We assign τ to all $\tilde{\pi}\otimes\omega_i$ and extend to $\tilde{\pi}\otimes\omega$ for any $\omega\in X$ by letting

$$\langle\cdot,\tilde{\pi}\otimes\omega_x\rangle_{\underline{\phi}}:=\langle x(\cdot)x^{-1},\tilde{\pi}\rangle_{\underline{\phi}} \tag{6.5}$$

for $x\in\mathcal{S}_{\underline{\phi}}$. This is well defined because of Hypothesis 1. By this construction, it is clear that (a) is satisfied. Moreover, this definition is independent of choice of $\pi(\rho)$. To see this, let us replace $\pi(\rho)$ by $\pi(\rho)^g$ for $g\in\widetilde{G}(F)$; by Lemma 3.13, we have

$$\langle\cdot,\pi^g\rangle_{\underline{\phi}}|_{\mathcal{S}_{\underline{\tilde{\phi}}}}=\omega_x(g)\langle\cdot,\pi\rangle_{\underline{\phi}}|_{\mathcal{S}_{\underline{\tilde{\phi}}}}=\langle\cdot,\pi\rangle_{\underline{\phi}}|_{\mathcal{S}_{\underline{\tilde{\phi}}}}.$$

Then (b) follows from (a) and Proposition 6.12. Finally, if $\tilde{\pi}$ is generic, there exists a generic representation π in its restriction, i.e., $\langle\cdot,\pi\rangle_{\underline{\phi}}=1$. It is easy to see that $\langle\cdot,\tilde{\pi}\rangle_{\underline{\phi}}=1$ by our construction. □

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6.3 Compatibility with θ-twist

Before we give the refinement of $\tilde{\Pi}_{\phi,\widetilde{\chi}_\phi}$, we want to show how the pairing in Proposition 6.14 can also be made to satisfy a special case of Conjecture 2.3. First, we would like to generalize Hypothesis 1 to the θ-twisted case.

HYPOTHESIS 2. Suppose that $\phi \in \Phi_{\mathrm{bdd}}(G)$ and let $\rho \in \mathrm{Irr}(\mathcal{S}_{\underline{\phi}})$ and $\tau \in \mathrm{Irr}(\mathcal{S}_{\underline{\tilde{\phi}}})$ be in the restriction $\rho|_{\mathcal{S}_{\underline{\tilde{\phi}}}}$. Let $\tilde{\pi}$ be an irreducible smooth representation of $\widetilde{G}(F)$ whose restriction to $G(F)$ contains $\pi(\rho)$; then, for any $x \in \mathcal{S}^{\theta}_{\underline{\phi}}$,

$$\tau^x \cong \tau \Longleftrightarrow \tilde{\pi}^{\theta} \cong \tilde{\pi} \otimes \omega_x.$$

Remark 6.15. (i) Fix $\tau_0 \in \mathrm{Irr}(\mathcal{S}_{\underline{\tilde{\phi}}})$; we can construct one to one correspondences between $\{\tau_0^y : y \in \mathcal{S}_{\underline{\phi}}\}$ and $\{\tilde{\pi}(\tau_0) \otimes \omega_y : y \in \mathcal{S}_{\underline{\phi}}\}$ through (6.5), where $\tilde{\pi}(\tau_0) \in \tilde{\Pi}_{\phi,\widetilde{\chi}_\phi}$. If we fix such a correspondence, and suppose that $\mathcal{S}^{\theta}_{\underline{\phi}}$ acts on $\{\tau_0^y : y \in \mathcal{S}_{\underline{\phi}}\}$, then it follows from this hypothesis that

$$\tilde{\pi}(\tau^x)^{\theta} \cong \tilde{\pi}(\tau) \otimes \omega_x$$

for any $\tau \in \{\tau_0^y : y \in \mathcal{S}_{\underline{\phi}}\}$ and $x \in \mathcal{S}^{\theta}_{\underline{\phi}}$. More generally, if $\tau_0' := \tau_0^{x_0} \notin \{\tau_0^y : y \in \mathcal{S}_{\underline{\phi}}\}$ for some $x_0 \in \mathcal{S}^{\theta}_{\underline{\phi}}$, then, by taking $\tilde{\pi}(\tau_0')$ such that $\tilde{\pi}(\tau_0')^{\theta} \cong \tilde{\pi}(\tau_0) \otimes \omega_{x_0}$, we can obtain a one to one correspondence between $\{(\tau_0')^y : y \in \mathcal{S}_{\underline{\phi}}\}$ and $\{\tilde{\pi}(\tau_0') \otimes \omega_y : y \in \mathcal{S}_{\underline{\phi}}\}$ again through (6.5). Note that $\tilde{\pi}(\tau_0') \in \tilde{\Pi}_{\phi,\widetilde{\chi}_\phi}$ (see Remark 6.17). In this way, one can construct a pairing from $\tilde{\Pi}_{\phi,\widetilde{\chi}_\phi}$ to $\mathrm{Irr}(\mathcal{S}_{\underline{\tilde{\phi}}})$ as in Proposition 6.14, which further satisfies

$$\tilde{\pi}(\tau^x)^{\theta} \cong \tilde{\pi}(\tau) \otimes \omega_x$$

for any $\tau \in \mathrm{Irr}(\mathcal{S}_{\underline{\tilde{\phi}}})$ and $x \in \mathcal{S}^{\theta}_{\underline{\phi}}$.

(ii) For $\rho \in \mathrm{Irr}(\mathcal{S}_{\underline{\phi}})$ and $\tau \in \mathrm{Irr}(\mathcal{S}_{\underline{\tilde{\phi}}})$ being in the restriction $\rho|_{\mathcal{S}_{\underline{\tilde{\phi}}}}$, it is easy to see for $x \in \mathcal{S}^{\theta}_{\underline{\phi}}$ that $\tau^x \cong \tau$ implies that $\rho^x \cong \rho \otimes \varepsilon$ for some $\varepsilon \in (\mathcal{S}_{\underline{\phi}}/\mathcal{S}_{\underline{\tilde{\phi}}})^*$. By the proof of Proposition 6.10, there exists $h \in \widetilde{G}(F)$ such that $\varepsilon = \varepsilon_h$. Since $X(\rho) = \{\varepsilon_g : g \in \widetilde{G}(\pi(\rho))\}$; then h is uniquely determined modulo $\widetilde{G}(\pi(\rho))$. It follows that

$$\pi(\rho)^{\theta^{-1}} \cong \pi(\rho^x) \cong \pi(\rho \otimes \varepsilon) \cong \pi(\rho)^h,$$

so $\pi(\rho)^{\theta_h} \cong \pi(\rho)$, where $\theta_h = h \rtimes \theta$. In the special case $\rho^x \cong \rho$, we can prove the hypothesis under Hypothesis 1.

PROPOSITION 6.16. *Suppose that $\phi \in \Phi_{\mathrm{bdd}}(G)$ and let $\rho \in \mathrm{Irr}(\mathcal{S}_{\underline{\phi}})$ and $\tau \in \mathrm{Irr}(\mathcal{S}_{\underline{\tilde{\phi}}})$ be in the restriction $\rho|_{\mathcal{S}_{\underline{\tilde{\phi}}}}$. Let $\tilde{\pi}$ be an irreducible smooth representation of $\widetilde{G}(F)$ whose restriction to $G(F)$ contains $\pi = \pi(\rho)$. We assume Hypothesis 1 and $\rho^x \cong \rho$ for $x \in \mathcal{S}^{\theta}_{\underline{\phi}}$; then, for any $x \in \mathcal{S}^{\theta}_{\underline{\phi}}$,*

$$\tau^x \cong \tau \Longleftrightarrow \tilde{\pi}^{\theta} \cong \tilde{\pi} \otimes \omega_x.$$

Proof. Since $\pi(\rho)^{\theta} \cong \pi(\rho^{x^{-1}})$ for $x \in \mathcal{S}^{\theta}_{\underline{\phi}}$, then by our assumption $\pi \cong \pi^{\theta}$. This means that we have $x_0 \in \mathcal{S}^{\theta}_{\underline{\phi}}$ such that $\langle x_0, \pi^+ \rangle_{\underline{\phi}} \neq 0$; in particular, $\tau^{x_0} \cong \tau$. By (3.15),

$$\langle x_0, (\pi^+)^g \rangle_{\underline{\phi}} = \omega_{x_0}(g) \langle x_0, \pi^+ \rangle_{\underline{\phi}}.$$

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Take $g \in \widetilde{G}(\pi)$; we get

$$\langle x_0, (\pi^+)^g\rangle_{\underline{\phi}} f_{G^+}(\pi^+) = \omega_{x_0}(g)\langle x_0, \pi^+\rangle_{\underline{\phi}} f_{G^+}(\pi^+) = \omega_{x_0}(g)\langle x_0, (\pi^+)^g\rangle_{\underline{\phi}} f_{G^+}((\pi^+)^g)$$

for $f \in C_c^\infty(G(F) \rtimes \theta)$. Hence,

$$f_{G^+}((\pi^+)^g) = \omega_{x_0}(g)^{-1} f_{G^+}(\pi^+). \tag{6.6}$$

Let G_F^1 be a maximal subgroup of $\widetilde{G}(F)$, to which one can extend π. Then we take the extension π^1 of π such that $\tilde{\pi} \cong \mathrm{Ind}_{G_F^1}^{\widetilde{G}(F)} \pi^1$. Since $\pi^1(g)$ intertwines between π and π^g, and $\pi^+(\theta)$ intertwines between π and π^θ, it follows from (6.6) that

$$\pi^1(\theta g \theta^{-1}) = \omega_{x_0}(g) \cdot \pi^+(\theta)\pi^1(g)\pi^+(\theta)^{-1}.$$

This means that $(\pi^1)^\theta \cong \pi^1 \otimes (\omega_{x_0}|_{G_F^1})$ and hence $\tilde{\pi}^\theta \cong \tilde{\pi} \otimes \omega_{x_0}$.

Now suppose that $\tau^x \cong \tau$ for some $x \in \mathcal{S}_{\underline{\phi}}^\theta$; then $x x_0^{-1} \in \mathcal{S}_{\underline{\phi}}(\tau)$. From Hypothesis 1, we have $\tilde{\pi} \cong \tilde{\pi} \otimes \omega_{x x_0^{-1}}$; then $\tilde{\pi}^\theta \cong \tilde{\pi} \otimes \omega_{x_0} \cong \tilde{\pi} \otimes \omega_x$. Conversely, if $\tilde{\pi}^\theta \cong \tilde{\pi} \otimes \omega_x$ for some $x \in \mathcal{S}_{\underline{\phi}}^\theta$, then $\tilde{\pi} \otimes \omega_x \cong \tilde{\pi} \otimes \omega_{x_0}$. It follows again from Hypothesis 1 that $x x_0^{-1} \in \mathcal{S}_{\underline{\phi}}(\tau)$ and hence $\tau^x \cong \tau$. □

Remark 6.17. Let $\tau_0 \in \mathrm{Irr}(\mathcal{S}_{\tilde{\underline{\phi}}})$ and $\rho_0 \in \mathrm{Irr}(\mathcal{S}_\phi)$ be both trivial. Then it is clear that $\rho_0^x \cong \rho_0$ for $x \in \mathcal{S}_{\underline{\phi}}^\theta$ and $m(\rho_0, \tau_0) = 1$. Let $\tilde{\pi}$ be an irreducible smooth representation of $\widetilde{G}(F)$ whose restriction to $G(F)$ contains $\pi = \pi(\rho_0)$. Note that $\pi = \pi(\rho_0)$ is generic, so, by Lemma 6.6, we have $m(\tilde{\pi}, \pi) = 1$. It follows from Proposition 6.13 that Hypothesis 1 is satisfied for such π and $\tilde{\pi}$. Therefore, the assumptions of this proposition are all satisfied in this case, and we have $\tilde{\pi}^\theta \cong \tilde{\pi} \otimes \omega_x$ for any $x \in \mathcal{S}_\phi^\theta$. Suppose that $\tilde{\pi} \in \tilde{\Pi}_{\phi, \widetilde{\chi}_\phi}$, i.e., $\tilde{\pi}$ has central character $\widetilde{\chi}_\phi$ (see Proposition 6.14); then this implies that $\widetilde{\chi}_\phi^\theta = \widetilde{\chi}_\phi \cdot \omega_x|_{Z_{\widetilde{G}(F)}}$ for any $x \in \mathcal{S}_\phi^\theta$.

6.4 Conjectural refinement

The refinement of L-packets of $\widetilde{G}$ should be a section of a certain choice of the pairing $\tilde{\Pi}_{\phi, \widetilde{\chi}_\phi} \to \widehat{\mathcal{S}_{\tilde{\underline{\phi}}}}$ given in Proposition 6.14, for which we make the following conjecture.

Conjecture 6.18. Suppose that $\phi \in \Phi_{\mathrm{bdd}}(G)$ and $\widetilde{\chi}_\phi$ is a character of $Z_{\widetilde{G}}(F)$ whose restriction to $Z_G(F)$ is χ_ϕ. Let $\widetilde{\chi} = \widetilde{\chi}_\phi|_{\widetilde{Z}_F}$. Then one can construct a pairing of $\tilde{\Pi}_{\phi, \widetilde{\chi}_\phi} \to \widehat{\mathcal{S}_{\tilde{\phi}}}$ as in Proposition 6.14 and a section $\Pi_{\tilde{\phi}}$, which satisfies the following properties.

(i)

$$\tilde{\Pi}_{\phi, \widetilde{\chi}_\phi} = \bigsqcup_{\omega \in X/\alpha(\mathcal{S}_{\underline{\phi}})} \Pi_{\tilde{\phi}} \otimes \omega.$$

(ii) For $\tilde{f} \in C_c^\infty(\widetilde{G}(F), \widetilde{\chi})$, the distribution

$$\tilde{f}(\underline{\tilde{\phi}}) := \sum_{\tilde{\pi} \in \Pi_{\tilde{\phi}}} \langle 1, \tilde{\pi}\rangle_{\underline{\phi}} \tilde{f}_{\widetilde{G}}(\tilde{\pi})$$

is stable.

(iii) Suppose that s is a semisimple element in $\bar{S}_{\underline{\tilde{\phi}}}$ and $(H_1, \underline{\phi}_{H_1}) \to (\underline{\phi}, s)$. Suppose that $\Pi_{\tilde{\phi}_{H_1}}$ exists and it satisfies (i) and (ii). Then we can choose some twist of $\Pi_{\tilde{\phi}_{H_1}}$ by $\mathrm{Hom}(\widetilde{H}_1(F)/H_1(F), \mathbb{C}^{\times})$, which is still denoted the same, such that

$$\tilde{f}^{\widetilde{H}_1}(\underline{\tilde{\phi}}_{H_1}) = \sum_{\tilde{\pi} \in \Pi_{\tilde{\phi}}} \langle x, \tilde{\pi} \rangle_{\underline{\phi}} \tilde{f}_{\widetilde{G}}(\tilde{\pi}), \quad \tilde{f} \in C_c^{\infty}(\widetilde{G}(F), \widetilde{\chi}), \tag{6.7}$$

where x is the image of s in $\mathcal{S}_{\underline{\tilde{\phi}}}$.

(iv) Suppose that s is a semisimple element in $\bar{S}^{\theta}_{\underline{\phi}}$ and $(H_1, \underline{\phi}_{H_1}) \to (\underline{\phi}, s)$. Let x be the image of s in $\mathcal{S}^{\theta}_{\underline{\phi}}$ and $\omega = \alpha(x)$. Suppose that $\Pi_{\tilde{\phi}_{H_1}}$ exists and it satisfies (i) and (ii). Then, for any $\tau \in \mathrm{Irr}(\mathcal{S}_{\underline{\tilde{\phi}}})$ such that $\tau^x \cong \tau$, and any extension τ_1 of τ to the group generated by $\mathcal{S}_{\tilde{\phi}}$ and x, one can associate it with an intertwining operator $A_{\tilde{\pi}(\tau)}(\theta, \omega) : \tilde{\pi}(\tau) \otimes \omega \to \tilde{\pi}(\tau)^{\theta}$ such that for some twist of $\Pi_{\tilde{\phi}_{H_1}}$ by $\mathrm{Hom}(\widetilde{H}_1(F)/H_1(F), \mathbb{C}^{\times})$, which is still denoted the same, we have

$$\tilde{f}^{\widetilde{H}_1}(\underline{\tilde{\phi}}_{H_1}) = \sum_{\substack{\tau \in \mathrm{Irr}(\mathcal{S}_{\underline{\tilde{\phi}}}) \\ \tau^x \cong \tau}} \mathrm{trace}(\tau_1(x)) \cdot \tilde{f}_{\widetilde{G}^{\theta}}(\tilde{\pi}(\tau), \omega), \quad \tilde{f} \in C_c^{\infty}(\widetilde{G}(F), \widetilde{\chi}). \tag{6.8}$$

It is clear that (6.8) generalizes (6.7). In the setup of this conjecture, for $x \in \mathcal{S}^{\theta}_{\underline{\phi}}$ and $\tau \in \mathrm{Irr}(\mathcal{S}_{\underline{\tilde{\phi}}})$ such that $\tau^x \cong \tau$, let π be an irreducible constituent in $\tilde{\pi}(\tau)|_G$; then $\pi^{\theta_h} \cong \pi$, where x determines $h \in \widetilde{G}(F)/\widetilde{G}(\pi)$ as in Remark 6.15(ii). We fix a representative of h in $\widetilde{G}(F)$; then $\tilde{\pi}(h) \circ A_{\tilde{\pi}(\tau)}(\theta, \omega)$ induces an intertwining operator $A_{I(\pi)}(\theta_h) : I(\pi) \to I(\pi)^{\theta_h}$ by restricting to the π-isotypic component $I(\pi)$ in $\tilde{\pi}(\tau)|_G$. So,

$$(\tilde{f}|_{\widetilde{Z}_F G(F) \cdot h})_{\widetilde{G}^{\theta}}(\tilde{\pi}(\tau), \omega) = \sum_{\pi \in \tilde{\pi}(\tau)|_G} f_{G^{\theta_h}}(I(\pi)), \tag{6.9}$$

where $f \in C_c^{\infty}(G(F), \chi)$ is obtained by letting $f(g) = \tilde{f}(gh)$ and $f_{G^{\theta_h}}(I(\pi))$ is the twisted character of $I(\pi)$ generalizing (3.12). We would like to restrict (6.8) to $\tilde{f} \in C_c^{\infty}(\widetilde{G}(F), \widetilde{\chi})$ supported on $\widetilde{Z}_F G(F) \cdot h$. To write down the formula, we make the following conjecture.

CONJECTURE 6.19. In the setup of Conjecture 6.18, let $\tau' = \tau^y, \tau'_1 = \tau_1^y$ for $y \in \mathcal{S}_{\underline{\phi}}$ and suppose that τ'_1 is associated with $A_{\tilde{\pi}(\tau')}(\theta, \omega) : \tilde{\pi}(\tau') \otimes \omega \to \tilde{\pi}(\tau')^{\theta}$. If we identify the representation space of $\tilde{\pi}(\tau')$ and $\tilde{\pi}(\tau)$ such that $\tilde{\pi}(\tau') = \tilde{\pi}(\tau) \otimes \omega_y$, then $A_{\tilde{\pi}(\tau')}(\theta, \omega) = A_{\tilde{\pi}(\tau)}(\theta, \omega)$.

As a result, we have

$$\tilde{f}^{\widetilde{H}_1}(\underline{\tilde{\phi}}_{H_1}) = \sum_{\substack{\pi \in \Pi_{\phi} \\ \pi \cong \pi^{\theta_h}}} \bigg(\sum_{y \in \mathcal{S}_{\underline{\phi}}/\mathcal{S}_{\underline{\phi}}(\tau)} \mathrm{trace}(\tau_1^y(x)) \cdot \omega_y(h) \bigg) f_{G^{\theta_h}}(I(\pi)), \tag{6.10}$$

where $A(\theta_h)$ is normalized according to τ_1 and $\tilde{f}$ is supported on $\widetilde{Z}_F G(F) \cdot h$. We should point out that when $\theta_h = \mathrm{id}$, $A_{I(\pi)}(\mathrm{id})$ is not necessarily trivial, although the notation for the twisted character then becomes the same as that for the ordinary one. Moreover, it is implied by this formula that if $f_{G^{\theta_h}}(I(\pi))$ is not zero, then the sum

$$\sum_{y \in \mathcal{S}_{\underline{\phi}}/\mathcal{S}_{\underline{\phi}}(\tau)} \mathrm{trace}(\tau_1^y(x)) \cdot \omega_y(h)$$

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must be well defined, i.e., for any $y' \in \mathcal{S}_{\underline{\phi}}(\tau)$,

$$\mathrm{trace}(\tau_1^y(x)) \cdot \omega_y(h) = \mathrm{trace}(\tau_1^{yy'}(x)) \cdot \omega_{yy'}(h).$$

Finally, we want to point out that (6.10) generalizes the formula (3.14) to the case where the automorphism of the group need not preserve an F-splitting.

6.5 Classical groups

The endoscopic hypothesis (see (2.1) and Conjectures 2.5 and 3.10) has been proven under slight modifications for quasisplit classical groups (cf. [Art13, Mok14]). In this section, we will look into the case of symplectic groups and special even orthogonal groups. So, from now on, G will always be a split symplectic group or a quasisplit special even orthogonal group, where the outer twist comes from the conjugation by the full orthogonal group. Let $\widetilde{G}$ be the corresponding similitude group. There is an exact sequence

$$1 \longrightarrow G \longrightarrow \widetilde{G} \xrightarrow{\lambda} \mathbb{G}_m \longrightarrow 1, \tag{6.11}$$

where λ is called the similitude character. We fix an automorphism θ_0 of G preserving an F-splitting. When G is symplectic, we require θ_0 to be trivial. When G is special even orthogonal, we require θ_0 to be the unique nontrivial outer automorphism induced from the conjugation of the full orthogonal group. Clearly, $\theta_0^2 = 1$, θ_0 extends to $\widetilde{G}$ by acting trivially on $Z_{\widetilde{G}}$ and λ is θ_0-invariant. Let $\Sigma_0 = \langle \theta_0 \rangle$. Note that Σ_0 acts on $\Pi(G(F))$ and its dual $\widehat{\Sigma}_0$ acts on $\Phi(G)$. So, we denote the set of Σ_0-orbits in $\Pi(G(F))$ by $\bar{\Pi}(G(F))$ and the set of Σ_0-orbits in $\Phi(G)$ by $\bar{\Phi}(G)$. Similarly, we can define $\bar{\Pi}_{\mathrm{temp}}(G(F))$, $\bar{\Phi}_{\mathrm{bdd}}(G)$ and analogues of these sets for $\widetilde{G}$. Now we will recall the conjectures in the introduction by stating them as theorems in the case of symplectic groups and special even orthogonal groups.

THEOREM 6.20 [Art13, Theorem 1.5.1]. *There is a canonical way to associate any $[\phi] \in \bar{\Phi}(G)$ with a finite subset $\bar{\Pi}_\phi$ of $\bar{\Pi}(G(F))$ such that*

$$\bar{\Pi}(G(F)) = \bigsqcup_{[\phi] \in \bar{\Phi}(G)} \bar{\Pi}_\phi$$

and

$$\bar{\Pi}_{\mathrm{temp}}(G(F)) = \bigsqcup_{[\phi] \in \bar{\Phi}_{\mathrm{bdd}}(G)} \bar{\Pi}_\phi.$$

THEOREM 6.21 [Art13, Theorem 1.5.1 and Proposition 8.3.2]. *We fix a Σ_0-stable Whittaker datum (B, Λ) for G, and suppose that $[\phi] \in \bar{\Phi}_{\mathrm{bdd}}(G)$.*

(i) *There is a Σ_0-orbit of (B, Λ)-generic representations in $\bar{\Pi}_\phi$.*

(ii) *There is a canonical pairing between $\bar{\Pi}_\phi$ and $\mathcal{S}_{\underline{\phi}}$, which induces an inclusion from $\bar{\Pi}_\phi$ to the characters $\widehat{\mathcal{S}_{\underline{\phi}}}$,*

$$\bar{\Pi}_\phi \longrightarrow \widehat{\mathcal{S}_{\underline{\phi}}},$$

$$[\pi] \longmapsto \langle \cdot, \pi \rangle_{\underline{\phi}},$$

such that it sends the (B, Λ)-generic representation to the trivial character. This becomes a bijection when F is nonarchimedean.

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Remark 6.22. When F is archimedean, it follows from [Kos78] that the Σ_0-orbit of (B,Λ)-generic representations in $\bar{\Pi}_\phi$ is unique. When F is nonarchimedean, one can deduce the uniqueness of the generic representation using the results from [JS03], [JS04], [Liu11] and [JL14] (see [Art13, Remark in 8.3]).

For $[\phi] \in \bar{\Phi}_{\mathrm{bdd}}(G)$, we can define $\Pi_\phi^{\Sigma_0}$ to be the set of all isomorphism classes of irreducible smooth representations of $G^{\Sigma_0}(F)$ whose restrictions to $G(F)$ belong to $\bar{\Pi}_\phi$. Note that $\mathcal{S}_{\underline{\phi}}^{\Sigma_0}$ is always abelian in the current case (see [Art13, § 1.4]).

Theorem 6.23 (Arthur). *We fix a Σ_0-stable Whittaker datum (B,Λ) for G, and suppose that $[\phi] \in \bar{\Phi}_{\mathrm{bdd}}(G)$.*

(i) *There is a canonical pairing between $\Pi_\phi^{\Sigma_0}$ and $\mathcal{S}_{\underline{\phi}}^{\Sigma_0}$, which induces an inclusion from $\Pi_\phi^{\Sigma_0}$ to the characters $\widehat{\mathcal{S}_{\underline{\phi}}}^{\Sigma_0}$,*

$$\Pi_\phi^{\Sigma_0} \longrightarrow \widehat{\mathcal{S}_{\underline{\phi}}}^{\Sigma_0},$$
$$\pi^{\Sigma_0} \longmapsto \langle \cdot, \pi^{\Sigma_0}\rangle_{\underline{\phi}}.$$

This becomes a bijection when F is nonarchimedean. Moreover, this pairing is an extension of that in Theorem 6.21 in the sense that

$$\langle \cdot, \pi^{\Sigma_0}\rangle_{\underline{\phi}}|_{\mathcal{S}_{\underline{\phi}}} = \langle \cdot, \pi\rangle_{\underline{\phi}},$$

where $\pi \in \pi^{\Sigma_0}|_G$.

(ii) *In case G is special even orthogonal, the following statements are equivalent:*

(a) *$\bar{\Pi}_\phi$ contains an element $[\pi]$ such that $\pi^{\theta_0} \cong \pi$;*
(b) *for any $[\pi] \in \bar{\Pi}_\phi$, $\pi^{\theta_0} \cong \pi$;*
(c) *$\mathcal{S}_{\underline{\phi}}^{\theta_0} \neq \emptyset$.*

Remark 6.24. Although this theorem is not stated in [Art13], one can view it as a consequence of Theorem 6.25. Moreover, we expect the (B,Λ)-generic representation in $\Pi_\phi^{\Sigma_0}$ to correspond to the trivial character of $\mathcal{S}_{\underline{\phi}}^{\Sigma_0}$.

If H is a θ-twisted endoscopic group of G for $\theta \in \Sigma_0$, Arthur showed that $H \cong M_l \times G_1 \times G_2$, where M_l is a product of general linear groups; G_i $(i = 1, 2)$ is also a symplectic group or special even orthogonal group. We define a group of automorphisms of H by taking the product of Σ_0 on each G_i, and we denote this group again by Σ_0. Then, by combining the local Langlands correspondence for $GL(n)$ (cf. [HT01, Hen00, Sch13]), all the previous theorems of Arthur can be extended to H. In particular, the L-packets for H are formed by tensor products of those of each factor. Let $\bar{\mathcal{H}}(G)$ (respectively $\bar{\mathcal{H}}(H)$) be the space of Σ_0-invariant smooth compactly supported functions on $G(F)$ (respectively $H(F)$). Then the twisted endoscopic transfer sends $\bar{\mathcal{H}}(G)$ to $\bar{\mathcal{H}}(H)$, and there is no need to consider z-pairs here.

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THEOREM 6.25 [Art13, Theorems 2.2.1 and 2.2.4]. *Suppose that* $[\phi] \in \bar{\Phi}_{\mathrm{bdd}}(G)$.

(i)

$$f(\underline{\phi}) := \sum_{[\pi]\in\bar{\Pi}_\phi} f_G(\pi), \quad f \in \bar{\mathcal{H}}(G) \tag{6.12}$$

is stable.

(ii) *Suppose that* $\theta \in \Sigma_0$, s *is a semisimple element in* $\bar{S}^{\theta}_{\underline{\phi}}$ *and* $(H, \underline{\phi}_H) \to (\underline{\phi}, s)$. *Then*

$$f^H(\underline{\phi}_H) = \sum_{[\pi]\in\bar{\Pi}_\phi} \langle x, \pi^+\rangle_{\underline{\phi}} f_{G^\theta}(\pi) \tag{6.13}$$

for $f \in \bar{\mathcal{H}}(G)$, *where* x *is the image of* s *in* $\mathcal{S}^{\theta}_{\underline{\phi}}$, *and* π^+ *is an extension of* π *to* $G^+(F) := G(F) \times \langle\theta\rangle$ *with* $\pi^+(\theta) = A_\pi(\theta)$. *If* G *is special even orthogonal and* $\mathcal{S}^{\theta_0}_{\underline{\phi}} \neq \emptyset$, *one can replace* $\bar{\mathcal{H}}(G)$ *by* $C_c^\infty(G(F))$.

It follows from the second part of Theorem 6.23 that $\bar{\Pi}_\phi = \Pi_\phi$ unless G is special even orthogonal and $\mathcal{S}^{\theta_0}_{\underline{\phi}} = \emptyset$. In the exceptional case, we have the following refined statement.

THEOREM 6.26 [Art13, Theorem 8.4.1]. *Suppose that* G *is special even orthogonal,* $[\phi] \in \bar{\Phi}_{\mathrm{bdd}}(G)$ *and* $\mathcal{S}^{\theta_0}_{\underline{\phi}} = \emptyset$.

(i) *There exists a unique subset* $\Pi_\phi \subseteq \Pi^{\Sigma_0}_\phi|_G$ *up to* θ_0*-twist such that:*

–

$$\Pi^{\theta_0}_\phi \sqcup \Pi_\phi = \Pi^{\Sigma_0}_\phi|_G,$$

–

$$f(\underline{\phi}) := \sum_{\pi\in\Pi_\phi} f_G(\pi), \quad f \in C_c^\infty(G(F)) \tag{6.14}$$

is stable.

(ii) *Suppose that* s *is a semisimple element in* $\bar{S}_{\underline{\phi}}$ *and* $(H, \underline{\phi}_H) \to (\underline{\phi}, s)$. *Then there exists* $\Pi_{\phi_H} \subseteq \Pi^{\Sigma_0}_{\phi_H}|_H$, *which can be constructed from part (i), such that*

$$f^H(\underline{\phi}_H) = \sum_{\pi\in\Pi_\phi} \langle x, \pi\rangle_{\underline{\phi}} f_G(\pi) \tag{6.15}$$

for $f \in C_c^\infty(G(F))$, *where* x *is the image of* s *in* $\mathcal{S}_{\underline{\phi}}$.

It follows from this theorem and Proposition 3.12 that the central character of Π_ϕ is well defined. Since Σ_0 acts trivially on Z_G, we can define the central character of $\bar{\Pi}_\phi$ to be that of Π_ϕ. Moreover, χ_ϕ only depends on $[\phi]$.

PROPOSITION 6.27. *For* $[\phi] \in \bar{\Phi}_{\mathrm{bdd}}(G)$, *the central character of* $\bar{\Pi}_\phi$ *is equal to* χ_ϕ.

Proof. Let π_0 be the generic representation in $\bar{\Pi}_\phi$. Since $Z_G(F) = \mathbb{Z}_2$, it suffices to show that $\chi_{\pi_0}(-1) = \chi_\phi(-1)$. Suppose that G is split; then Deligne [Del76] showed that $\chi_\phi(-1) = \varepsilon(1/2, \rho_{\mathrm{std}} \circ \phi, \psi_F)$ (defined by Langlands) and Lapid [Lap04] showed that $\chi_{\pi_0}(-1) = \varepsilon(1/2, \pi_0, \rho_{\mathrm{std}}, \psi_F)$ (defined by Shahidi). In both formulas ρ_{std} is the standard representation of ${}^LG$.

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It is now known that the local Langlands correspondence for G preserves these epsilon factors (see [JS03, Liu11, JL14]); in particular,

$$\varepsilon(1/2, \rho_{\mathrm{std}} \circ \phi, \psi_F) = \varepsilon(1/2, \pi_0, \rho_{\mathrm{std}}, \psi_F).$$

So, $\chi_{\pi_0}(-1) = \chi_\phi(-1)$. Suppose that G is not split; then G has to be special even orthogonal. We can view G as an endoscopic group of the split symplectic group G_+ of the same $\bar{F}$-rank, and let $[\phi]$ map to $[\phi_+] \in \bar{\Phi}_{\mathrm{bdd}}(G_+)$ through the endoscopic embedding. Let $\pi_{+,0}$ be the generic representation in $\bar{\Pi}_{\phi_+}$. From the proof of Lemma 4.1, we have

$$\chi_{\phi_+}(-1)/\chi_{\pi_{+,0}}(-1) = \chi_\phi(-1)/\chi_{\pi_0}(-1).$$

Note that we have $\chi_{\phi_+}(-1) = \chi_{\pi_{+,0}}(-1)$ from the split case. Therefore, $\chi_\phi(-1) = \chi_{\pi_0}(-1)$. This finishes the proof. □

As a consequence of these results, the results in §§ 6.2 and 6.3 are unconditional. In fact, we could obtain stronger results, which will be summarized below.

Proposition 6.28. *Suppose that $[\phi] \in \bar{\Phi}_{\mathrm{bdd}}(G)$ and $[\pi] \in \bar{\Pi}_\phi$. If $\tilde{\pi}$ is an irreducible smooth representation of $\widetilde{G}(F)$ whose restriction to $G(F)$ contains π, then, for $\theta \in \Sigma_0$ and $\omega \in \mathrm{Hom}(\widetilde{G}(F)/G(F), \mathbb{C}^\times)$,*

$$\tilde{\pi}^\theta \cong \tilde{\pi} \otimes \omega \Longleftrightarrow \omega \in \alpha(\mathcal{S}_{\underline{\phi}}^\theta).$$

In particular, $X(\tilde{\pi}) = \alpha(\mathcal{S}_{\underline{\phi}})$.

Proof. If $\theta = \mathrm{id}$, this follows from Proposition 6.13, and we will have $X(\tilde{\pi}) = \alpha(\mathcal{S}_{\underline{\phi}})$. So, we can assume that G is special even orthogonal and $\theta = \theta_0$. Note that the direction '$\Leftarrow$' follows from Proposition 6.16. For the other direction, we suppose that $\tilde{\pi}^{\theta_0} \cong \tilde{\pi} \otimes \omega$. Then $\pi^{\theta_0} \cong \pi^g$ for some $g \in \widetilde{G}(F)$. If $\mathcal{S}_{\underline{\phi}}^{\theta_0} = \emptyset$, by Theorem 6.26, we can assume that $\pi \in \Pi_\phi$. Then $\pi^{\theta_0} \in \Pi_{\phi}^{\theta_0}$, $\pi^g \in \Pi_\phi$ and we get a contradiction. So, $\mathcal{S}_{\underline{\phi}}^{\theta_0} \neq \emptyset$ and, by Theorem 6.25, $\pi^{\theta_0} \cong \pi$. Let $\omega_0 \in \alpha(\mathcal{S}_{\underline{\phi}}^{\theta_0})$; we know that $\tilde{\pi}^{\theta_0} \cong \tilde{\pi} \otimes \omega_0$. Therefore, $\tilde{\pi} \cong \tilde{\pi} \otimes \omega\omega_0^{-1}$, which means that $\omega\omega_0^{-1} \in \alpha(\mathcal{S}_{\underline{\phi}})$. Hence, $\omega \in \alpha(\mathcal{S}_{\underline{\phi}}^{\theta_0})$. □

For $[\phi] \in \bar{\Phi}_{\mathrm{bdd}}(G)$, let us fix a character $\widetilde{\chi}_\phi$ of $Z_{\widetilde{G}}(F)$ such that $\widetilde{\chi}_\phi|_{Z_G(F)} = \chi_\phi$. We define $\bar{\tilde{\Pi}}_{\phi,\widetilde{\chi}_\phi}$ to be the subset of $\bar{\Pi}(\widetilde{G}(F))$ with central character $\widetilde{\chi}_\phi$, whose restriction to $G(F)$ is contained in $\bar{\Pi}_\phi$. Let $X = \mathrm{Hom}(\widetilde{G}(F)/Z_{\widetilde{G}}(F)G(F), \mathbb{C}^\times)$.

Proposition 6.29. *Suppose that $[\phi] \in \bar{\Phi}_{\mathrm{bdd}}(G)$ and $\widetilde{\chi}_\phi$ is chosen as above.*

(i) *The orbits in $\bar{\Pi}_\phi$ under the conjugate action of $\widetilde{G}(F)$ all have size $|\mathcal{S}_{\underline{\phi}}/\mathcal{S}_{\underline{\tilde{\phi}}}|$. If F is nonarchimedean, there are exactly $|\mathcal{S}_{\underline{\tilde{\phi}}}|$ orbits.*

(ii) *There is a natural fibration*

$$X/\alpha(\mathcal{S}_{\underline{\phi}}^{\Sigma_0}) \longrightarrow \bar{\tilde{\Pi}}_{\phi,\widetilde{\chi}_\phi} \xrightarrow{\mathrm{Res}} \bar{\Pi}_\phi/\widetilde{G}(F).$$

(iii) *There is a unique pairing*

$$[\tilde{\pi}] \longrightarrow \langle \cdot, \tilde{\pi} \rangle_{\underline{\phi}}$$

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from $\tilde{\bar{\Pi}}_{\phi,\widetilde{\chi}_\phi}/X$ *into* $\widehat{\mathcal{S}_{\underline{\tilde{\phi}}}}$, *satisfying*

$$\langle x, \tilde{\pi}\rangle_{\underline{\tilde{\phi}}} = \langle \iota(x), \pi\rangle_{\underline{\phi}},$$

where $\iota : \mathcal{S}_{\underline{\tilde{\phi}}} \hookrightarrow \mathcal{S}_{\underline{\phi}}$; π *is in the restriction of* $\tilde{\pi}$. *It sends the generic representation to the trivial character. Moreover, this map from* $\tilde{\bar{\Pi}}_{\phi,\chi}/X$ *to* $\widehat{\mathcal{S}_{\underline{\tilde{\phi}}}}$ *is injective and, when* F *is nonarchimedean, it is in fact a bijection.*

Proof. The proof essentially follows from that of Proposition 6.14, and the uniqueness of this pairing is due to the fact that $\mathcal{S}_{\underline{\phi}}$ is abelian. The last property follows from the same property of the pairing between $\bar{\Pi}_\phi$ and $\mathcal{S}_{\underline{\phi}}$. □

Finally, for the conjectural refinement of $\tilde{\bar{\Pi}}_{\phi,\widetilde{\chi}_\phi}$, we would like to state it in the following two theorems.

THEOREM 6.30. *Suppose that* $[\phi] \in \bar{\Phi}_{\mathrm{bdd}}(G)$, *and* $\widetilde{\chi}_\phi$ *is a character of* $Z_{\widetilde{G}}(F)$ *whose restriction to* $Z_G(F)$ *is* χ_ϕ. *Let* $\widetilde{\chi} = \widetilde{\chi}_\phi|_{\widetilde{Z}_F}$. *Then there exists a subset* $\bar{\Pi}_{\tilde{\phi}}$ *of* $\tilde{\bar{\Pi}}_{\phi,\widetilde{\chi}_\phi}$, *unique up to twisting by* X, *and it is characterized by the following properties:*

(i)

$$\tilde{\bar{\Pi}}_{\phi,\widetilde{\chi}_\phi} = \bigsqcup_{\omega \in X/\alpha(\mathcal{S}_{\underline{\phi}}^{\Sigma_0})} \bar{\Pi}_{\tilde{\phi}} \otimes \omega;$$

(ii) *for* $\tilde{f} \in \bar{\mathcal{H}}(\widetilde{G}, \widetilde{\chi})$, *the distribution*

$$\tilde{f}(\underline{\tilde{\phi}}) := \sum_{[\tilde{\pi}] \in \bar{\Pi}_{\underline{\tilde{\phi}}}} \tilde{f}_{\widetilde{G}}(\tilde{\pi})$$

is stable.

Recall that a θ-twisted endoscopic group H of G for $\theta \in \Sigma_0$ takes the form $H \cong M_l \times G_1 \times G_2$. Let $\widetilde{G}_i$ $(i = 1, 2)$ be the similitude group of G_i with similitude character λ_i. Suppose that $\widetilde{H}$ is the (θ, ω)-twisted endoscopic group of $\widetilde{G}$ lifted from H under Proposition 3.1; then (cf. [Mor11])

$$\widetilde{H} = \{(x, g_1, g_2) \in M_l \times \widetilde{G}_1 \times \widetilde{G}_2 : \lambda_1(g_1) = \lambda_2(g_2)\},$$

where M_l is a product of general linear groups, and $\lambda_H(x, g_1, g_2) := \lambda_1(g_1)$. For $[\phi_H] \in \bar{\Phi}_{\mathrm{bdd}}(H)$, we can assume that $\phi_H = \phi_l \times \phi_1 \times \phi_2$, where $\phi_l \in \Phi_{\mathrm{bdd}}(M_l)$, $[\phi_i] \in \bar{\Phi}_{\mathrm{bdd}}(G_i)$ $(i = 1, 2)$. Fix a character $\widetilde{\chi}_{\phi_H}$ of $Z_{\widetilde{H}}(F)$, which is the restriction of some character $\chi_{\phi_l} \otimes \widetilde{\chi}_{\phi_1} \otimes \widetilde{\chi}_{\phi_2}$ of $M_l \times \widetilde{G}_1 \times \widetilde{G}_2$; then, by Theorem 6.30, we can define $\bar{\Pi}_{\tilde{\phi}_H}$ to be the restriction of $\Pi_{\phi_l} \otimes \bar{\Pi}_{\tilde{\phi}_1} \otimes \bar{\Pi}_{\tilde{\phi}_2}$, which is unique up to twisting by $\mathrm{Hom}(\widetilde{H}(F)/Z_{\widetilde{H}}(F)H(F), \mathbb{C}^\times)$.

THEOREM 6.31. *Suppose that* $[\phi] \in \bar{\Phi}_{\mathrm{bdd}}(G)$, *and* $\widetilde{\chi}_\phi$ *is a character of* $Z_{\widetilde{G}}(F)$ *whose restriction to* $Z_G(F)$ *is* χ_ϕ. *Let* $\widetilde{\chi} = \widetilde{\chi}_\phi|_{\widetilde{Z}_F}$. *Suppose that* $\theta \in \Sigma_0$, s *is a semisimple element in* $S_{\underline{\phi}}^\theta$ *and* $(H, \underline{\phi}_H) \to (\underline{\phi}, s)$. *Let* x *be the image of* s *in* $\mathcal{S}_{\underline{\phi}}^\theta$, *and* $\omega = \alpha(x)$. *Fix a packet* $\bar{\Pi}_{\tilde{\phi}_H}$ *with* $\widetilde{\chi}_{\phi_H}|_{Z_{\widetilde{G}}} = \widetilde{\chi}_\phi \chi_{\widetilde{C}}$ *(cf. § 3.4); then we can choose* $\bar{\Pi}_{\tilde{\phi}}$ *in Theorem 6.30 such that*

$$\tilde{f}^{\widetilde{H}}(\underline{\tilde{\phi}}_H) = \sum_{[\tilde{\pi}] \in \bar{\Pi}_{\tilde{\phi}}} \tilde{f}_{\widetilde{G}^\theta}(\tilde{\pi}, \omega), \quad \tilde{f} \in \bar{\mathcal{H}}(\widetilde{G}, \widetilde{\chi}), \tag{6.16}$$

where $A_{\tilde{\pi}}(\theta, \omega)$ is normalized in a way so that if $f \in \bar{\mathcal{H}}(G, \chi)$ is the restriction of $\tilde{f}$ on $G(F)$, then

$$(\tilde{f}|_{\widetilde{Z}_F G(F)})_{\widetilde{G}^\theta}(\tilde{\pi}, \omega) = \sum_{\pi \in \tilde{\pi}|_G} \langle x, \pi^+ \rangle_{\underline{\phi}} f_{G^\theta}(\pi), \tag{6.17}$$

where π^+ is an extension of π to $G^+(F) := G(F) \times \langle\theta\rangle$ with $\pi^+(\theta) = A_\pi(\theta)$.

Remark 6.32. (i) Theorems 6.30 and 6.31 are the main local results in [Xu15]. Their proofs involve global methods, and the main tool is the stabilization of the twisted Arthur–Selberg trace formula due to Mœglin and Waldspurger.

(ii) If F is archimedean, both theorems will follow from Theorem 6.25 directly. This is clear when $F = \mathbb{C}$ for $\widetilde{G}(\mathbb{C}) = Z_{\widetilde{G}}(\mathbb{C})G(\mathbb{C})$. When $F = \mathbb{R}$, it is known by results of Harish-Chandra (cf. [Har75, Theorem 27.1]) that if $\bar{\Pi}_\phi$ consists of discrete series representations of $G(\mathbb{R})$, then $X(\tilde{\pi}) = X$ for any $[\tilde{\pi}] \in \bar{\tilde{\Pi}}_{\phi, \widetilde{\chi}_\phi}$. So, $\bar{\Pi}_{\tilde{\phi}} = \bar{\tilde{\Pi}}_{\phi, \widetilde{\chi}_\phi}$. Moreover, for $\widetilde{Z}_{\mathbb{R}} = Z_{\widetilde{G}}(\mathbb{R})$ and $\tilde{f} \in \bar{\mathcal{H}}(\widetilde{G}, \widetilde{\chi}_\phi)$,

$$\tilde{f}(\underline{\tilde{\phi}}) = \frac{1}{|X|} \sum_{\omega \in X} \sum_{[\tilde{\pi}] \in \bar{\Pi}_{\tilde{\phi}}} \tilde{f}_{\widetilde{G}}(\tilde{\pi} \otimes \omega) = \frac{1}{|X|} \sum_{\omega \in X} (\tilde{f} \otimes \omega)(\underline{\tilde{\phi}}) = (\tilde{f}|_{Z_{\widetilde{G}}(\mathbb{R})G(\mathbb{R})})(\underline{\tilde{\phi}}) = f(\underline{\phi}),$$

where $f \in \bar{\mathcal{H}}(G, \chi_\phi)$ is the restriction of $\tilde{f}$. So, the stability of $\bar{\Pi}_{\tilde{\phi}}$ follows from that of $\bar{\Pi}_\phi$. For general tempered L-packets, they can be constructed by parabolic induction from the discrete series L-packets of Levi subgroups of $\widetilde{G}$. For (6.16), by a standard descent argument we can reduce it to the case that H is *elliptic* (i.e., $H = G_1 \times G_2$) and $\bar{\Pi}_{\phi_H}$ consists of discrete series representations of $H(F)$. In this case, by Proposition 6.28, one can check that $X(\tilde{\pi}) = X$ for any $[\tilde{\pi}] \in \bar{\tilde{\Pi}}_{\phi, \widetilde{\chi}_\phi}$ (cf. [Xu15, Proposition 6.9]). Let $\widetilde{Z}_{\mathbb{R}} = Z_{\widetilde{G}}(\mathbb{R})$; then the right-hand side of (6.16) becomes

$$\sum_{\tilde{\pi} \in \bar{\Pi}_{\tilde{\phi}}} \tilde{f}_{\widetilde{G}^\theta}(\tilde{\pi}, \omega) = \sum_{\tilde{\pi} \in \bar{\Pi}_{\tilde{\phi}}} (\tilde{f}|_{Z_{\widetilde{G}}(\mathbb{R})G(\mathbb{R})})_{\widetilde{G}^\theta}(\tilde{\pi}, \omega) = \sum_{\pi \in \bar{\Pi}_\phi} \langle x, \pi^+ \rangle_{\underline{\phi}} f_{G^\theta}(\pi).$$

One can also check that $\lambda_H(Z_{\widetilde{H}}(\mathbb{R})) = \lambda(Z_{\widetilde{G}}(\mathbb{R}))$. As a result, under $Z_{\widetilde{G}} \hookrightarrow Z_{\widetilde{H}}$, we have $Z_{\widetilde{H}}(\mathbb{R})H(\mathbb{R}) = Z_{\widetilde{G}}(\mathbb{R})H(\mathbb{R})$. So, the left-hand side of (6.16) becomes

$$\tilde{f}^{\widetilde{H}}(\underline{\tilde{\phi}}_H) = (\tilde{f}^{\widetilde{H}}|_{Z_{\widetilde{H}}(F)H(\mathbb{R})})(\underline{\tilde{\phi}}_H) = (\tilde{f}^{\widetilde{H}}|_{Z_{\widetilde{G}}(\mathbb{R})H(\mathbb{R})})(\underline{\tilde{\phi}}_H) = (\tilde{f}|_{Z_{\widetilde{G}}(\mathbb{R})G(\mathbb{R})})^{\widetilde{H}}(\underline{\tilde{\phi}}_H).$$

By Lemma 3.9, $(\tilde{f}|_{Z_{\widetilde{G}}(\mathbb{R})G(\mathbb{R})})^{\widetilde{H}}(\underline{\tilde{\phi}}_H) = \widetilde{f^H}(\underline{\tilde{\phi}}_H) = f^H(\underline{\phi}_H)$. Therefore, (6.16) follows from (6.13) in this case.

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Appendix A

Let F be a local field of characteristic zero and let G be a quasisplit connected reductive group over F. In this appendix, we would like to recall Langlands' construction of

$$H^1(W_F, Z(\widehat{G})) \longrightarrow \mathrm{Hom}(G(F), \mathbb{C}^\times), \tag{A.1}$$

and we will also show that it is an isomorphism. To define this homomorphism, we first need to take a z-extension of G

$$1 \longrightarrow Z \longrightarrow \widetilde{G}' \longrightarrow G \longrightarrow 1,$$

where $G' := \widetilde{G}'_{\mathrm{der}}$ is simply connected and $H^1(F, Z) = 1$. Let $\widetilde{G}'/G' = D$; we have an exact sequence

$$1 \longrightarrow G' \longrightarrow \widetilde{G}' \xrightarrow{\lambda'} D \longrightarrow 1.$$

Since $\widehat{G}'$ is adjoint, $\widehat{D} \cong Z(\widehat{\widetilde{G}'})$ and hence $H^1(W_F, Z(\widehat{\widetilde{G}'})) \cong H^1(W_F, \widehat{D}) \cong \mathrm{Hom}(D(F), \mathbb{C}^\times)$ by the local Langlands correspondence for tori. By pulling back quasicharacters of $D(F)$ to $\widetilde{G}'(F)$, we then get a homomorphism

$$H^1(W_F, Z(\widehat{\widetilde{G}'})) \to \mathrm{Hom}(\widetilde{G}'(F), \mathbb{C}^\times). \tag{A.2}$$

Next, we consider the following Γ_F-equivariant exact sequence:

$$1 \longrightarrow Z(\widehat{G}) \longrightarrow Z(\widehat{\widetilde{G}'}) \longrightarrow \widehat{Z} \longrightarrow 1.$$

It induces a long exact sequence

$$\pi_0(\widehat{Z}^{\Gamma_F}) \longrightarrow H^1(W_F, Z(\widehat{G})) \longrightarrow H^1(W_F, Z(\widehat{\widetilde{G}'})) \longrightarrow H^1(W_F, \widehat{Z}).$$

By Tate–Nakayama duality, we have $\pi_0(\widehat{Z}^{\Gamma_F}) \cong H^1(F, Z)^* = 1$. So, we get an inclusion $H^1(W_F, Z(\widehat{G})) \hookrightarrow H^1(W_F, Z(\widehat{\widetilde{G}'}))$. On the other hand, $\widetilde{G}'(F)/Z(F) \cong G(F)$, so we also have an inclusion $\mathrm{Hom}(G(F), \mathbb{C}^\times) \hookrightarrow \mathrm{Hom}(\widetilde{G}'(F), \mathbb{C}^\times)$. Then (A.1) is defined to satisfy the following commutative diagram:

$$\begin{array}{ccccccc}
1 & \longrightarrow & H^1(W_F, Z(\widehat{G})) & \longrightarrow & H^1(W_F, Z(\widehat{\widetilde{G}'})) & \longrightarrow & H^1(W_F, \widehat{Z}) \\
\| & & \downarrow{\scriptstyle (A.1)} & & \downarrow{\scriptstyle (A.2)} & & \downarrow{\scriptstyle \simeq} \\
1 & \longrightarrow & \mathrm{Hom}(G(F), \mathbb{C}^\times) & \longrightarrow & \mathrm{Hom}(\widetilde{G}'(F), \mathbb{C}^\times) & \longrightarrow & \mathrm{Hom}(Z(F), \mathbb{C}^\times)
\end{array}$$

To show that (A.1) is an isomorphism, from this diagram it is enough to know that (A.2) is an isomorphism. Since G' is semisimple simply connected, $\mathrm{Hom}(G'(F), \mathbb{C}^\times) = 1$, which implies that (A.2) is surjective. For the injectivity, we need to show that $\lambda'(\widetilde{G}'(F)) = D(F)$. We choose a maximal torus $\widetilde{T}'$ of $\widetilde{G}'$ and let $T' = \widetilde{T}' \cap G'$. The short exact sequence

$$1 \longrightarrow T' \longrightarrow \widetilde{T}' \xrightarrow{\lambda'} D \longrightarrow 1$$

induces the following exact sequence:

$$\widetilde{T}'(F) \xrightarrow{\lambda'} D(F) \xrightarrow{\delta_{T'}} H^1(F, T').$$

By Tate–Nakayama duality, $H^1(F,T') \cong \pi_0(\widehat{T}'^{\Gamma})^*$. Now let T' be the Levi component of a Borel subgroup B' of G'; we fix a Γ-splitting $\{\widehat{B}', \widehat{T}', \{\mathcal{X}'_\alpha\}\}$ for $\widehat{G}'$. Then there is a Γ-equivariant isomorphism

$$\widehat{T}' \longrightarrow \prod_\alpha \mathbb{C}^\times_\alpha,$$

$$t \longmapsto (\alpha^\vee(t)),$$

where $\mathbb{C}^\times_\alpha = \mathbb{C}^\times$, $\alpha^\vee$ are simple coroots of (G', T'), and the Γ-action on $\prod_\alpha \mathbb{C}^\times_\alpha$ is given by permutations on the indexing set of simple roots. Clearly, $\widehat{T}'^{\Gamma} \cong (\prod_\alpha \mathbb{C}^\times_\alpha)^\Gamma$ is connected, i.e., $\pi_0(\widehat{T}'^{\Gamma})^* = 1$. This implies that $\lambda'(\widetilde{T}'(F)) = D(F)$ and hence $\lambda'(\widetilde{G}'(F)) = D(F)$.

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Bin Xu bin.xu2@ucalgary.ca

Department of Mathematics and Statistics, University of Calgary,
2500 University Drive NW, Calgary, Alberta, Canada, T2N 1N4